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VaR and Liquidity Risk

Impact on Market Behaviour and Measurement Issues

Luca Erzegovesi(*)

Abstract

Current trends in international banking supervision following the 1996 Amendment to the Basel Accord emphasise market risk control based upon internal Value-at-risk (VaR) models. This paper discusses the merits and drawbacks of VaR models in the light of their impact on market liquidity. After a preliminary review of basic concepts and measures regarding market risk, market friction and liquidity risk, the arguments supporting the internal models approach to supervision on market risk are discussed, in the light of the debate on the limitations and possible enhancements of VaR models. In particular, adverse systemic effects of widespread risk management practices are considered. Risk measurement models dealing with liquidity risk are then examined in detail, in order to verify their potential for application in the field. We conclude that VaR models are still far from effectively treating market and liquidity risk in their multi-faceted aspects. Regulatory guidelines are right in recognising the importance of internal risk control systems. Implementation of those guidelines might inadvertently encourage mechanic application of VaR models, with adverse systemic effects.

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1. Introduction

Until the second half of 1998, more than ten years after the 1987 stock market crash, financial markets were supposed to have learnt how to ensure their own stability. Financial instability with widespread effects in the foreign currency, stock, credit and real estate markets could still be a menace for developing economies, as the 1997 crisis in the Far East had demonstrated. On the contrary, industrial countries with sound monetary and budget policies and developed financial markets were deemed capable of resisting even severe shocks without serious real consequences. According to the vision expressed by the financial industry (see Group of Thirty (1993)), supported by eminent financial economists (e.g. Miller (1992)), awareness of risk and availability of sophisticated tools and methods for its effective management had opened the way to a new era of financial stability where markets could be capable of self regulating their exposures through risk transfer by means of derivative instruments and insurance contracts. In this perspective the impressive growth of derivative markets, far from being a cause of destabilising speculation, had been making the world a safer place.

If this optimistic view is shared, the scope for intervention by monetary and supervisory authorities tends to be narrowly defined. Monetary control is responsible for maintaining on orderly macro-financial context, where the evolution of key factors (inflation, interest rates, currency rates, the stock market) exhibits a "healthy" volatility and follows manageable stochastic processes. This is the required environment for modern risk management to be effective. Prudential regulation is responsible for setting minimum quantitative standards for capital adequacy and qualitative standards for the internal controls framework. Only in exceptional circumstances routine control and supervision must be reinforced by ad hoc measures in order to avoid disaster scenarios (e.g. the easing of monetary conditions that was decided in the US after the Stock Market crash of October 1987). The occurrence of losses that cannot be absorbed by the capital cushion of financial institutions should become an event with negligible probability.

Current trends in international banking supervision (see Basel Committee on Banking Supervision (1996)) emphasise risk control based upon estimates of potential losses by means of internal Value-at-risk models. Bank supervisors are responsible of monitoring and validating bank internal models, which implies evaluating not only algorithms, procedures and systems, but also the adequacy of the entire framework for internal risk controls in its organisational, technological and professional aspects.

The events preceding and following the 1998 Russian crisis have brought to the fore a number of issues that had been neglected before. Excessive volatility and liquidity risk have become key concerns of international banks and supervisory authorities. In the common opinion, the débacle of the Long Term Capital Management hedge fund has demonstrated that assertive application of sophisticated financial models can lead to disaster. Even in less critical situations, the robustness and reliability of risk management models has been challenged, and flaws of those models have made themselves apparent, together with worrying side-effects on market liquidity and volatility.

The key questions discussed in this paper are the following: in the wake of the recent crisis episodes, do risk management systems remain a viable tool, or do they promise to become a better tool, in order to ensure solvency of traders and investors and foster liquidity in financial markets? Are supervisory authority right in extending the scope of application of internal models within the new capital adequacy framework?

These questions are discussed here with special regard to liquidity risk. The paper is organised as follows. In the first introductory section, we review the basic concepts and measures used in the paper regarding market risk (Value-at-Risk, or *VaR*, models), market friction and liquidity risk. In the second section we discuss the arguments supporting the internal models approach to supervision on market risk, in the light of the debate on the limitations and possible enhancements of Value-at-Risk models. In particular, adverse systemic effects of widespread risk management practices are considered. In the third section, we analyse that particular branch of risk measurement models dealing with our key focus—liquidity risk. We discuss their potential to overcome the shortcomings of standard *VaR* in the estimation and prevention of potential losses originated by market friction. In the final section, concluding remarks are presented.

2. Basic Concepts and Measures

2.1 Value-at-Risk (*VaR*)

As stated in McNeil (1999), the standard mathematical approach to modelling market risks uses the language of probability theory. Risks are random variables *X* mapping unforeseen future states of the world into values *x* representing profits and losses. The potential values of a risk have a probability distribution, mathematically represented as a distribution function, $F(x) = P(X \leq x)$, which we will never observe exactly although past losses due to similar risks, where available, may provide partial information about that distribution.

Market risk measurement consists of the estimation of potential losses given assumptions on the distribution of returns of relevant risk factors (interest rates, currency rates, stock index values, option volatilities, risk premia) Losses for a given asset are mapped onto the distribution(s) of returns of its underlying risk factor(s) by means of an appropriate payoff and pricing function. In current practice, risk is measured as **Value-at-Risk** (***VaR***), defined as the maximum potential loss for a given degree of probability. More formally, *VaR* is obtained from the quantile function $F^{-1}(p)$ [1] of the distribution of losses.

- Suppose *X* models the losses on a certain financial position over a certain time horizon. ***VaR*** can then be defined as the *p*-th quantile of the distribution of the possible losses, i.e. the worst case loss that can be obtained with probability *p*:

$$VaR_p = F^{-1}(p) \tag{1}$$

In risk management jargon, the value of the *p* probability is called the *degree of confidence* for which *VaR* is computed.

> According to the usual convention in the Basel regulatory capital framework, *X* is measured as the negative of returns (losses have positive value) and p is the probability of obtaining a loss smaller than VaR_p.

a) Standard Parametric VaR

Standard VaR, also known as *Parametric*, or *Variance-Covariance VaR*, is computed with a procedure assuming a multivariate normal joint distribution of *asset* returns. In the parametric approach a variance-covariance matrix **V** is specified for the random vector of asset returns. Variances and covariances are estimated on recent observations of the time-series of returns, i.e. they are related to historical volatility and correlation. Standard practice is based on exponentially weighted historical volatilities and correlations, or estimates obtained from multivariate GARCH models. In the parametric setting, *VaR* is simply a multiple of the standard deviation of the distribution of portfolio returns, with the multiple factor increasing with the assumed degree of confidence. Given **x** as the vector of asset weights in the portfolio, we obtain standard *VaR* for a given probability *p* as

$$SVaR_p = W\phi_p\sigma, \tag{2}$$

[1] Let *X* be a random variable with distribution function *F*. The quantile function is the inverse of *F*, giving the smallest value *x* of *X* for which *F* is greater than or equal to a given argument value *p*:

$$F^{-1}(p) = \inf\{x \mid F(x) \geq p\} \quad with \quad p \in (0,1)$$

where W is current market value of the portfolio, ϕ_p is the "confidence interval multiplier" given by the p-quantile of a standardised normal distribution and σ is the standard deviation of portfolio returns over the given horizon, obtained with the familiar Markowitz formula

$$\sigma = \sqrt{\mathbf{x}'\mathbf{V}\mathbf{x}} \qquad \textbf{(3)}$$

where $\mathbf{x}'$ denotes the $\mathbf{x}$ vector transposed.

> The weight for asset i, x_i, is the sensitivity of market value of positions exposed to the price of asset i scaled on the market value of the portfolio W, i.e. divided by W. Such sensitivities are equal to market value for cash positions. The sum of the weights is 1 if the portfolio is composed by cash positions only, it can be different from 1 if the portfolio includes derivatives. The expression used here for standard VaR is different from the usual formulation to be found in books on the subject (see e.g. Jorion (2000)). It is used here because it makes apparent the link between standard VaR and the standard deviation of portfolio returns.

Parametric *VaR* may also be computed assuming other parametric distributions, such as the Student-t distribution, in order to account for fat-tailed returns.

Standard *VaR* procedure gained popularity when JP Morgan made publicly available its RiskMetrics™ model, described in JP Morgan and Reuters (1996). Among banks all over the globe, standard *VaR* has been, or still is, the first step in the implementation of market risk management systems. The formula is terse and can be computed with easily obtainable data on historical volatilities and correlations. Simplicity comes at the expense of extensibility: parametric *VaR* can be safely applied only to linear instruments, i.e. instruments whose payoff exhibit a linear relationship with the underlying asset price, such as cash, forward and future positions. Option non-linear price risk—or Gamma risk—can be treated with cumbersome extensions (the so called Delta-plus method). Option volatility risk—or Vega risk—could in theory be included in the model, if only data on volatility of volatilities were available, together with additional correlations (prices vs. volatilities and volatilities vs. volatilities). In practice, parametric *VaR* is computed only for general price risk, stemming from broad market factors such as key interest rates, stock indices, currencies and commodities.

b) Historical Simulation VaR

When the observed distribution of returns cannot be fitted by any parametric model, we can resort to computing *VaR* with historical simulation. In that case we sample historical returns for the assets in the portfolio and, given its current composition, we simulate a time series of *portfolio* returns. *VaR* is then computed as the p-quantile[2] of the empirical distribution obtained in this way. Historical *VaR* is assumption-free: we do not need neither specifying a shape for the multivariate distribution, nor estimating volatilities for individual asset returns and correlation coefficients, since the dependence across returns is implicitly represented by their past joint behaviour. Such a method is not feasible when we lack an adequately long price history, e.g. for new markets and instruments. It is not recommendable when current market scenario could evolve along lines that have no correspondence in past market conditions

c) Static Monte Carlo VaR

When we lack adequate historical data on asset returns, and/or manage portfolios with sizeable positions in non-linear instruments, or assume arbitrary shapes for the distributions of asset returns, we must resort to Monte Carlo *VaR*. Here we describe static Monte Carlo *VaR*, i.e. *VaR* that is computed, as in the standard case, for the current portfolio composition over a one-period horizon. There are also dynamic procedures, but we will not deal with them in this

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[2] The *empirical quantile* is a simple non-parametric estimator of the p-quantile from an ordered sample of $X = (X_{1,n}, ..., X_{n,n})$ (see Bassi, Embrechts and Kafetzaki (1997:7)):

$$\hat{x}_{p,n} = \hat{F}_n^{-1}(p) = X_{k,n}, \quad 1 - \frac{k}{n} < p \le 1 - \frac{k-1}{n}$$

paper[3]. Given our assumptions on the joint distribution of returns over a given horizon (shape, expected values, variances, higher moments, correlations or other measures of dependence frequency of outliers, etc.) we generate a high number of simulation runs. For each run, a random vector of asset returns over the assumed horizon is computed and each asset in the portfolio, as well as the whole portfolio, are re-evaluated accordingly. At the end of the simulation process, we obtain a sample of the distribution of portfolio returns over which *VaR* can be computed as the empirical quantile for the given degree of confidence. Monte Carlo methods are computer-intensive, but they allow a precise estimation of the distribution of returns for assets whose payoff is a non-linear function of the underlying asset return, as is the case of options.

2.2 Liquidity and Liquidity Risk

a) Definitions of Liquidity and Friction

The concept of liquidity can be applied either to markets or to firms. Liquidity of firms has to do with their ability to maintain a prospective equilibrium between cash inflows and outflows, ensuring smooth coverage of payments on the firm's liabilities. Liquidity management is at the heart of financial intermediation. A financial institution facing difficulties in meeting its financial obligations experiments the so-called *funding risk*. Management of funding risk requires projection of future cash flows, together with identification of possible causes of unexpected future imbalances, as detailed in Basel Committee on Banking Supervision (2000).

In this paper we shall focus on liquidity in financial markets. Financial markets' liquidity, on the one hand, and banks' and firms' liquidity, on the other hand, are clearly intertwined. Investigation of such relationship is beyond the scope of the present work, where market liquidity risk in trading activity is the key issue.

Market liquidity is an elusive and multi-faceted concept. A broad definition is the following:

- a **liquid market** is a market where participants can rapidly execute large-volume transactions with a small adverse impact on prices.

In market traders' view, liquidity is defined by the ease with which an operator can enter and exit for a given block of securities. According to Taleb (1997:68)

> "If one were to summarise what trading (as opposed to investing) is about, the best answer would be adequate management (and understanding) of liquidity. Liquidity is the source of everything related to markets".

In order to identify and measure costs and risks arising from imperfect liquidity, we can borrow an insightful statement from Stoll (2000), defining the opposite to liquidity, i.e. *friction*:

> **"Friction** in financial markets measures the difficulty with which an asset is traded. Friction could be measured by how long it takes optimally to trade a given amount of an asset (Lippman and McCall (1986)). Alternatively, friction can be measured by the price concession needed for an immediate transaction (Demsetz (1968)). The two approaches converge because the immediate price concession can be viewed as the payment required by another trader, such as a dealer, to buy (or sell) the asset immediately and then dispose of (acquire) the asset according to the optimal policy".

In order to measure liquidity risk as a source of potential losses the approach by Demsetz (1968) is more practical.

[3] Dynamic scenario simulation, including dynamic Monte Carlo models, has been pioneered by Algorithmics, a leading vendor of risk management software, with its *Mark-to-future*^{TM} framework. Quoting from the company's material, "Mark-to-Future is a robust and forward-looking framework that integrates disparate sources of risk. By explicitly incorporating the passage of time, the evolution of scenarios over time, and the dynamics of portfolio holdings over time, Mark-to-Future provides a flexible and unifying platform for assessing future uncertainty" (see http://www.mark-to-future.com). We could add that a framework of this kind is just an enabling technology, allowing flexible modelling of the future behaviour of a portfolio management activity. Feeding appropriate information about future market conditions and knowledge about the repositioning behaviour to be simulated requires a huge amount of time and effort, and one may be sceptical, with reason, about the feasibility of a comprehensive representation of the relevant future states of the world.

Friction is related to the compensations paid by demanders of immediacy (active traders who place market orders to trade immediately) to suppliers of immediacy (passive traders, such as market makers, who stand ready to trade at prices they quote). The types of intermediaries supplying liquidity and the “pricing” of immediacy changes with the microstructural features of markets. We shall refer to a general case with generic market makers quoting a mid price and a bid-ask spread (static component) for a limited amount and adjusting quotes (price impact, dynamic component) in response to the flow of orders. The cases of pure and mixed order driven markets can be considered as a variation on the dealer market case, where the role of market-makers is performed by a varying population of traders, who may perform that function explicitly—e.g. locals in open-outcry derivatives markets—or occasionally—as is the case of operators placing limit orders. However, in a pure order driven market there are no bid and offer quotes backed by dealers who stand ready to trade at those prices, and therefore a measure for the price of immediacy is not readily available.

For references on the impact on friction of market design (dealer vs. order-driven auction markets) see Stoll (2000:1485).

> According to Persaud (2000), a factor draining liquidity since 1999 has been the rise in the number of different ways of transacting, such as electronic alternative trading systems (ATS) and electronic communication networks (ECNs). ECNs such as Instinet and Island have recently taken a large part of the market share of trading on Nasdaq (the world’s second largest equity trading market) and Electronic Broking System has established a significant share of the foreign exchange market. These systems operate well when markets are large and participants have different views. As a result, they draw liquidity away from outside those systems. However, when markets are small or participants adopt the same view, no dealer is obliged to make a market on the crossing network or broking systems, so liquidity vanishes and little is left outside those systems to help.

In market microstructure research, market liquidity is assessed along three possible dimensions:

- **Tightness** is how far quotes and transaction prices diverge from mid-market prices, and can generally be measured by the bid-ask spread.
- **Depth** denotes either the volume of trades possible without affecting prevailing market prices, or the amount of orders on the order-books of market makers at a given time; the deeper a market, the lesser the price impact of trades on that market.
- **Resiliency** refers to the speed with which price fluctuations resulting from trades are dissipated, or the speed with which imbalances on the order flows are adjusted.

b) Determinants of Friction

Liquidity can be modelled by a price-quantity function. Suppliers of immediacy stand ready to trade at bid-ask quotes up to a given amount. Larger purchase orders are fulfilled at an increasing ask price, and sale orders at a decreasing bid price.

Practitioners are well-aware of this phenomenon. It can be defined and measured as price impact.

- **Price impact** (or **slippage**), as defined in Taleb (1997:68), is computed (for a given quantity to execute) by taking the variation between the average execution price and the initial middle point of the bid and the offer[4]. Slippage in not a precise measure of liquidity for a particular commodity, but it provides a reliable comparative measurement of liquidity between markets.

The **quoted half-spread** can be seen as an *ex ante* price impact, fixed by market makers and added to, or subtracted from, the mid quote on limited traded amounts. Ex post, it is part of the realised impact. The shape of the price impact function beyond a normal transaction size is not observable ex ante. It may only be inferred ex post from the behaviour of prices with respect to traded volumes.

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[4] The product (slippage × position size) gives the *implementation shortfall*, a measure of transaction cost widely adopted in practice, see below p.29, The Almgren-Chriss Model.

The economic determinants of transaction costs have been extensively analysed in market microstructure literature. Broad consensus has been reached as to the determinants of cross-sectional differences in the quoted spread—the objectively measurable component of friction—across securities and markets. Empirical evidence for stock markets commented in Stoll (2000:1480-83), extensible to other asset markets, shows that quoted proportional spreads (i.e. the ratio of half the bid-ask spread over the closing price) are negatively related to measures of trading activity, such as volume and positively related to a stock's volatility. The rationale for this explaining variables is based primarily on order processing and inventory considerations. Increases in volume, number of trades and firm size increase the probability of locating a counterparty, thereby reducing the risk of accepting inventory. The stock's return variance measures the risk of adverse price change of a stock put into inventory.

Following Stoll (2000), we consider two classes of factors coming into play in explaining this strong empirical relation: real versus informational sources of friction.

As to *real factors*, first, the supply of immediacy, like any business activity, requires *real economic resources*—labour and capital—to route orders, to execute trades, and to clear and settle trades. Second, dealers assume unwanted *inventory risk* for which compensation must be provided. A third factor, *market power*, may allow dealers to increase the spread relative to their costs. Trading friction, in this approach, depends on the amount of real resources used up (or extracted as monopoly rents) to accomplish trades. Theoretical papers underlying the real friction view of the spread include Garman (1976a), Stoll (1978), Amihud and Mendelson (1980), Cohen, Maier, Schwartz and Whitcomb (1981), Ho and Stoll (1981), Ho and Stoll (1983) and Laux (1995).

Later views of friction relied on *informational factors*. Under this view, the spread is the value of the information lost by dealers to more timely or better informed traders. The spread, or a part of it, exists to provide protection against losses, offsetting the redistribution of wealth from dealers to better informed traders.

The informational view of the spread has two intellectual branches. One branch (see e.g. Copeland and Galai (1983)) views the spread as the value of the free trading option offered by those posting quotes. The second and more prevalent informational branch assumes the presence of asymmetric information. A supplier of immediacy faces the danger that a bid or ask will be accepted by some one with superior—or adverse—information. Informed traders buy at the ask if they have information justifying a higher price, they sell at the bid if they have information justifying a lower price. When the information becomes known, informed traders gain at the expense of the suppliers of immediacy. As Treynor (1971) noted, the equilibrium spread must at least cover such losses.

In Bangia, Diebold, Schuermann and Stroughair (1999:4), determinants of market liquidity are classified into two broad categories: exogenous and endogenous factors.

- **Exogenous factors** are related to characteristics of the market microstructure. It is common to all market players and unaffected by the actions of any one participant (although it can be affected by the joint action of all or almost all market participants as happened in several markets in the summer of 1998). Liquidity costs are stable and may be negligible for liquid assets, characterised by heavy trading volume, small bid-ask spreads, stable and high levels of quote depth. In contrast illiquid markets are characterised by high volatility of spread, quote depth and trading volume.
- **Endogenous factors**, in contrast, are specific to one's position in the market, varies across market participants, and the exposure of any one participant is affected by his actions. It is mainly driven by the size of the position: the larger the size, the greater the endogenous illiquidity.

The distinction between exogenous and exogenous factors is for expositional purposes only. As argued in Treynor (1989), what really counts is the interplay between them, i.e. the correlation between a trader's position sign and size and the sign and size of imbalances between aggregate demand and supply in the market. If aggregate excess demand (or supply) is negligible, then the individual trader's position is key to the determination of the price impact from endogenous factors, taking exogenous factors as given. If the market expects a persistent imbalance between aggregate demand and supply, a so-called "one-way market", then even a small position on the wrong side of market activity can suffer heavy liquidation costs. This argument is key to the analysis of liquidity crises (see below p.13, Liquidity Paradoxes and Pathologies).

c) Measures of Friction

In order to build a measurement framework, transaction costs can be decomposed into three components:

- the *quoted spread*;
- a *temporary price impact* that adds to the half-spread, but is subsequently absorbed, with a lag dependent on the resiliency of the market;
- a *permanent price impact,* reflected in permanent revisions of the mid-quote and/or the tightness of the spread after the trade.

Each component can be related to a class of microstructural factors: quoted spread and temporary price impact are compensations for the real resources consumed by dealers, while the permanent impact is affected by informational factors. Separate measurement of the three components is not as straightforward as their identification. Their relative importance can be gauged from appropriate measures of transaction costs, which are briefly described hereafter following Stoll (2000). With the exception of the quoted spread, all the measures considered are very demanding in terms of the data required, since they need information on prices for every trade, as well as quotes prevailing immediately before the trade.

The **quoted and effective spreads** are static measures observable at the moment of the trade. They measure total friction. Because the spread is the cost of a round trip—two trades—the friction associated with one trade is measured by half the spread. The **quoted half-spread** is defined as

$$S = (A - B)/2\,, \qquad \textbf{(4)}$$

where A is the ask price and B is the bid price.

Because many transactions take place inside the quoted spread—because of limit orders or when the dealer guarantees the current quoted and seeks to improve on it—the quoted half-spread overstates the actual level of friction. An alternative measure of friction is the effective spread. The **effective half-spread** is defined as

$$ES = | P - M |\,, \qquad \textbf{(5)}$$

where P is the trade price and M is the quote midpoint just prior to the trade.

Market makers assess their daily performance by comparing the average price of purchases during the day to the average price of sales. If inventory does not change, this is a measure of market makers' profits. In this perspective, a new measure of trading friction can be computed: the daily traded half-spread.

The **traded half-spread** is half the difference between the average price of trades at the ask side and the average price of trades at the bid side. A trade is at the ask side if its price is closer to the ask than to the bid, and vice versa. Trades at the quote midpoint are allocated equally between the bid and ask side. Two versions of the traded spread, differing in the weighting of trades, are calculated. The first weights each trade equally. The second weights by trade volume.

The traded spread:

- is a measure of *real friction* because it reflects real earnings for suppliers of immediacy, net of losses resulting from adverse information effects;

- is an estimate of what the supplier of immediacy earns on a round trip of two trades, whereas the traded half-spread is half this amount and reflects what a trader can expect to earn on one trade;
- is equal to the quoted spread if quotes do not change in response to trades, i.e. there is only the order processing component; it is less than the quoted spread if quotes respond to trades because of the inventory and adverse information components.

Normally, the quoted spread reflects all components. Over a longer time interval, the information component can be approximated by the difference between the quoted and traded spread if the two inventory effects at the start and at the end of the observation period offset each other.

The quoted spread is a measure of what a trader placing a market order must pay when seeking immediate execution. It is a static measure in the sense that it is measured at a moment in time. Another approach is to measure the temporary price change associated with trading. For example, what is the price impact associated with a trade, or how much the price bounces back after a trade? Such approaches are dynamic—they depend on price changes through time. In fact, suppliers of liquidity earn revenues only dynamically—from favourable changes in the prices of their positions. Conversely, demanders of immediacy pay costs only dynamically—from adverse realised price changes.

By analysing the **serial correlation of price changes**, we can detect the relative importance of real versus informational factors. Roll (1984) shows that the serial covariance of price changes in an informationally efficient market with real frictions is negative and given by $\mathrm{cov} = -\frac{1}{4}S^2$.

In such a market setting, dealers succeed in cashing-in their quoted spread on pairs of trades of opposite sign, selling at the ask price and purchasing at the bid price. Assuming that mid quotes are promptly adjusted to new fundamental information, trade price changes will exhibit negative autocorrelation, the so called "bid-ask bounce". Under such assumptions, a theoretical value of the spread—**Roll implied spread—**can be inferred as $S = 2\sqrt{-\mathrm{cov}}$. If the source of the spread is totally informational, the bid-ask bounce, as Glosten and Milgrom (1985) first showed, will not be observed, for in that case the transaction price is a martingale. Thus the friction measured by the Roll measure reflects primarily non-informational (real) factors.

Stoll (1989) showed that, like traded price changes, *quote* changes exhibit negative serial covariance when the spread reflects inventory costs. This is true because suppliers of immediacy adjust quotes to induce inventory equilibrating trades. As inventory equilibrating trades occur, the quotes return to their former level. In the absence of inventory effects, *quote* changes would not exhibit negative serial covariance, although *price* changes would. Consequently, a finding that quote changes exhibit negative serial correlation would be evidence of inventory effects. Inventory effects are key to the model of dealers' mid-quote and spread described in Treynor (1987).

Negative serial covariance reflects temporary price impact. In order to discriminate between temporary and permanent impact, Stoll (2000) adopts an approach to measure price impact over the day in response to the trading imbalance (or excess demand/supply) for the day. Stoll's methodology differs from those followed in other studies, where the price impact function is fitted on high-frequency data.

- The **imbalance** for the day t, I_t, is defined as the sum of the signed trade quantities during the day expressed as a percentage of daily volume. A trade is classified as a sale if the trade price is closer to the bid than to the ask. It is classified as a purchase if the trade price is closer to the ask. Trades at midpoint are allocated half to sales and half to purchases.

The price change for the day t, ΔP_t, is measured as the change in the *quote* midpoint from close to close, adjusted for a benchmark return (e.g. return on the S&P 500 index in the case of a US stock)

$$\Delta P_t = C_t - C_{t-1}\left(1 + R_{It}\right), \tag{6}$$

where C_t is the closing midpoint on day t and R_{It} is the daily return on the benchmark index. The midpoints are used to abstract from the bid-ask bounce.

- The **permanent price impact coefficient** is λ in the following regression:

$$\Delta P_t = \lambda_0 + \lambda I_t + \lambda_2 I_{t-1} + e_t \tag{7}$$

where I_t is the percentage imbalance on day t, defined as

$$I = \frac{\sum_{i=1}^{m} w_i^A - \sum_{i=1}^{n} w_i^B}{\sum_{i=1}^{m} w_i^A + \sum_{i=1}^{n} w_i^B}(100) \tag{8}$$

and w_i^A, w_i^B are the volume of the ith purchase and sale respectively. The prior day's imbalance is included to determine if prices bounce back the day after an imbalance.

The price impact coefficient, λ, in equation (7) measures the sensitivity of the quote change over a day to the daily imbalance. Insofar as the quote change is permanent, λ measures the information content of the day's imbalance. If prices bounce back the next day, one would conclude that the price impact also reflects real factors.

> Evidence presented in Stoll (2000) from a regression analysis on transactions data for a sample of 1,706 NYSE/AMSE stocks and 2,184 Nasdaq stocks in the three months ending on February 28, 1998 indicate that there is a significant price impact. The "reversal" coefficient λ_2 is statistically significant in fewer than 5% of the individual regressions, and has mixed signs. The lack of reversal implies that the price impact coefficient reflects the information content of the net imbalance for the day. The price sensitivity to a given percentage imbalance is larger for large stocks, which exhibit also higher prices. However, the price impact is not necessarily higher, because large (and more actively traded) stocks experience smaller percentage imbalances.

Stoll (2000) takes as an **unconditional measure of price impact** for stock i the predicted price impact for the average imbalance defined as $\lambda_i \,\mathrm{Avg}\left|I_{it}\right|$, where $\mathrm{Avg}\left|I_{it}\right|$ is the average absolute imbalance in stock i over the days in the sample. This variable has dollar dimensions and, unlike other studies (e.g. Chan and Lakonishok (1993)), is based on quotes, not on trade prices, and does not include the effect of the bid-ask bounce.

Persaud (2000) illustrates a procedure for calculating a liquidity index based on the price impact of cross-border investment flows, which is a refinement of Stoll's approach (two λ_i are estimated, one for positive and the other for negative imbalances). The liquidity index for a market or security is obtained with a regression of the time series of returns for that market or security on the series of net purchases and sales by cross-border investors. Another paper analysing the same database of international portfolio flows, maintained by State Street Bank is Froot, O'Connell and Seasholes (1999).

d) Liquidity Paradoxes and Pathologies

The measures of friction described in the previous paragraph can detect evidence of cross-sectional differences in friction across markets, market segments, and securities. The same models of market microstructure are less useful in explaining changes in friction over time, together with anomalies that are frequently observed in financial markets.

As noted in Bank for International Settlements (1999), the evaporation of liquidity from some markets and the spread of illiquid conditions to other, seemingly unrelated markets following the recent Asian (1997) and Russian (1998) crises have reminded observers that the determinants and dynamics of market liquidity have yet to be fully understood. Three phenomena, in particular, are of interest:

- The **concentration** of liquidity in specific markets or instruments, often at the expense of liquidity in closely related markets.

- The **evaporation** of liquidity from markets.
- The **flight to liquidity**, with a rise in the premium investors are willing to pay to hold liquid assets, or assets perceived as having low levels of all kinds of risks. It can be regarded as a migration of activity into markets which are expected to continue to provide quotes even in times of stress. This usually happens as an aspect of a broader “flight to quality”. While *activity* may move to more liquid markets, it is not clear why *liquidity* per se increases in them.

There are several reasons why market liquidity can dry up suddenly and unexpectedly. The possible mechanisms include a re-evaluation of the credit risk of an important class of counterparties, the mutually reinforcing effects of broad shock to credit quality and market prices, and doubt about the integrity of settlement systems. Another way that liquidity may dry up would be if new information or a shock to prices leads to a severe imbalance between buyers and sellers, i.e. a “one-way market”. This is thought to have been a key factor in certain liquidity-shortage episodes in the past, such as the October 1987 equity market reversal.

Pritsker (1997:148-149) sheds light on the conditions triggering a regime shift from normal liquidity conditions to a liquidity crisis. At any time, there are two main sources of liquidity in the underlying markets. The first source of liquidity is the liquidity provided by market-makers for fixed, typically small quantities of underlying assets. As shown before, the quoted spread or the price impact associated with making a trade with a market maker is an appropriate measure of liquidity in normal market conditions. However, it is probably not reasonable in abnormal conditions. Market-makers provide immediate temporary liquidity to the market to absorb short-term order imbalances which they believe will disappear when the other side of the market eventually (hopefully soon) emerges. On abnormal market conditions, this other side of the market may be small or non-existent; in these abnormal circumstances market-makers will provide very little liquidity to the market. The most important determinant of market liquidity in the event of abnormal market conditions is not market-makers, but value investors who will presumably be willing to take the other side of the positions that market-makers are holding temporarily. Value investors willingness to provide liquidity is a function of their propensity to push back prices towards what they perceive as fundamental values when prices appear to deviate from fundamentals. When this propensity is weak, there is a distinct absence of value investors. In this case, variations in price are due to noise traders, derivatives hedgers and short-term arbitrageurs, but are not due to fundamentals. This creates a scenario where prices could wander far from long-run asset value.

> Muranaga and Shimizu (1999a) explore this issue using simulation techniques. They find that market liquidity can affect price discovery in times of stress in at least two different ways. In one simulation, it is found that the loss of market liquidity in response to a market shock sometimes performs the function of a built-in stabiliser in the market, by preventing a precipitous secondary drop in prices that would have not be warranted by fundamentals. As uncertainty increases in response to the shock, market participants become less willing to trade, and the decline in the number of orders generated, in turn, results in a loss of market liquidity. In other words, when market liquidity is low, price discovery is not conducted as often, so a crash in prices is less likely to lead to an endogenous (secondary) crash in prices that does not reflect fundamentals. In a sense, the withdrawal of liquidity breaks the self-reinforcing dynamics of market crashes and allows time for fundamentals to reassert themselves. In a second simulation, however, resting on somewhat different set of assumptions, conditions are found under which secondary crashes might develop. If market participants amend their expectations of future prices in response to a price shock and uncertainty remains low, order streams do not diminish but instead, reflecting sharply lower expected future prices, become one-way, resulting in secondary crashes.

Treynor (1987) assumes an analogous division of labour between market-makers and value investors in the market for immediacy. In Treynor’s model, the uncertainty of value investors about the true equilibrium price is reflected in the wideness of the so-called “external spread”, i.e. the difference between the ask and bid prices at which they are willing (a) to take the other side of positions “laid off” by market makers reaching their position limits, or (b) to execute block trades submitted by other investors through brokers or electronic trading systems.

A general definition of a pathological liquidity condition is given by Taleb (1997:69):

- "A **liquidity hole** or a black hole is a temporary event in the market that suspends the regular mechanics of equilibrium attainment. It is an informational glitch in the mechanism of free markets, one that can cause considerable damage to firms. In practice, it can be seen when lower prices bring accelerated supply and higher prices bring accelerated demand."

Typically, liquidity holes occur when operators are aware of a major piece of information such as a fundamental event, a political announcement, or the release of an economic figure, or a size order in the market (e.g. a stop-loss order), but cannot gauge its size and possible impact.

Liquid markets can go through liquidity holes with no particular damage. Liquidity holes become dangerous when in the market there is a large open interest in trading or investment strategies originating large orders contingent on price movements, especially when those orders need to be executed regardless of the price. Examples of these strategies are technical trading with stop-loss points, portfolio insurance, dynamic hedging of short positions in options, and, to a lesser extent, value-at-risk limits or "relative-return" investment styles following risk-adjusted performance benchmarks. Given their relevance in recent episodes, we will now consider those strategies in more detail in a later section (see p.37,

Assessing Liquidity Risk under Stress).

3. Market Risk Management and Regulatory Capital Requirements

3.1 The Amendment to the Basel Accord and the Internal Models Approach

The internal models approach to market risk was first presented in Basel Committee on Banking Supervision (1996), an Amendment to the "Basel Accord" of 1988 on capital requirements for credit risk. The 1996 Amendment establishes capital charges for the banks' trading book and, in this framework, defines a set of standards for the market risk management process which apply to banks basing their capital requirements on internal models. This provision responds to the industry' s request to allow banks to use proprietary in-house models for measuring market risks as an alternative to a standardised (or "building-block") measurement framework originally put forward in April 1993, and broadly accepted in the EU Capital Adequacy Directive[5] and by national bank supervisors.

The internal models approach requires that "value-at-risk" be computed daily, using a 99th percentile, one-tailed confidence interval; that a minimum price shock equivalent to ten trading days (holding period) be used; and that the model incorporate a historical observation period of at least one year. The capital charge for a bank that uses a proprietary model is set be the higher of:

1. the previous day' s value-at-risk;
2. the average of the daily value-at-risk of the preceding sixty business days multiplied by a factor of three.

The multiplication factor is designed to account for potential weaknesses in the modelling process. Such weaknesses exist for the following reasons:

- Market price movements often display patterns (such as fat tails) that differ from the statistical simplifications used in modelling (such as the assumption of a normal distribution).

[5] We refer to directives by the European Community on capital adequacy of banks: 95/15/EEC of 31 May 1995, 96/10/EEC of 21 March 1996, and 98/32/EC - 98/33/EC, both of 22 June 1998.

- The past is not always a good approximation of the future (for example volatilities and correlations can change abruptly).
- Value-at-risk estimates are typically based on end-of-day positions and generally do not take account of intra-day trading risk.
- Models cannot adequately capture event risk arising from exceptional market circumstances.

Ad said before, required capital is the higher of two quantities, the first being the worst case daily loss, which could, in exceptional situation, trespass the second, i.e. the 60-days average *VaR* multiplied by three. In order to account for losses arising from extreme events, internal models must complement the measurement of standard *VaR* by stress testing. According to the 1996 Amendment, banks' stress tests should be both of a quantitative and qualitative nature, incorporating both market risk and liquidity aspects of market disturbances. Quantitative criteria should identify plausible stress scenarios to which banks could be exposed. Qualitative criteria should emphasise that two major goals of stress testing are to evaluate the capacity of the bank's capital to absorb potential large losses and to identify steps the bank can take to reduce its risk and conserve capital. Banks should combine the use of supervisory stress scenarios with stress tests developed by the banks themselves to reflect their idiosyncratic risk characteristics.

Banks adopting internal models for regulatory purposes have to “backtest” their system in order to asses the accuracy of the daily profit and loss distribution assumed for computing *VaR*. More specifically, the number of “violations”, i.e. daily profits or losses larger, in absolute value, than the 99 percent-quantile measured by *VaR*, must not exceed 4 in 250 business days, i.e. approximately the two-sided tail-probability corresponding to the chosen quantile, i.e. 2 percent. In the presence of relevant violations the multiplication factor can be increased by supervising authorities from the minimum level of three up to a maximum of four, depending on the severity of violations.

> Note that internal models have to prove accurate in estimating losses over a 1-day horizon. The Basel rules impose a minimum capital requirement that is a multiple of the 1-day VaR. More precisely, average daily 1-day VaR is first rescaled to a 10-day horizon. Usually this is done applying the “square root of time” rule, i.e. the 1-day VaR is multiplied by $\sqrt{10} = 3.\overline{3}$. The 10-day VaR is then multiplied again by the prudential factor of three. The resulting number for minimum capital is simply 10 times daily VaR.

The Basel Committee leaves banks free as regards the choice of a preferred model for computing *VaR*, provided that it passes backtesting. Banks may choose a model based on standard variance-covariance *VaR*, or historical simulation, or Monte Carlo methods. For instance, the model may account for a variance-reducing portfolio diversification effect based on estimated correlations among broad risk factors. However, if such a model does not account for the generalised increase in correlations among markets that occurs under market stress, it could underestimate *VaR* in critical situations. The bank adopting such a flawed model would fail the backtesting exam and be penalised accordingly.

The Amendment sets some technical standards regarding the types and number of risk factors, the treatment of non-linear risks in options, the minimum length of the observation period assumed for estimating volatilities and correlations, the updating of risk parameters, and the treatment of specific risk, i.e. risk stemming from factors that are specific of a single position in a security or asset class. Moreover, qualitative standards are set regarding the approval of risk management guidelines by the Board of Directors, the adequacy of internal supervision performed by an independent risk control unit, and the external validation of the internal models, to be performed by independent consultants and/or supervisory authorities.

The endorsement of the internal models approach for market risk measurement by the Basel Commission, and its probable future extension to other risks, especially credit risk, is in the spirit of the new framework set forth more recently in Basel Committee on Banking

Supervision (1999a), where a major revision of the 1988 Accord is designed, with particular emphasis on capital charges for credit risk. The new framework rests upon three pillars:

1. *Minimum capital requirements*, mainly based on standardised ratios, but allowing for an internal models approach by some sophisticated banks for appropriate risks, subject to supervisory approval;
2. *Supervisory review of capital adequacy*, in order to ensure that a bank's capital position is consistent with its overall risk profile and strategy, giving supervisors discretionary ability to require banks to hold capital in excess of minimum regulatory capital ratios when deemed necessary. The object of supervisory controls is not only the quantity of capital set aside by banks, but also the quality of the overall internal capital assessment and risk control process.
3. *Market discipline*, fostered by high disclosure standards and by enhancing the role of market participants in encouraging banks to hold adequate capital, e.g. promoting the issuance of subordinated debt and therefore stricter monitoring of solvency risk by subordinated lenders.

3.2 Who Favours Internal Models, and Why?

The ideal risk management framework sketched in the 1996 Amendment broadly matches what is now unanimously considered "best practice" in market risk management. Such an approach can be related to the principles set forth in Group of Thirty (1993), a very influential document inspired by a panel of élite international banks, mainly based in the United States. Subsequent advances in internal business practices, as well as in regulation, can be viewed as a gradual implementation of those recommendations. It is not surprising to observe that big US banks with strong interest in investment banking were leaders in the adoption of internal models, while European banks lagged behind.

Early adopters claim that the shift to the internal models approach has saved them relevant amounts of regulatory capital. However, the savings in capital charges are not pursued in order to justify a reduction in the total capital of the bank. Those savings are rather appreciated as a way to increase the surplus capital over minimum regulatory requirements. Ample free capital is a necessary condition in order to exploit growth opportunities in a fierce competitive environment. Big global banks agree on the point that adequate capital from a bank's internal perspective should be much higher than the minimum amount required by supervisory authorities (see for instance Shepheard Walwyn and Litterman (1998)).

Such a potential advantage does not look so attractive to many banks in continental Europe, where market risk in the trading book absorbs a relatively small amount of regulatory capital if compared with what is commanded by credit risk in the banking book.

Besides of the measurable savings in regulatory capital, other more compelling arguments support the internal models approach as a better alternative to standardised capital ratios. By adopting the internal models approach, regulatory capital becomes a function of capital allocated according to internal criteria. In both cases, required capital is computed as a function of the distribution of future losses. Banks can thus unify their risk management activities under one framework, avoiding duplication of models, methods and procedures. Models computing capital are validated by supervisors and internal risk controllers alike according to criteria that share the same conceptual and technical foundations.

Within this philosophy, a bank is potentially in the best position to assess its risks and to quantify how much capital is needed to absorb them. The aim of prudential regulation is to ensure that sound methodologies and practices are followed by banks in such an assessment. But who defines what is to be considered best practice? It's the banking profession itself, although under close scrutiny by supervisory authorities. If we accept the criterion that knowledgeable assessment of risk is, despite all the known limitation of risk models, always better than applying an arbitrary figure set by regulators, extending the scope of application of internal models is only a matter of time, the time needed to experiment new models by banks plus the time needed by supervisory bodies to be convinced of their soundness.

Internal models are now a reality for a relatively small number of sophisticated banks, but the acceptance of the internal models approach by regulators has far-reaching consequences, well beyond the group of banks who have been allowed to adopt them. Regulatory approval has contributed to dignify the financial theory behind risk management models. With *VaR* models, textbook portfolio theory has officially entered bank management culture[6].

Only a few banks are allowed to compute regulatory capital from *VaR*, but all the banks are required to set up a risk control infrastructure, and *VaR* is an essential component of that infrastructure. More and more bank managers are persuaded to treat *VaR* as a better estimate of aggregate capital, or a better measure to set position limits, or a better basis for computing risk-adjusted returns. The *VaR* concept is propagating well beyond its original field of application, i.e. wholesale financial trading. Banks offering on-line trading services have been ready to sell the idea to their retail customers, as demonstrated by the huge marketing investment on KILOVAR™ by the Unicredito Italiano Group in Italy. *VaR* is increasingly popular in asset management as a metric for setting risk-adjusted performance targets.

3.3 Limitations of *VaR* Models and Possible Remedies

Is the dissemination process described above a desirable trend? In order to answer this question, a brief survey of the state-of-the-art of *VaR* implementation will be useful.

Internal models, as they are described in regulatory guidelines, are not intended as a naïve application of standard *VaR*. The 1996 Amendment makes a list of possible shortcomings, and points to several remedies. Stress testing is one of them, and the exercise of informed judgement is an even more general one. In practice, however, the art of reasonable risk management is still proprietary knowledge, shared by a limited group of élite institutions. Current practice outside that exclusive circle, i.e. what is incorporated in the systems and models that can be purchased off-the-shelf from consultancies and software vendors, is far from having implemented ideal systems. Many conceptual and practical problems have still to be resolved. So we can suppose that the average risk management system currently in use by the average bank may be partial, or flawed.

We may concede that the average risk manager in charge of running that system is fully aware of those flaws. However, when deciding how to cope with their model's limitations, risk managers cannot avoid facing a dilemma: are those shortcomings manageable with enhancements coming from research, experience and judgement, or do they challenge the viability of the entire analytical framework? In order to answer this question, two points must be made: one is about the reliability of *VaR* models in the view of the single bank adopting them. The second point regards the aggregate impact of widespread adoption of risk management models based on *VaR*.

a) The Micro-View: Technical Limitations of Standard VaR Models

"The risk of the whole bank in just one number": that was the promise of the 4:15 system adopted internally by JP Morgan, named after the time of the day when the consolidated risk figure was submitted to the bank's general manager. The same concept became popular among a much wider audience when the same bank made publicly available RiskMetrics™, a model based on, but different from, its internal system. That promise is far from being maintained still now. Reliable solutions are still to be found with regard to a number of basic problems, which are the following:

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[6] To a disenchanted observer, the concept of standard parametric *VaR* and its applications (e.g. RAROC, i.e. risk-adjusted return on capital computed as the ratio of profits over *VaR*) may appear as a smart repackaging of two basic textbook concepts: the formula of the variance of the returns on a portfolio in the Markowitz model, and the return-to-volatility ratio introduced by Sharpe as a risk-adjusted performance measure for diversified portfolios. Such parallel is highlighted in Shimko (1997b).

- *Observation frequency and time scaling of returns*. Historical volatilities are estimated on daily or a higher-frequency return series in order to obtain adequately long samples. In common practice, volatility for horizons longer than one day is obtained by applying the "square-root of time" rule, i.e. assuming stationarity and serial independence of returns. Unfortunately, observed returns exhibit patterns of serial dependence, which differ across markets, and are difficult to model assuming usual stochastic processes of returns. As a consequence, broad consensus has not been reached yet on the methodology for computing volatility over longer investment horizons.
- *Assumption of static position management*. Even for longer time-horizons, *VaR* is usually computed for current positions under the assumption that they are liquidated with a block trade at the end of the horizon. Assumption of dynamic liquidation strategies would be more reasonable and realistic.
- *Fat-tails of asset return distribution*. The distribution of realised asset returns takes an irregular shape. One worrying anomaly is the thicker than normal tails of empirical distributions. Under such conditions, *VaR* may underestimate the quantile of the distribution for high probability values (e.g. beyond 99 percent). Moreover, in the presence of thicker than normal tails, extreme losses, i.e. losses beyond *VaR*, may have a significant probability of occurrence, and should be considered explicitly[7].
- *Asymmetry of asset and portfolio returns*. Some assets (e.g. emerging countries currencies) exhibit a biased distribution, with a lower tail much higher than the upper one. The distribution of portfolio returns can take an even more irregular shape due to the non-linearity of its aggregate payoff, especially in the presence of options or option-replicating strategies. Risk measured with parametric *VaR* can in these situations lead to a distorted view of the risk-return trade-off.
- *Instability of volatilities and VaR*. Historical volatilities change with the passage of time. If *VaR* is computed on "short-memory" volatilities, it better reflects current conditions, but tends to vary more rapidly. Instability of *VaR* is disturbing for traders that must meet *VaR* position limits or are evaluated on the basis of *VaR*-adjusted performance measures[8]. Moreover, the ensuing mechanic link between market volatility and position limits has also undesirable systemic effects, which are considered in the following paragraph.
- *Instability of correlations*. Correlations that are estimated across a high number of risk factors may exhibit a strong sampling error. More worryingly, their estimates tend to swing wildly in highly volatile markets (correlations between "unsafe" markets tend to increase under stress, despite the apparent absence of fundamental linkages, as noted in

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[7] In the presence of extreme event risk, a better measure of potential loss is **Expected shortfall** , i.e. the estimate of the potential size of the loss exceeding *VaR*, for a given probability *p*. For a continuous distribution we have:

$$ES_p = \mathrm{E}\left[X \middle| X > VaR_p \right]$$

A more genera formula adaptable to discrete empirical distributions includes an adjustment in order to account for the difference between the *p* probability level and the nearest empirical frequency that is computable for a subsample of the observations.

As demonstrated in Artzner, Delbaen, Eber and Heath (1997) the expected shortfall meets the requirements of a coherent risk measure for any distribution of returns, while *VaR* is a coherent measure only under restricted distributional assumptions. Coherent risk measures meet desirable requisites, first of all the sub-additivity condition, respected when the sum of risk exposures for individual positions is always larger then the aggregate measure at the portfolio level, the latter taking diversification into account.

[8] According to Daníelsson and Morimoto (2000:1-2): "Current risk measures based on standard *VaR* prove excessively volatile as a result of their dependency of high instability of historical estimates of the variance-covariance matrix assuming multivariate normality in the presence of non-normal observed returns. […] When risk measurement methods are used, for example, to allocate position limits to individual traders, or set mandate letters for fund managers, high volatility of risk measures is a serious problem because it is very hard to manage individual positions or to accomplish risk-adjusted performance targets, with highly volatile position limits or allocated capital".

Persaud (2000)). Moreover, correlation is an appropriate measure of dependence only with elliptical joint distributions of returns, of which the multivariate normal is a special case (see Embrechts, McNeil and Straumann (1999a)).

The list of open issues could be longer, as could the list of solutions devised in order to tackle them. Solutions differ depending on the scope of *VaR* application. With an extreme simplification, we distinguish two main approaches

First, *Standard VaR*, extended to account for non-linear and specific risks, is applied as a measure of potential loss under *normal market conditions* (usually at a 95 percent confidence) for internal purposes, mainly for the purpose of allocating position limits to trading desks, or set mandate letters for fund managers. The method has its known limitations, but it is considered "better than what we had before", i.e. better than a collection of heterogeneous and arbitrary metrics for different exposures. The ideal case for standard *VaR* applications is that of a trading desk: operating in liquid instruments (low event risk), traded in one or a few closely related markets (low correlation risk), on a short time horizon (no time-scaling problems). As we go farther off these ideal setting—e.g. from a single desk up to the trading division, or from trading to asset management—inaccuracies creep in, and reliability declines. At any rate, relevant shortcomings, such as instability of *VaR* as market volatility changes, neglect of extreme losses, and possible adverse systemic effects on market volatility, still remain, even in the ideal case.

Second, formulas of *Enhanced VaR* are applied in order to estimate potential losses in *extremely adverse scenarios* (i.e. for larger than 99 percent confidence), as required in *stress testing*, and in computing *regulatory capital*. To that purpose, a number of technical improvements has been proposed in recent research, with the most promising results coming from applications of *extreme value theory*.

The main goal of stress testing is to estimate the potential loss arising from improbable but plausible events, in order to protect our portfolio with an adequate capital cushion against exceptional risks such as those associated with currency crises, stock market crashes, and large bond defaults. Stress testing, as the 1996 Basel rules describe it (see above p. 15, The Amendment to the Basel Accord and the Internal Models Approach), as well as it is applied in practice, is an unstructured simulation exercise focused on the outcome of discrete events, be they the replay on current portfolios of past episodes of crisis (e.g. the 1987 market crash, the 1994 US interest rate rise, the1998 Russian crisis), or subjective scenarios designed by the risk manager. The estimation of the entire joint distribution function of extreme losses, by means of appropriate statistical methods, would be a substantial improvement on such exercises. Unfortunately, traditional econometric methods, typically based on estimation of entire density functions, are intended to produce a good fit in regions where most of the data fall, potentially at the expense of good fit in the tails, where, by definition, few observations fall. With "extreme value theory" (EVT), one can estimate extreme quantiles and probabilities by fitting a "model" to the empirical tail probability function of a set of data using only the extreme event data rather than all the data, thereby fitting the tail, and only the tail. Even in the absence of useful historical data, EVT provides guidance on the kind of distribution we should select so that extreme risks are handled conservatively.

The use of EVT in financial risk management is a fairly recent innovation, but there is a much longer history of its use in the insurance industry. For a thorough survey of the theory underlying EVT oriented towards its application in insurance and finance see the comprehensive book by Embrechts, Klüppelberg and Mikosch (1997). A concise survey is given in the papers by McNeil (1999) and Diebold, Schuermann and Stroughair (1998).

A detailed description of EVT models and estimation methods is beyond the scope of this paper. We only report that EVT-enhanced risk measures are obtained through semi-parametric procedures: the central part of the distribution of losses is estimated via historical simulation, while the tails are modelled parametrically fitting empirical data to an *a priori* specification of the distribution function in the tail. A *tail index*, measuring thickness in extreme regions, is the crucial statistic obtained. *VaR* estimated in this way is more accurate for high *p*-values, and also generally more stable than standard *VaR*. Two arguments explain these desirable features. First, estimation of tail parameters demands large samples, which in turn favours long-memory estimation periods. As a result, the influence of short term fluctuations on fitted distributions is weaker. Second, the parametric correction of the shape of the distribution in the tails (where the sampling error is stronger) dampens fluctuations due to outliers and changing volatility. The reduction in *VaR* volatility becomes less impressive if we take into account the wide confidence intervals, especially on the upside, that surround estimates of *VaR* obtained with EVT.

EVT could help in assessing the multiplier factor that must be applied to standard *VaR* in order to account for model limitations and event risk. Basel rules prescribe an arbitrary multiplier of three, increasable to four, for any market or portfolio. Through EVT, one could differentiate such prudential add-on accounting for different characteristics of tail distributions across markets and their dependence structure.

Estimation procedures based on EVT are an example of *VaR*-enhancing techniques. Implementation of new methods is challenging. We have to set and resolve all the problems encountered in the standard case, i.e. time-scaling and aggregation at the portfolio level given appropriate estimates of dependence among risk factors. Complexity of computing models increases by an order of magnitude with respect to standard *VaR*. Solutions have been proposed—e.g. multivariate EVT with copula functions[9], Monte Carlo with EVT simulation. Current research is constantly improving on them.

> As stated in de Haan, Embrechts and Huang (1999:1), current multivariate EVT, from an applied point of view, only allows for the treatment of fairly low-dimensional problems. The latter suffice in insurance where two- or three-line products already are fairly advanced. In finance, however, a typical investment portfolio involves several hundred if not a thousand instruments. The truly multivariate extreme value analysis of such problems is well outside the reach of available theory. The same conclusion is reached in Smith (1999:14), where multivariate EVT is said to be impractical for portfolios with more than three risk factors.

However, day-to-day application of enhanced *VaR* models is still beyond the reach of the majority of financial institutions.

b) The Macro-View: the Impact of VaR models on Market Behaviour

Taking a micro-view, *VaR* is calculated from the perspective of a single trader confronting a given distribution function of market returns and a price impact function that are exogenous with respect to his behaviour. Awareness of risk exposure may well induce traders to reposition themselves as risk-return opportunities change in the market, or as risk limits are trespassed. If a substantial share of market participants adopts *VaR* models and holds similar portfolios in the same markets, they may move in herds in response to market shocks. Their response would create a one-way market, vulnerable to liquidity crises and crashes. If this contention were true, *VaR* would be a part of the problem, not of the solution, with regard to market illiquidity.

Actually, after the 1998 Long Term Capital Management bail out, a more sceptical opinion of *VaR* has emerged, insisting on systemic effects of widespread risk management activity based on *VaR* models.

Following Bank for International Settlements (2000a:29-32), to understand how a one-way market might arise, and how *VaR* may contribute to that occurrence, we can separate market participants into two groups according to how they respond to a shock. To simplify exposition,

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9 A *copula function* may be thought as a multivariate distribution function that takes as arguments the marginal probabilities of a series of random variables for a vector of values of those variables and returns the joint probability for the same vector of values. It can be used as a generalised representation of dependence across random variables. A clear introductory textbook on copulas is Nelsen (1999). For examples of application to risk measurement see McNeil (1999:15-17).

we will consider the case where traders and investors assume long positions on a given market. We label the two groups “negative feedback traders” and “positive feedback traders.” Negative feedback traders buy when the price falls, while positive feedback traders sell when the price falls. Market liquidity is then a function of the aggregation of negative and positive feedback trading. In a liquid market, negative feedback traders outweigh positive feedback traders, so that market price fluctuations are dampened and liquidity does not dry up in the wake of a stressful shock. A collapse of market liquidity occurs when the “market share” of positive feedback traders increases enough to outweigh that of negative feedback traders. The market becomes one-sided and market liquidity vanishes.

Five factors, related to trading and risk control strategies, may contribute to the dominance of positive feedback traders. We now briefly define those factors.

Stop-loss rules. A stop-loss rule requires assets to be sold once portfolio value falls by some amount (the stop-loss limit). Positive feedback trading driven by stop-loss rules will grow in the wake of a sharp decrease in price. “Stops” are used in technical trend-following trading, but non only there. Relative-value strategies and dynamic hedging of short option positions (see below) all require systematic placing of stop orders so as to adjust portfolio composition to changing prices.

Leverage and funding risk. Asset managers and arbitrageurs may be leveraged. They manage their leverage (or more generally, their risk profile, including funding risk) so they will not have to liquidate assets in normal times. In response to a large shock, they may have to liquidate assets to avoid further losses, or because their creditors withdraw credit or impose tighter margin deposits or collateralisation. The greater the leverage of those invested in the market, the higher the probability of triggering positive feedback trading in response to a stressful shock.

Limited arbitrage. Some asset managers and relative value traders normally engage in negative feedback trading, i.e. investing in assets whose price has fallen below the trader’s perception of its future value. Because investors cannot distinguish low returns due to bad luck from low returns due to bad investing strategies, fund managers with poor performance will face withdrawals by investors. As a result, following a negative shock to asset returns, the funds available to negative feedback traders will decline, so negative feedback trading falls.

Sharpe-ratio-based trading. Some asset managers and relative value traders choose among alternative investment strategies by comparing Sharpe ratios, i.e. the ratio of expected return to volatility of returns. A prolonged bull market, with returns high relative to risk, will induce traders to accumulate positions in that market, in search for high Sharpe ratios. If large negative moves, followed by an increase in expected volatility, occur in that market, a mechanical decline in a portfolio’s Sharpe ratio will result. Traders who base their asset allocation on Sharpe ratios will tend to rebalance away from stressed markets, leading them to be positive feedback traders.

Dynamic hedging or portfolio insurance. A net seller of options—or a fund manager insuring against shortfall risk—will replicate a long position in options through dynamic hedging, and will be a positive feedback trader. For instance, a dealer selling a put in a stock could hedge by selling short a number of shares equal to the option’s Delta. Since the option’s Delta varies with the price of the underlying stock, the amount of the hedge must be continually adjusted. Because the option’s Delta rises as the stock price falls, dynamic hedging leads to additional selling of the stock during price declines. The amount of underlying assets sold as part of a dynamic hedging strategy following a negative shock grows with the size of the shock, so positive feedback trading of such dealers rises in stress.

VaR models may have contributed to reinforce and generalise the tendency to positive feedback trading in markets where agents imitating fashionable strategies tend to prevail on independent-minded operators.

This argument is made by Persaud (2000), with reference to international asset management after the 1998 Russian crisis. International capital flows are driven by institutional investors seeking better returns than their competitors against common benchmark indices. As assumed before, relative-return investors may behave like a herd, rushing in and out of markets together.

> One evidence of herd behaviour is given in the same paper considering portfolio flows across 22 emerging markets between January 1996 and March 2000. Although these 22 markets exhibit very different economic fundamentals, politics and markets, there are times when all 22 markets are receiving net inflows, as in most of 1996, while in late 1998 almost all 22 were receiving net outflows.

To some extent, tighter risk management systems combined with herding investors, have aggravated liquidity conditions. Investors with similar portfolios hit they risk management limits together, unload their undesired positions at the same time reducing liquidity and causing those markets to exhibit even greater volatility than in the past, which in turn feeds back into risk management systems and leads to further sales. Thus the cycle becomes self-feeding. Similar arguments were made with foresight in Taleb (1997:450) and, after the fact, they became common wisdom among practitioners, as reported in the financial press, see, e.g., The Economist (1999):

> "[…] But the more these models [i.e. *VaR* models] are used, the more likely it is that markets will suffer in the way that they did in 1987 [due to portfolio insurance]—and not just in one market but in many. The models are profoundly affected by rises and falls in volatility. Less volatile markets mean a lower *VaR*, implying that for the same apparent risk, banks can pile up more assets. But if markets become more volatile, *VaR* goes up by at least a proportional amount (and much more if an institution has sold masses of options). The bank is then faced with two choices: put in extra capital or reduce its positions, whatever and wherever they may be. This is what happened last autumn [1998]. But what made that crisis especially acute was that it produced a vicious circle. As banks dumped assets, markets fell further and volatility rose, forcing them to sell still more. As the head of risk at one bank told at a meeting at the Federal Reserve shortly after last year's crisis: "The recent market turmoil has highlighted a 'generic' form of dynamic hedging which […] tends to amplify the direction and speed of market change." Regulators themselves may have contributed to the problem. By international agreement, banks put capital aside for risks on their trading books, but are allowed to use their own models to calculate how much. Banks feed into these the prevailing levels of volatility. Since volatility rises and falls, so does the *VaR* . So in a crisis, to comply with capital requirements, banks must unload risky positions when the markets are at their least liquid. It might be better, some argue, to make banks use a consistently high, but stable, level of volatility in their models."

According to Persaud (2000), in these environments, markets need contrarian investors able to buy at prices depressed by liquidity evaporation, in the hope that they will be the first to benefit from the turnaround. Interestingly, unregulated hedge funds oriented towards relative-value arbitrage (the LTCM sort, to be clear) are more likely to do this than regulated investors and banks. The job of this kind of relative-value investor is to bet on misalignments of price/value ratios across markets and instruments, and on the correction of such misalignments. This entails entering depressed or poorly considered markets and take big exposures to liquidity risk. Such strategy commands a strong capital position in order to shore possible stresses from funding risk (as in the case of arbitrage between OTC vs. exchange traded derivatives, with the latter subject to margining). Return targets and risk control often impose, as in the case of LTCM, a level of leverage, investment horizons and risk limits that are incompatible with the very nature of value arbitrage. We must not forget the risk of squeezes arising from huge positions in less liquid markets financed with debt by a restricted number of banks: malicious as it may sound, we cannot rule out opportunistic behaviour by banks knowing everything about one's huge and vulnerable exposures. Last but not least, also contrarian strategies may become fashionable and attract, as they have in the hedge fund sector, an excessive amount of capital with respect to arbitrage opportunities. In that case, virtuous—negative feedback—contrarian trading, may be tainted by vicious—positive feedback—herding behaviour.

> On the case of Long Term Capital Management a rich literature has flourished, see e.g. the books by Dunbar (2000) and Lowenstein (2000), and the articles by Jorion (1999), Stonham (1999a) and Stonham (1999b).

4. Incorporating Liquidity Risk into VaR Models

Critique of *VaR* in the light of its adverse macro-impact commands a response that goes beyond technical improvements. What is needed is a change from a static to a dynamic view, i.e. treating trading losses not as exogenous random events, as if they were natural catastrophes, but as phenomena caused by the interaction between the single agent and the market. Research on liquidity risk and liquidity-adjusted *VaR* has gone in that direction. A survey of contributions coming from that strand of research will be presented hereafter.

The following quotation from Taleb (1997:70) stresses the importance of liquidity in risk management

> "It cannot be stressed enough that liquidity is the most serious risk management problem. A substantial part of unforeseen losses is due either to market jumps caused by illiquidity or by liquidation costs that substantially move the market against one's position. Liquidation costs tend to be underestimated since operators usually "fade" when some one is forced into market action. […] The market is merciless with operators who start closing down a position, particularly when the liquidating party has no choice".

In the first place, we could ask what standard *VaR* is missing about liquidity risk. The point, according to Bangia, Diebold, Schuermann and Stroughair (1999), is that market risk management under normal conditions traditionally has focused on the distribution of portfolio value changes resulting from moves in the mid-price, i.e. the price currently used for marking-to-market portfolios. Hence the market risk is really in a "pure" form: risk in an idealised market with no "friction" in obtaining the fair price. However, as we argued extensively above (see p.8, Liquidity and Liquidity Risk), many markets possess an additional liquidity component that arises from a trader not realising the mid-price when liquidating his position, but rather the mid-price minus half the bid-ask spread, minus an additional price impact. It follows that liquidity risk associated with the uncertainty of the spread-plus-impact component, particularly for thinly traded or emerging market securities under adverse market conditions, is an important part of overall risk and is therefore an important component to model.

Standard VaR estimates the maximum potential shortfall that a trader can suffer on a losing position before it can be completely liquidated on the market. Standard VaR assumes the following ideal conditions: the position is unwinded with one trade at a fixed market price, equal to the current quoted mid-price, within a fixed period of time (usually one trading day), regardless of the size of the position.

In order to account for the various factors responsible of deviations of liquidation price from the mid quote, several methods have been proposed to incorporate market liquidity and transaction costs into the VaR measurement framework. Those methods will be briefly surveyed hereafter. All the approaches examined here return a measure for liquidation risk to be added to (or compounded into) "mid-price" VaR. They can be classified into four main categories, each differing in the way the liquidity effect is calculated, i.e. as

- an empirical adjustment;
- the current or expected value of the bid-ask spread;
- a quantile, for a given probability, of the bid-ask spread distribution;
- costs and price risk estimated assuming an optimal liquidation strategy given a price impact function.

For a review of factors affecting liquidity risk in the context of *VaR* measurement see also Muranaga and Ohsawa (1997).

4.1 Empirical adjustments

In the empirical approach to liquidity risk, *VaR* is corrected with ad hoc adjustments. In their simplest form, the time horizon for the return distribution is extended so as to match an orderly liquidation period t_l for the position. This adjustment varies across asset categories and may also depend on position size. *VaR* is then computed on the distribution of t_l-days returns.

In order to estimate the historical volatilities of *n*-days returns, we face the well-know small sample problem (need of very long historical series). We can otherwise apply some time scaling rule to the one-day *VaR*. As stated above (see p.18, The Micro-View: Technical Limitations of Standard VaR Models), the standard choice is represented by "t-square" scaling assuming serial independence. The empirical behaviour of returns over different time scale exhibit complex patterns of dependence. Alternative approaches to time scaling of volatility have been proposed in Dacorogna, Müller, Pictet and de Vries (1998), Christoffersen, Diebold and Schuermann (1998) and Diebold, Hickman, Inoue and Schuermann (1998).

However, this approach provides an estimate of a worst case move of the mid price, and that is not exactly what we are looking for—i.e. an estimate of execution costs related to the difference between the average traded price and the initial mid-price.

A second approach consists in artificially increasing volatilities for less liquid assets. Standard *VaR* is then computed on a modified variance-covariance matrix. Criteria for volatility add-ons are subjectively specified. This approach as well has some relevant limitations. It gives no guidance on how to quantify corrections to volatility and it is not focused on the very problem of liquidity risk.

> The Basel guidelines on the use of internal models considered above provide an authoritative example of ad hoc adjustment of volatilities motivated also by liquidity risk. However, such an adjustment is uniform across assets with differing liquidity.

4.2 Adjustments Based on Quoted Spread

a) Current or Expected Value of the Bid-Ask Spread

If we assume that our position can be smoothly closed at quoted prices, execution costs can be approximated as a linear function of current or expected bid-ask spread. Liquidity-adjusted *VaR* can be estimated by adding to standard *VaR* the expected costs from the quoted spread component, i.e. the product of half the percentage spread by an appropriate measure of position size (market value for equities, notional value for derivatives). We obtain for the *j*-th position:

$$LVaR_j = VaR_j + Size_j \times \frac{\mathrm{E}(S_j)}{2} \tag{9}$$

where *LVaR* is liquidity-adjusted *VaR*, *VaR* is the standard measure of market risk, E(.) is the expected value operator, and *S* is the percentage quoted spread.

In this way the liquidity adjustment for an asset class varies according to the tightness of the spread in the market for that class. We refer to normal or expected market conditions. Important liquidity risk factors, namely variability of spreads and the impact of position size on execution price, are not accounted for.

b) Quantiles of the Bid-Ask Spread Distribution

Extending the previous methodology, we can compute the liquidity component of *VaR* substituting a quantile—for a given probability—of the spread distribution for the current or expected value of the spread. This approach, presented in Bangia, Diebold, Schuermann and Stroughair (1999), adopts a worst-case estimate of liquidation costs with a given degree of confidence. Only exogenous liquidity is considered. The position size, the main factor affecting endogenous liquidity risk, does not enter the estimated cost of execution.

The formula for liquidity adjusted *VaR* becomes the following

$$LVaR_j = VaR_j + Size_j \times \frac{\mathrm{Q}_{S_j}(p)}{2}, \tag{10}$$

where $\mathrm{Q}_{S_j}(.)$ is the quantile function of the quoted spread distribution (i.e. the inverse function of the spread distribution function) and *p* the probability value at which *LVaR* is computed.

Though simple to compute for an individual position, *LVaR* is difficult to implement at the portfolio level. At that level heavy data requirements must be met and delicate decisions must be taken.

An historical database of spread information for different products is required in order to infer the probability distributions. An historical simulation approach is to be preferred to a parametric (e.g. normal) model, because of the irregular shape of the empirical distribution of the spread that is observed in less liquid markets. Spread distributions exhibit fat tails and bimodality, hinting at a regime-switching process of spread innovations where market conditions can shift between normal and "hectic" regimes.

> In turbulent markets abnormal widening of the spread acts as a circuit breaker, signalling the unwillingness to trade by market makers. In those situations, quoted spread loses much of its information content.

Estimates of quantile based on assumptions of normality could lead to measures that are seriously misspecified. The time series sample should be long enough to account for both normal and stressed markets. Estimation could be enhanced adopting the mixed (historical + parametric) techniques developed in the extreme value theory literature for computing *VaR* at extreme quantiles, as described in Daníelsson and de Vries (1997) and McNeil (1999).

In order to overcome data availability and econometric problems, it is advisable to consider distributions of average spreads for a class of assets, instead of individual security spread, computed as weighted averages of quoted spreads for a representative sample of products in that class.

The liquidity component of *LVaR* can be aggregated at the portfolio level in several ways. We can compute a conservative estimate through simple addition of liquidity components, without accounting for diversification effects. We obtain for a portfolio *P* with *n* asset classes

$$LVaR_P = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} VaR_i VaR_j \rho(P_i, P_j)} + \sum_{j=1}^{n} LA_j \text{ ,} \qquad \textbf{(11)}$$

where $LA_j = Size_j \times Q_{S_j}(p)/2$ and $\rho(P_i,P_j)$ is the correlation coefficient between mid-returns on asset classes *i* and *j*. The first term in the sum is the usual parametric *VaR*, i.e. the measure of mid-price risk.

Diversification can be accounted for as in the variance-covariance *VaR* treating the spreads S_j as additional factors integrated in the correlation matrix, as in Cherubini (1997). That requires the estimation of linear correlation coefficient for each pair price$_i$/spread$_j$ and spread$_i$/spread$_j$. We obtain

$$LVaR_P = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} VaR_i VaR_j \rho(P_i, P_j) + \sum_{i=1}^{n}\sum_{j=1}^{n} LA_i LA_j \rho(S_i, S_j) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} VaR_i LA_j \rho(P_i, S_j)} \text{ ,} \qquad \textbf{(12)}$$

A positive correlation between volatility of mid-quotes and spreads is observed and is also theoretically justified. Spreads tend to exhibit also positive correlation across clusters of markets for contagion effects, especially in times of crisis.

The spread quantile approach has one main advantage: it relies on observable market data, providing the risk control function with objective information that does not depend on estimates by controlled traders. We may question the availability and reliability of spread series in less active markets, where publicly quoted prices are scarce, and in new markets, where a price history of adequate length is missing.

However, the merits of this approach, based on informed judgement of potential losses from execution based on the empirical behaviour of quoted spread, cannot be dismissed. Market information on the price for immediacy—that is what quoted spread are—are an objective foundation for estimating total execution costs: such a price is what market makers require to assume the risk we want to measure.

> The alternative approach requires models of the price impact function of individual trading decisions (see below 4.3), which are partial and difficult to validate empirically.

Application of this approach is hindered in the markets where it is more badly needed, i.e. in thin or highly volatile markets and in stress times, when spread data required by the model are not available as market makers refrain from quoting a cost for immediacy around an equilibrium price. Anyway, such an approach exposes transparently its limits in critical situations when it is not applicable. In normal times, it allows independent risk controllers, with no direct experience of market behaviour, to reasonably assess liquidity risk.

4.3 Optimisation models of execution cost and price risk

As stated in Jorion (2000:346-349), the extensions of *VaR* based on bid-ask spreads, while an improvement over traditional *VaR* calculation, very much ignores the market impact factor—i.e. the main source of endogenous liquidity risk—which can be very significant. To some extent, this can be mitigated by suitable execution strategies. These should be taken into account when computing a liquidity-adjusted *VaR*.

The optimal execution strategy in the presence of information asymmetry and its relationship with the behaviour of prices has been the focus of many important theoretical papers in market microstructure theory, including Kyle (1985), Easley and O'Hara (1987), Admati and Pfleiderer (1988), and Glosten (1994). A recent strand of research, rooted in those studies, models the decisions to be taken by a rational trader in order to optimise the expected utility of the revenue from liquidating a trading position, given assumptions about the time horizon and the price process, corrected for trade impact. Models in this stream lead to the joint definition of an optimal strategy and the distribution of losses—including transaction costs and adverse price movements—on that strategy. Interest by practitioners in these kind of models is not new: current applications to market risk management build upon previous applied research in the field of investment management, such as Perold (1988) and Collins and Fabozzi (1991), and application by financial research firms, as the model by BARRA (see Torre (1998)).

In Lawrence and Robinson (1995) a conceptual model is presented for incorporating endogenous liquidity risk into *VaR* measurement. Lawrence and Robinson (1997) propose a simple solution for a utility-maximising execution strategy, which leads to the optimal number of days n for liquidating a position with a series of equal-sized daily trades. The optimal strategy minimises the sum of costs from price impact and potential price losses from delayed liquidation, the latter adjusted for risk aversion. Bertsimas and Lo (1998) derive more articulated optimal strategies that minimise the expected cost of trading a large block of equity over a given time horizon. Specifically, given a fixed block of shares S to be executed within a fixed finite number of periods T, and given a price-impact function that yields the execution price of an individual trade as a function of shares traded and market conditions, they obtain the optimal sequence of trades as a function of market conditions—closed form expressions in some cases—that minimises the expected cost of executing S within T periods. Their analysis is extended to the portfolio case in which a price impact across stocks can have an important effect on the total cost of trading a portfolio. The authors discuss the possibility of incorporating risk into the objective function but do not provide an explicit model. The latter extension is to be found in the model by Almgren and Chriss (1999a) (summarised in Almgren and Chriss (1999b)). Almgren and Chriss, as Lawrence and Robinson (1997), work in the more general framework of minimising the expected loss of utility from trading costs, where loss of utility is a certainty equivalent measure equal to the expected cost of trading plus a constant times the variance of cost. Hisata and Yamai (2000) extend on Almgren and Chriss (1999a) in several directions, comparing discrete vs. continuous time processes, and trying to make the optimal liquidation period an endogenous variable. Other models of endogenous liquidity risk are provided in Longstaff (1998) and Krakovsky (1999).

In this section we give a non-technical overview of dynamic models of trading costs and their applications to risk measurement. The intuition behind them and the mathematical formulation will be presented following Jorion (2000:346-349).

To simplify, let us assume a linear price-quantity function. For a sale,

$$P(q) = P_0(1 - kq) \tag{13}$$

Where P_0 is the initial mid-price, q is the traded quantity, k is a linear price impact coefficient, and $P(q)$ the execution price given the traded quantity. We assume a constant mid price and a temporary impact only, i.e. no influence of the trade size and timing on the mid price.

Measuring execution costs as the variation of the traded price with respect to the initial mid, we obtain that *immediate liquidation* creates quadratic costs

$$C_I = q \times \left[P_0 - P(q)\right] = kq^2 P_0, \tag{14}$$

whereas *uniform liquidation* creates lower costs:

$$C_U = q \times \left[P_0 - P(q/n)\right] = k(q^2/n)P_0. \tag{15}$$

The drawback of liquidating more slowly is that the portfolio remains exposed to price risks over a longer period. Risk exposure is a function of the sequence of residual position size and of the variance of mid-quotes. Exposure is brought to zero in one shot with immediate liquidation, whereas it decreases linearly with uniform liquidation. The key is to choose a strategy that offers the best cost-risk trade-off.

To analyse the price-risk profile of these strategies, define σ as the daily volatility of the share price, in dollars. We assume that sales are executed at the close of the business day, in one block. Hence, for the immediate sale, the price risk or variance of wealth is zero. For the uniform sale, the portfolio variance can be computed assuming independent returns over n days as

$$\begin{aligned} V_U &= \sum_{j=1}^{n} \sigma^2 \times position_j = \sigma^2 q^2 \left\{ \left(1 - \frac{1}{n}\right)^2 + \left(1 - 2\frac{1}{n}\right)^2 + \cdots + \left[1 - (n-1)\frac{1}{n}\right]^2 \right\} = \\ &= \sigma^2 q^2 \left[n \frac{1}{3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{2n}\right) \right] \end{aligned} \tag{16}$$

For example, with n = 5 days, the correction factor between braces is 1.20. Thus the price risk of a constant liquidation over 5 days is equivalent to that of the initial position held over 1.2 days (variance is linear in the holding horizon from the assumption of independent returns). It is interesting to note that in this setting the 10-day fixed horizon dictated by the Basel Committee is equivalent to a constant liquidation over 31 days.

Execution strategies need not be limited to these two extreme cases. Assumptions regarding the price impact function and, more generally, the stochastic process of prices can also be released. More generally, we can choose a wealth maximising strategy X defined by a sequence of trade amounts, adding up to the initial position, which leads to an optimal trade-off between execution costs and price risk

$$\min_X \left[C_X + \lambda V_X\right] \tag{17}$$

where λ reflects the risk aversion to price risk.

This leads to another formulation of liquidity adjusted *VaR*, or *implementation shortfall*, which is

$$LVaR_X = \alpha\sqrt{V_X} + C_X \tag{18}$$

where α is a multiple depending on the required probability level. Note that $LVaR_X$ does not account for the risk of varying market liquidity unless the assumed price process includes a stochastic impact function. Almgren and Chriss (1999a) consider the case of time-varying parameters of the price impact function (see below The Almgren-Chriss Model).

As noted by Jorion (2000:349), in practice, the computational requirements to adjust the conventional *VaR* numbers are formidable. The method requires a price-quantity function for all securities in the portfolio. Combined with the portfolio composition, this yields an estimate of the price impact of a liquidation, as well as the optimal time to liquidation. This approach is not feasible under market stress, just like the one based on the behaviour of quoted spreads. During a liquidity crisis, normal price-response mechanisms break down. Modelling the ensuing pathologies—liquidity holes, crashes, regime switching—with a price-quantity function is an impossible task, let alone measuring its parameters and feeding it into an optimisation procedure.

a) The Almgren-Chriss Model

In Almgren and Chriss (1999a) a model for optimising execution costs is specified in detail. Their analysis gives deep insight into the merits of this approach to liquidity risk measurement, and will be summarised here.

The Trading Model. Suppose we hold a block of X units of a security that we want to completely liquidate before time T. We divide T into N intervals of length $\tau = T/N$ and define the discrete times $t_k = k\tau$, for $k = 0, \dots, N$. We define a *trading trajectory* to be a list $x_0, \dots, x_N$, where x_k is the number of units that we plan to hold at time t_k. Our initial holding is $x_0 = X$, and liquidation at time T requires $x_N = 0$. We may equivalently specify a strategy by the *trade list* $n_0, \dots, n_N$. The paper consider the case of liquidating a long position by selling X. The case of buying against a short position lends itself to the same treatment.

- A **trading strategy** is defined as a rule for determining n_k in terms of information available at time $t_{k\text{-}1}$. There are two basic types of trading strategies: **static strategies** are determined in advance of trading; in **dynamic strategies**, conversely, each n_k depend on all information up to and including the last time before $t_{k\text{-}1}$.

Suppose that the initial security price is S_0, so that the initial market value of our position is $X\,S_0$. The security's price evolves according to two exogenous factors (due to random forces independent from our trading): volatility and drift, and one endogenous factor (due to our trading): market impact. We distinguish *temporary* impact, and *permanent* impact, affecting the "equilibrium price", which remains at least for the life of our liquidation.

We assume that the security price evolves according to the discrete arithmetic random walk

$$S_k = S_{k-1} + \underbrace{\sigma\sqrt{\tau}\xi_k}_{\substack{\text{exogenous}\\ \text{volatility}}} - \underbrace{\tau g\left(\overbrace{\frac{n_k}{\tau}}^{\substack{v=\\ \text{average}\\ \text{rate of}\\ \text{trading}}}\right)}_{\substack{\text{permanent}\\ \text{impact}}}, \text{ for } k=1, \dots N. \tag{19}$$

Here σ represents the volatility of the asset, the ξ_k are draws from an independent random walk (with zero mean and unit variance), and $g(v)$ is a permanent impact function of the *average rate* of trading $v = n_k/\tau$ during the k-th interval.

> The assumption of arithmetic random walk is acceptable over the short-term "trading" time horizons. In (19) there is no drift term: we assume that we have no information about the direction of future price movements. The paper at a later point relaxes some of the hypotheses, introducing serial correlation and drift.

Our trader gets an execution price for trading n_k units that is given by S_k corrected for a temporary market impact, modelled as a function $h(v)$ of the average rate of trading. Given this, the actual price per share received on sale k is

$$\tilde{S}_k = S_{k-1} - \underbrace{h\left(\frac{n_k}{\tau}\right)}_{\substack{\text{temporary}\\ \text{impact}}}, \tag{20}$$

- The **capture** of a trajectory is defined as the full trading revenue upon completion of all trades. This is the sum of the product of the number of units n_k that we sell in each time interval times the effective price per share $\tilde{S}_k$ received on that sale.

We readily compute

$$\underbrace{\sum_{k=0}^{N} n_k \tilde{S}_k}_{\text{capture}} = \underbrace{XS_0}_{\substack{\text{initial}\\ \text{market}\\ \text{value}}} + \underbrace{\sum_{k=1}^{N}\left[\sigma\sqrt{\tau}\xi_k - \tau g\left(\frac{n_k}{\tau}\right)\right]x_k}_{\substack{\text{sum of variations in market value,}\\ \text{due to exogenous volatility}\\ \text{and permanent impact,}\\ \text{function of residual positions}}} - \underbrace{\sum_{k=1}^{N}\left[n_k h\left(\frac{n_k}{\tau}\right)\right]}_{\substack{\text{sum of temporary}\\ \text{impact effects,}\\ \text{function of trades}}}. \quad \textbf{(21)}$$

- The **total cost of trading** is the difference $XS_0 - \sum_{k=0}^{N} n_k \tilde{S}_k$ between the initial value from marking-to-market and the capture. This is the standard *ex-post* measure of transaction costs used in performance evaluations, and is essentially what Perold (1988) calls **implementation shortfall**.

In this model, prior to trading, implementation shortfall is a random variable. We write $E(x)$ for the expected shortfall and $V(x)$ for the variance of the shortfall. We readily compute

$$E(x) = \sum_{k=1}^{N} \tau x_k g\left(\frac{n_k}{\tau}\right) + \sum_{k=1}^{N} n_k h\left(\frac{n_k}{\tau}\right) \quad \textbf{(22)}$$

$$V(x) = \sigma^2 \sum_{k=1}^{N} \tau x_k^2 = \sigma^2 T \sum_{k=1}^{N} \frac{\tau}{T} x_k^2 \quad \textbf{(23)}$$

The variance of the implementation shortfall depends on the exogenous volatility only. The impact functions are assumed deterministic. Note that V*(*x*)* is computed as the potential loss for a shock equal to the volatility on a time horizon T, on a position whose size is the time-weighted average of the position amounts squared held between t_0 and t_N.

The Efficient Frontier of Optimal Execution. We can show that for each level of risk aversion there is a uniquely determined optimal execution strategy.

A rational trader will always seek to minimise the expectation of shortfall for a given level of variance of shortfall. We may construct efficient strategies by solving for the constrained optimisation problem

$$\min_{x: V(x) \le V_*} E(x). \quad \textbf{(24)}$$

We solve the constrained optimisation problem (24) by introducing a Lagrange multiplier λ, solving the unconstrained problem

$$\min_{x} \left[E(x) + \lambda V(x)\right]. \quad \textbf{(25)}$$

The parameter λ is a measure of risk-aversion.

For given values of the parameters, problem (25) can be solved by various numerical techniques, which are not explained here. The solution includes a parameter θ, defined as the trade's "half-life". The value θ is related to the amount of time it takes do deplete the portfolio**.** The definition of θ is independent of the exogenously specified execution time *T*; it is only determined by the security price dynamics and the market impact factors. The half-life could be perhaps considered a guide to the proper amount of time over which to execute a transaction.

Extensions of the model. Almgren and Chriss (1999a) obtain their results using static optimisation procedures, which lead to globally optimal trading trajectories, determined in advance of trading. They regard statically optimal strategies as a benchmark for comparison against dynamic strategies. Considering static strategies as strategies that ignore the arrival of new, possibly relevant information, they ask what gains are available to strategies that incorporate all relevant information.

Three types of information are identified.

1. *Serial correlation and drift* has small potential for marginal gains over static strategies. Such gains are independent, in absolute value, of portfolio size, i.e., they proportionally decrease with portfolio size.

2. *Scheduled news events* have a significant temporary impact on the parameters governing price movements. It is shown that that the optimal execution strategy entails following a static strategy up to the moment of the event, followed by another static strategy that can only be determined when the outcome of the event is known. Predictability of the occurrence of a future event modifies the strategy followed in the first time interval.
3. *Unanticipated events* cause a shift in the parameters of the price dynamics that is unpredictable in its time of occurrence and possible outcomes. Facing these events, the best approach is to be actively watching the market for such changes, and react swiftly should they occur. One approximate way to include such completely unexpected uncertainty into the model is to artificially raise the value of the volatility parameter.

Analysis by Almgren and Chriss is rigorous and insightful. Their approach includes an estimation procedure of liquidity-adjusted *VaR* given a price impact function with stable parameters. The effects arising from unanticipated changes in the market impact function are discussed, but they do not fit in the model's framework. Findings about patterns of optimal execution strategies are of greater practical interest than what the model says about liquidity risk and its measurement.

b) The Jarrow-Subramanian Model

The model described in Jarrow and Subramanian (1997a) has a place of its own in the literature on optimal execution strategies and liquidity. Their model aims at two objectives:

- define a formula for the expected liquidation price, to be used in marking-to-market portfolios as a substitute for current price;
- define a formula for the liquidity adjusted *VaR*.

Traders are assumed to maximise the expected liquidation value of S shares of a risky asset under a maximum liquidation time constraint T given a permanent price impact of trades, modelled as a random quantity discount applied to current market price, and a random lag in trade completion, both of which depend on the amount traded. The model determines an optimal liquidation strategy. Given that strategy, the desired distribution of the liquidation value is obtained. The main finding of the model is that under economies of scale in trading conditions—i.e. when splitting an amount of shares in two immediately consecutive trades is always more costly that trading the same amount as a single block—block trading is always the optimal strategy. On that finding, a closed formula for the stochastic liquidation price is derived. Hence, fair liquidation price and liquidity-adjusted *VaR* are derived. Values derived under the "economies of scale in trading" assumption can be made good as conservative estimates under the alternative hypothesis.

Description of the market structure. The risk asset will be called a stock. To simplify the analysis, risk neutrality of traders is assumed. However, the model can be easily extended to treat risk averse traders.

The market price for the stock is defined as the last traded price for one round lot, a unit sale (or purchase). Let $p(t)$ denote the stock's market price at time t. We assume that between trades—i.e. when the trader is not in the market— $p(t)$ follows a geometric Brownian motion, i.e.:

$$dp(t) = p(t)\left[\alpha dt + \sigma dW(t)\right] \qquad \textbf{(26)}$$

where α is the stock's expected return, σ is the standard deviation of returns, both constant and referred to the same time unit, and $W(t)$ is a standard Brownian motion. When the trader sells $s \leq S$ shares at time t, given a market price of $p(t)$, the price he receives, per share, is:

$$c(s)\,p(t) \qquad \textbf{(27)}$$

where $c(s)$ represents a *quantity discount* coefficient, assumed non-decreasing in s and valued between 0 and 1. The quantity discount is assumed random (distribution is not specified) and (for simplicity) independent of the market price process $p(t)$. It is random because the size of the discount can be unknown prior to the trade.

The effect of the shares sold on the market price is cumulative, i.e. permanent in our terminology. After a sale is executed, the new market price begins at a magnitude determined by the quantity discount, i.e $p(t^+) = c(s)\, p(t)$, where t^+ means an instant after time t.

In addition, given that the trader's sell order is placed at time t, it is assumed to be executed at time $t + \Delta(s)$, where $\Delta(s) \geq 0$ is an *execution lag*, assumed non-decreasing in s, i.e. the larger the sales, the longer the time to execute, holding everything else constant. The execution lag is random—it can be unknown prior to the trade— and independent of the market price process $p(t)$ and the quantity discount $c(s)$.

Proceeds from liquidation are deposited in a money market account yielding a rate r.

To ensure that liquidation has a cost, the following condition is also imposed:

$$c(s)\exp\left[(\alpha - r)\Delta(s)\right] \leq 1 \text{ for all } s \tag{28}$$

This states that the impact of the quantity discount is greater than the expected appreciation of the stock prior to execution, discounted to the present

Derivation of optimal trading strategies. As already stated, the trader wants to liquidate S shares over the horizon t=0 to T. He can sell the shares as he wishes, either as a block or slowly in smaller quantities.

Formally the trader sells shares using a *trading strategy*, defined as a collection of dates (t_1, t_2, … , t_n) and shares sold (s_1, s_2, … , s_n) such that $s_1 + s_2 + \ldots + s_n = S$. The last trade can take place at T; if so, it will be executed at $t + \Delta(s_n)$.

The trader's liquidation problem is to choose a trading strategy to maximise the expected value of the discounted proceeds from the sales of the S shares by time T:

$$\max_{(s_i,t_i)}\left[\mathrm{E}_0\left(\sum_{i=1}^{n} s_i c(s_i)\, p\left(t_i + \Delta(s_i)\right)\exp\left(-r\left[t_i + \Delta(s_i)\right]\right)\right)\right] \tag{29}$$

where

$$p\left(t_i + \Delta(s_i)\right) = p(0)\exp\left\{\left[\alpha - \frac{\sigma^2}{2}\right]\left(t_i + \Delta(s_i)\right) + \sigma\left[W\left(t_i + \Delta(s_i)\right) - W(t_i)\right]\right\}$$

If there is **no liquidity risk** the problem can be restated in the following way:

$$\max_{(s_i,t_i)}\left[\mathrm{E}_0\left(\sum_{i=1}^{n} s_i\, p(t_i)\exp\left(-r\left[t_i\right]\right)\right)\right] \tag{30}$$

where

$$p(t_i) = p(0)\exp\left\{\left[\alpha - \frac{\sigma^2}{2}\right]t_i + \sigma\left[W(t_i) - W(0)\right]\right\}$$

Solving for the optimum liquidation policy in expression (30) we determine $u^*(p,S)$, the maximum discounted proceeds from the sale of S shares when the current market price is p with no liquidity risk. Under risk neutrality, the optimal strategy depends in a straightforward way of the excess expected return of the stock over the deposit rate: if $\alpha > r$, then it is optimal to wait until T to liquidate S as a block so as to earn a positive expected excess return; when $\alpha > r$, an immediate block sale of S gives the best expected value. In both cases, the optimal strategy consists of a single trade, and there is no gain from split execution. We obtain the following expression for the *optimal expected liquidation value*

$$u^*(p,S) = \begin{cases} Sp & \text{if } \alpha \leq r \\ Sp\exp\left[(\alpha - r)T\right] & \text{if } \alpha > r \end{cases} \tag{31}$$

♦ *This finding shows that when there is no liquidity risk, marking-to-market ($S \times p$) always provides a fair prudential liquidation value for a portfolio. Current practice of marking-to-market assumes a value equal to the liquidation value with no liquidity risk defined above under the prudential hypothesis $\alpha \leq r$.*

With liquidity risk, a unique solution to the trader's liquidation problem cannot be obtained. It could be either a block sale of *S*, as with no liquidity risk, or slow liquidation with any sequence of trades adding up to *S*.

Surprisingly, it can be demonstrated that the trade-off between these two alternatives depend on a single condition, called the *economies of scale in trading* condition. This conditions holds when, given a lot to sell, the liquidation value, given by the expected market price at $t + \Delta(s)$ corrected for the quantity discount and discounted to t at a money market rate, is always greater for a block sale than for a pair of split sales of equal total amount, the second executed *immediately after* the first. In simple words, the economies of scale in trading condition states that two trades are more costly than one.

Under this condition, Jarrow and Subramanian (1997b) demonstrate that there is no gain from splitting the execution in a series of lots. As a consequence, the optimal strategy is always a block sale, and expected proceeds from the optimal liquidation of the *S* shares can be easily quantified.

Determining the Liquidation Value of the Portfolio. Solving for the optimum liquidation policy in expression (29) we determine $u(p,S)$, the maximum discounted proceeds from the sale of *S* shares when the current market price is *p* and in the presence of liquidity risk. Given liquidity risk and economies of scale in trading, the optimal policy depends on the excess expected return: a positive excess returns leads to delayed liquidation at *T*, while a negative excess return justifies immediate liquidation. The following expression for the *optimal expected liquidation value* can be obtained

$$u(p,S) = \begin{cases} Spc(S)\exp\left[\left(\alpha - r\right)\Delta(S)\right] & \text{if } \alpha \le r \\ Spc(S)\exp\left[\left(\alpha - r\right)\left(T + \Delta(S)\right)\right] & \text{if } \alpha > r \end{cases} \tag{32}$$

The trading strategy is identical to that without liquidity risk. But, in contrast with the situation without liquidity risk, the maximum proceeds received from liquidation differ due to the quantity discount and the execution lag. By condition (28), the discounted expected proceeds from liquidation, given liquidity risk, are always less.

Given the above solution, we can now question whether the optimal liquidation value obtained is a proper measure for marking-to-market a portfolio. This would be acceptable for the case where the price appreciation on the stock is less than the discount rate, but not if it exceeds the discount rate: in the latter case, a trader would have an incentive to classify assets with positive expected return as on sale, and to delay their liquidation, in order to book immediately the resulting increase in value from marking-to-market.

An elegant valuation procedure proposed in the paper overrides this problem with a straightforward generalisation of the marking-to-market approach used in the case without liquidity risk. The idea is to determine a hypothetical initial price in a market without liquidity risk that would provide the same proceeds given under the liquidity risk solution. Since without liquidity risk the initial market price is a fair estimate of liquidation value, then there is some reason behind adopting this "perfect-liquidity-equivalent" price (in analogy with certainty-equivalency that we use under price-risk neutral valuation) as a fair expected liquidation value.

We define the per-share liquidation value of the stock to be that initial market price p^* such that a trader facing no liquidity risk would receive the same expected proceeds as a trader facing liquidity risk and the current market price *p*, i.e., the per-share liquidation value is p^* such that:

$$u^*(p^*,S) = u(p,S), \tag{33}$$

that, under the economies of scale in trading condition can be expanded into

$$\begin{cases} Sp^* = Spc(S)\exp\left[(\alpha - r)\Delta(S)\right] & \text{if } \alpha \le r \\ Sp^* c(S)\exp\left[(\alpha - r)T\right] = Spc(S)\exp\left[(\alpha - r)(T + \Delta(S))\right] & \text{if } \alpha > r \end{cases} \quad \textbf{(34)}$$

Solving in (34) for the per share liquidation value p^*, we obtain the same expression in both cases—$\alpha \le r$ and $\alpha > r$, i.e.

$$p^* = pc(S)\exp\left[(\alpha - r)\Delta(S)\right] \quad \textbf{(35)}$$

Hence the *liquidation value of our portfolio* can be obtained by marking-to-market using p^* and not p, i.e., the liquidation value of our portfolio is p^*S. From (28), p^* is surely lesser than p, so that the estimated liquidation price is lower than market price. The difference between p and p^* allows for fair liquidation costs in marking-to-market.

When the economies of scale in trading condition do not hold, (35) is no longer a fair estimate of expected liquidation price from optimal execution, as an initial or terminal block sale is no longer an optimal strategy: splitting trades we can save on adverse price impact. Nonetheless, we can still use the above to provide us with a *conservative* estimation of the liquidation value of the portfolio: the liquidation value from a "split trades" strategy will be at least as large as those received from a block trade, measured by(35)

Determining Liquidity-Adjusted *VaR*. Let's now look at our quantification of liquidity risk for the computation of *VaR*. For comparison, we first compute the standard *VaR* measure for the trader's portfolio. Let δ be the horizon over which the change in the portfolio's value is considered. Notice that the horizon is independent of the shares sold. Setting a confidence interval of two standard deviations, the standard *VaR* measure is easily computed as:

$$VaR = pS\left|\mathrm{E}\left[\ln\left(p(\delta)/p\right)\right] - 2\mathrm{std}\left[\ln\left(p(\delta)/p\right)\right]\right| \quad \textbf{(36)}$$

where $p = p(0)$ and std[.] represents the standard deviation. Given the price process (26), a simple calculation yields:

$$VaR = pS\left|\left[\alpha - \frac{\sigma^2}{2}\right]\delta - 2\sigma\sqrt{\delta}\right| \quad \textbf{(37)}$$

This represents the loss in the dollar value of the portfolio due to a two-standard deviation move below the mean.

Using the conservative estimate of the liquidation value as given by expression (35) [which is a stochastic variable function of the three independent stochastic variables p, $c(S)$ and $\Delta(S)$], we can compute the liquidity adjusted *VaR* (*LVaR*) as follows:

$$LVaR = pS\left|\mathrm{E}\left[\ln\left(p(\Delta(S))c(S)/p\right)\right] - 2\mathrm{std}\left[\ln\left(p(\Delta(S))c(S)/p\right)\right]\right| \quad \textbf{(38)}$$

Using expression (29), the authors, with a non-trivial calculation, obtain:

$$\begin{aligned} LVaR = pS\Bigg|&\left[\alpha - \frac{\sigma^2}{2}\right]\mathrm{E}\left[\Delta(S)\right] + \mathrm{E}\left[\ln c(S)\right] \\ &-2\left[\sigma\sqrt{\mathrm{E}\left[\Delta(S)\right]} + \left|\alpha - \frac{\sigma^2}{2}\right|\mathrm{std}\left[\Delta(S)\right] + \mathrm{std}\left[\ln c(S)\right]\right]\Bigg| \end{aligned} \quad \textbf{(39)}$$

The dollar loss in the value of the portfolio including liquidity risk is greater than that implied by the standard *VaR* measure. It differs from the standard calculation in three ways:

- First, the liquidation horizon δ is replaced by the expected execution lag in selling the S shares, $\mathrm{E}[\Delta(S)]$. This may differ due to the size of the shares in the portfolio.
- Second, the initial discount on the shares sold must be included. This is the term $\mathrm{E}[\ln c(S)]$. It is negative because $c(S) \le 1$.

- Third, the volatility of the changes in value needs to be increased to include the volatility of the execution time, $|\alpha - \sigma^2/2|$std[$\Delta(S)$], as well as the volatility of the quantity discount, std[ln $c(S)$].

Applying the LVaR formula in practice. This liquidity adjusted *VaR* measure is easy to calculate, in theory. It requires an estimate of the mean and standard deviation of the market price's movements (α,σ), an estimate of the mean and standard deviation of the quantity discount (E[ln $c(S)$],std[ln $c(S)$]), and an estimate of the mean and standard deviation of the execution time for a block of S shares (E[$\Delta(S)$],std[$\Delta(S)$]). In principle, these should be easy to estimate.

This is certainly the case for the mean and standard deviation of the market price, which is obtained with standard techniques. Computing the remaining parameters is more problematic. In order to infer the parameters of the distributions of $c(S)$ and $\Delta(S)$, firms (or traders) need to collect time-series data on the shares traded, prices received, and times to execution. In the short run, before an historical database can be built, however, subjective estimates must be used, based on trading experience.

Alternatively, one could calculate the standard deviation of the market price, conditional upon a serious market decline. This conditional standard deviation may be a reasonable proxy for the sum of the standard deviations of the market prices and the liquidity discount. The intuition is that in a market crash, sales are dominating purchases. So the prices observed are due to the joint movements of the market price and the quantity discount combined.

The model can be extended to multiple assets, stochastic volatilities with jumps, and risk aversion (see Jarrow and Subramanian (1997b))

The model by Jarrow and Subramanian is an intriguing endeavour to apply a risk-neutral-like approach to liquidity risk measurement. Liquidation value depends on position size S and on "objective" market variables only, i.e. expected execution lag $\Delta(S)$ and quantity discount $c(S)$ Subjective policy parameters or constraints, such as the upper limit on liquidation time T, are not relevant. In this model, the dynamic optimisation framework provides only the procedure to obtain a compact fair estimate of liquidation value and *LVaR*. The model is structured in order to make such measures totally independent of arbitrary (i.e. "split") execution strategies.

4.4 A Synopsis of Model Features

The following table summarises the main features of the three most relevant models examined before. The model by Bangia, Diebold, Schuermann and Stroughair (1999) is relatively straightforward to implement both at asset and portfolio levels, provided that reliable data on quotes are available. In Jarrow and Subramanian (1997a), unlike Almgren and Chriss (1999a), the time to liquidation is not an endogenous (optimised) variable: the model does not advise on optimal time to liquidation, nor on the optimal sequence of trades. The "format" of the resulting formulae resembles that of the bid-ask quantile approach of Bangia, Diebold, Schuermann and Stroughair (1999). In the model by Jarrow and Subramanian liquidation costs require the estimation of a permanent price impact function (quantity discount) and an execution lag, while in Bangia et al. quoted spreads are assumed to incorporate such information, and size is not relevant. In order to apply the model by Jarrow and Subramanian, one faces the formidable tasks of (a) modelling the distributions of the execution lag and the quantity discount and (b) fitting the required parameters to data. In the suggestions for implementations, the authors suggest an alternative, and more practical, approach that goes directly to the estimation of the volatility of the liquidation price, washing away the three-factor process. Elegant as this construction may appear, it hardly offers a viable approach to liquidity risk measurement . As to the relevance of the Almgren and Chriss model, we may argue that findings about patterns of optimal execution strategies are of greater practical interest than what the model says about liquidity risk and its measurement.

Finally, it is to be remarked that none of the approaches considered attempts to model the dynamics of liquidity risk under stress, in particular how the price impact function responds to the correlation between trading by the individual operator and aggregate excess demand (or supply), i.e. the effect on transaction costs of positive-feedback trading vis-à-vis negative-feedback trading.

Table 1 *Comparison of Alternative Models for Measuring Liquidity-Adjusted VaR*

		Models		
Features		*Bangia, Diebold, Schuermann and Stroughair (1999)*	*Jarrow and Subramanian (1997)*	*Almgren and Chriss (1999)*
risk components	expected bid-ask spread	yes	no	yes
	volatility of bid-ask spread	yes	no	no
	expected price impact	no	yes	yes
	volatility of price impact	no	yes	no
	type of price impact	not relevant	permanent	temporary and permanent
assumptions	liquidation horizon	not relevant	exogenous	exogenous
	market conditions	any	any	normal
	price process	not relevant	continuous time geometric random walk	discrete time arithmetic random walk (continuous time as limiting case)
	shape of impact function	not relevant	not specified	any (linear case analysed)
	trade size considered in impact function	none	amount traded by the individual trader	amount traded by the individual trader
	revision of initial strategy	not relevant	no	no (dynamic strategies discussed)
Required data and parameters		historical series of mid-prices and bid-ask spreads	mean and standard deviation of: return, quantity discount, execution time	price volatility, temporary and permanent impact parameters
Variable to be optimised		none	expectation of proceeds from liquidation (not risk-adjusted when assuming risk neutrality)	expectation of execution cost adjusted for price risk
Output variables		*LVaR*	liquidation value *LVaR*	efficient frontier of optimal trading trajectories, optimal trajectory given risk aversion natural execution time (half life) liquidation value *LVaR*
Extensibility to portfolios		yes	possible but complex	possible but complex

4.5 Assessing Liquidity Risk under Stress

Assessment of liquidity in abnormal conditions relies mainly on qualitative information about market behaviour. In theory, risk managers may better anticipate and monitor liquidity crisis if aggregate information about prevailing exposures and trading strategies in the market were available. In Bank for International Settlements (2000a) a framework for simulating and interpreting aggregate market dynamics under stress is presented. When a risk manager makes an *ex ante* assessment of market liquidity risk under stress, he must attempt to gauge the amount of rebalancing and rehedging that each type of trading strategy would call for. Ideally, he would use information such as the trading demand resulting from each type of strategy. The market-wide total would allow the risk manager to see whether market liquidity risk under this stress scenario is high or low. Market liquidity risk under stress would be inversely related to the market-wide total change in exposure, other factors held constant.

An accurate assessment of each strategy's rebalancing and rehedging demand would call for different information for each of the five factors discussed above (see p.21, The Macro-View: the Impact of VaR models on Market Behaviour).

- **Stop-loss rules.** Information on how far away from stop-loss limits traders in the market are, along with the size of each stop-loss trader's exposure in the market.
- **Leverage and funding risk.** Information on the leverage of asset managers and relative value traders, along with information on their exposures to the market.
- **Limited arbitrage.** Information on the size of arbitrageur's exposures.
- **Sharpe-ratio-based trading.** Information on the size of such traders' exposures.
- **Dynamic hedging.** Information on the size of dealers' short option positions in the market and on the distribution of positions across strikes.

Of course, for all five factors such precise information on all of the relevant players in a market is not available to a risk manager. However, the quantity of agents' rebalancing and rehedging in stress is likely to be positively related to the size of their exposures. Risk managers use a variety of sources of information to sketch a map of prevailing strategies in order to anticipate the possible emergence of a one-way market. These include public sources, proprietary information sold by outside data providers—e.g. State Street data on cross-border investment flows—and private information gathered while observing customers' order flow and discussing with the firm's sales people, customers and counterparties. However, the obstacles to put together timely and reliable information in a format that allows comparisons across time are formidable.

In order to assess liquidity risk in a crisis scenario, the same study proposes an *aggregate stress testing* exercise, to be performed by supervisory authorities. Such an exercise should aggregate the data on exposures, exposure change and losses under several stress scenarios provided by a group of most active financial institutions. In the future, disclosure and standardisation of risk models could make such an endeavour feasible. In the present situation, it can only provide a check-list for qualitative assessment based on experience, intuition, and guesswork. Also from the point of view of the individual bank, the assessment of the impact of a widespread liquidity crisis on its profit and loss statement remains a highly subjective exercise.

5. Concluding Remarks

Risk management models are one of the most impressive achievements of applied research in finance. Their implementation borrows from both theoretical and empirical analysis. Standard *VaR* models are rooted in portfolio theory. Liquidity-enhanced *VaR* models, presented in this paper, are rooted in research on financial market microstructure and transaction costs.

Important benefits come from the widespread adoption of risk management models among financial institutions. Their use fosters knowledge acquisition, promotes the highest standards of efficiency in information systems, allows clear attribution of operational limits and performance targets, provides sharper assessment of the risk/reward opportunities confronting banks, and stimulates insightful analysis of market past and current behaviour.

However, a clear point has emerged from recent experience: risk management is not an automated control system allowing a sort of “management by numbers”, and by numbers only, of market exposures. It cannot be for three reasons.

First, financial engineering is not an exact science. Several of the parameters of portfolio risk models—distribution shapes, volatilities, time scaling coefficients and correlations, to name a few—cannot be quantified safely with mechanical, “black-box” procedures. An indeterminacy principle, much stronger than the one in quantum mechanics, challenges all our attempts to gauge financial risks. Their reasonable estimation is a matter of craftsmanship, requiring financial knowledge, market experience and judgement.

Second, relying on a set of fuzzy and unstable risk indicators to set the route may be dangerous, both at the individual firm level, and even more at the market level, as anecdotical evidence on the systemic effects of *VaR* strongly suggests.

Third, we contend that we cannot hope too much from technical extensions of computing models in order to manage some crucial components of risk, such as the risk we have considered in more detail in the paper, i.e. liquidity risk. Liquidity risk in abnormal market conditions stems from endogenous uncertainty about aggregate market behaviour. Event studies and artificial market models may well explain factors conducive to herd behaviour and liquidity holes. We agree with Daníelsson (2000), affirming that current, state-of-the-art, liquidity adjusted *VaR* models are very far from including all the factors considered in the models of market crises. One could question whether new versions of such models will be ever be able to model crises so as to measure a significant statistic of potential losses, individual or aggregate, conditional on their occurrence.

The positive aspect is that awareness of these shortcomings is growing. Risk management professional practice has been able to self-correct some of the flaws that may arise, and have indeed arisen, from a naïve application of *VaR* models. Promising developments are in place. The tendency towards the use of more stable measures for volatility, differentiated according to the long term behaviour of markets, is one comforting example. Growing caution in accounting for correlation effects in the measurement of aggregate risk capital is another. As a result, the scope of application of *VaR* has been refocused. A new framework has emerged where many of the problems that were supposed to be resolved by mechanic rules based on *VaR*—capital allocation, optimisation of risk-adjusted performance—are being tackled with more qualitative approaches, e.g. design of incentive systems, negotiation, constructive discussion between risk managers and traders. Designers of risk control systems have learnt to be discreet in applying operational *VaR* limits at higher organisational levels, where *VaR* figures are unstable and difficult to manage. Aggregate *VaR*, allowing for diversification effects, has become an informational indicator, more useful for seeking opportunities in the strategic allocation of exposures than for enforcing controls.

Learning from these findings, we can turn to the key aspect of our analysis, i.e. the suitability of *VaR* models as regulatory tools, and the impact of regulation on capital adequacy on risk management practice. Growing awareness of risk is good news for the stability of the banking systems, as is the diffusion of better technological infrastructures and internal control systems.

However, four hidden dangers can be detected.

First, endorsement of internal risk models by regulators may warrant an undeserved lease on life to obsolete and flawed technical approaches. Enactment at the national level of a directive from a supranational body usually takes years, and compliance by regulated entities takes an even longer time. Due to inertia, a long time after being dismissed by the smartest banks, obsolete approaches might work their way through a much wider population of medium-sized institutions. This could happen not only in the small minority of banks applying for approval of their internal models for calculating capital charges, but in all regulated banks, as an effect of the enforcement of the *qualitative* standards concerning internal risk control systems.

Second, a larger population of inexperienced users of *VaR* models may induce positive-feedback trading behaviour, increasing the risks of adverse systemic impact of risk management. In this respect, forcing smaller banks to adopt a risk control framework they do not fully understand may be worse than being satisfied with standardised capital ratios only.

Third, excessive emphasis on *VaR* as a tool for regulatory control, or internal control by independent units or external auditors, would ask *VaR* an impossible endeavour, i.e. correct assessment of risk exposure by outsiders with no direct presence on the markets, and little or no time to discuss (and argue) with the traders who live on the markets.

Fourth, linking required capital to a multiple of daily *VaR*, makes capital charges endogenous to market dynamics. If short-memory volatilities are fed into the models, in low-volatility markets with positive trends banks might build up huge exposures with associated seemingly low *VaR*. A market correction could trigger a surge in volatility, and *VaR* would suddenly explode, prompting banks to unload massive positions in a rush.

The words "is based on, but differs significantly from, the methodology developed by JP Morgan" have been put on the front page of the RiskMetrics™ Technical Document. That statement may be rephrased as "there is more to real risk management than what can be wrapped in a computing model borrowed from a third party". "There is more to supervision on bank risk than standardised capital requirements", we may add, shifting to the supervisors' perspective. However, with all the esteem for the banks' expertise in managing their risks, supervision on market risk must preserve its unique feature, in the sense that supervisors should impose *exogenous* constraints on business activity. Making this point clear, one can reasonably guess where the boundary line must be traced between risk figures that deserve endorsement by regulators, and risk figures that bank risk managers would better keep for themselves.

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