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Channel Assignment with Separation for Interference Avoidance in Wireless Networks *

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Abstract

Given an integer $\sigma > 1$, a vector $(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$ of nonnegative integers, and an undirected graph $G = (V, E)$, an $L(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$-coloring of G is a function f from the vertex set V to a set of nonnegative integers such that $|f(u) - f(v)| \geq \delta_i$, if $d(u, v) = i,\ 1 \leq i \leq \sigma - 1$, where $d(u, v)$ is the distance (i.e. the minimum number of edges) between the vertices u and v. An optimal $L(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$-coloring for G is one using the smallest range λ of integers over all such colorings. This problem has relevant application in channel assignment for interference avoidance in wireless networks, where channels (i.e. colors) assigned to interfering stations (i.e. vertices) at distance i must be at least δ_i apart, while the same channel can be reused in vertices whose distance is at least σ. In particular, two versions of the coloring problem – $L(2, 1, 1)$, and $L(\delta_1, 1, \ldots, 1)$ – are considered. Since these versions of the problem are NP-hard for general graphs, efficient algorithms for finding optimal colorings are provided for specific graphs modeling realistic wireless networks including rings, bidimensional grids, and cellular grids.

Key Words: Wireless Networks, Channel Assignment, Interferences, Rings, Cellular Grids, Bidimensional Grids, $L(2, 1, 1)$-coloring, $L(\delta_1, 1, \ldots, 1)$-coloring.

1 Introduction

The tremendous growth of wireless networks requires an efficient use of the scarce radio spectrum allocated to wireless communications. However, the main difficulty against an efficient use of radio spectrum is given by *interferences*, caused by unconstrained simultaneous transmissions, which result in damaged communications that need to be retransmitted leading to a higher cost of the service. Interferences can be eliminated (or at least reduced) by means of suitable *channel assignment* techniques. Indeed, co-channel interferences caused by frequency reuse is one of the most critical factors on the overall system capacity in the wireless networks. The purpose of

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channel assignment algorithms is to make use of radio propagation loss characteristics in order to increase the radio spectrum reuse efficiency and thus to reduce the overall cost of the service.

The channel assignment algorithms partition the given radio spectrum into a set of disjoint channels that can be used simultaneously by the stations while maintaining acceptable radio signals. By taking advantage of physical characteristics of the radio environment, the same channel can be reused by two stations at the same time without interferences (*co-channel stations*), provided that the two stations are spaced sufficiently apart. The minimum distance at which co-channels can be reused with no interferences is called *co-channel reuse distance* σ.

The interference phenomena may be so strong that even different channels used at near stations may interfere if the channels are too close. Since perfect filters are not available, interference between close frequencies is a serious problem, which can be handled either by adding guard frequencies between adjacent channels or by imposing channel separation. In this latter approach, followed in the present paper, channels assigned to near stations must be separated by a gap on the radio spectrum – counted in a certain number of channels – which is inversely proportional to the distance between the two stations. In other words, the channels $f(u)$ and $f(v)$ assigned to the stations u and v at distance i, with $i < \sigma$, must verify $|f(u) - f(v)| \geq \delta_i$ when a minimum *channel separation* δ_i is required between stations at distance i. The purpose of channel assignment algorithms is to assign channels to transmitters in such a way that the co-channel reuse distance and channel separation constraints are verified and the difference between the highest and lowest channels assigned is kept as small as possible.

Formally, let the channels be modeled as colors, i.e. nonnegative integers, and let the stations of the network be modeled as the vertices of an undirected graph $G = (V, E)$ where the edges correspond to pairs of stations whose transmission regions intersect. Defined the distance $d(u, v)$ between stations u and v of the network as the number of edges on a shortest path between the corresponding vertices u and v of G, and given the co-channel reuse distance σ and the channel separation vector $(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$ of nonnegative integers, an $L(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$-coloring of the graph G is a function f from the vertex set V to the set $\Lambda = \{0, \ldots, \lambda\}$ of colors, such that

$|f(u) - f(v)| \geq \delta_i$ if $d(u, v) = i$, for $i = 1, 2, \ldots, \sigma - 1$. The *channel assignment problem with separation (CAPS)* is defined as the problem of finding an optimal $L(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$-coloring of G, that is one which minimizes the largest color $\lambda \in \Lambda$. Note that, since the set Λ contains 0, the overall number of colors involved by a coloring f is in fact $\lambda + 1$ (although, due to the channel separation constraint, some colors in $\{1, \ldots, \lambda - 1\}$ might not be actually assigned to any vertex).

Up to now, most works have considered a small reuse distance because cells were made very large to minimize their number (since site surveys, renting space for the antenna and transceiver, the antenna and transceiver itself, all added up to a high price.) For example, for the well-known "Philadelphia" channel assignment instances, realistic values of σ are 3 or 4, while actual separation values are $\delta_1 = 2$, and $\delta_i = 1$ for $2 \leq i \leq \sigma - 1$ [17]. However, in the next 4th generation of wireless access systems, due to the decreasing cost of infrastructures and to the need of wide bandwidth, a large number of small cells, each with significant power, is expected. In such a scenario, a small re-use distance will not be feasible anymore, and σ will be expected to be much larger [20].

The CAPS problem has been shown to be NP-hard for general graphs, and therefore it is computationally intractable. When the channel separation vector $(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$ is equal to $(1, 1, \ldots, 1)$, there is only the co-channel reuse constraint, but no channel separation constraint [12]. In such a case, the channel assignment problem has been widely studied in the past, e.g. see the paper by Chlamtac and Pinter [7]. In particular, the intractability of optimal $L(1, 1, \ldots, 1)$-coloring, for any positive integer σ, has been proved by McCormick [14]. For $\sigma = 3$, Battiti, Bertossi and Bonuccelli [1] found optimal $L(1, 1)$-colorings for rings, hexagonal, cellular, and bidimensional grids, as well as efficient heuristics for geometric graphs. Optimal $L(1, 1, \ldots, 1)$-colorings, for any positive integer σ, have been proposed by Bertossi and Pinotti [3] for rings, complete trees, and bidimensional grids. Sen et al. [16] provided lower and upper bounds for $L(1, 1, \ldots, 1)$-colorings on cellular grids. However, their solution is optimal only for $\sigma = 2$ and $\sigma = 3$.

When the channel separation constraint is present, the problem has been studied only for small values of σ. The intractability of the $L(2,1)$-coloring has been shown by Griggs and Yeh [10] along with bounds on the number of channels for buses, rings and hypercubes. Later, bounds for this problem on chordal graphs and arbitrary trees have been found, respectively, by Sakai [15] and Chang and Kuo [6]. Approximated solutions for outerplanar, permutation and split graphs are presented by Bodlaender et al. [5]. Optimal solutions for the $L(\delta_1, \delta_2)$-coloring problem on bidimensional grids and cellular grids have been given by Van Den Heuvel et al. [19], who provided also an optimal $L(2,1,1)$-coloring for bidimensional grids. An optimal $L(2,1,1)$-coloring for complete binary trees has been shown in [4].

As a related case, when $\sigma = 3$ and $(\delta_1, \delta_2) = (0,1)$, the $L(0,1)$-coloring problem models that of avoiding only the so-called *hidden interferences*, due to stations which are outside the hearing range of each other and transmit to the same receiving station. Optimal $L(0,1)$-colorings have been provided by Makansi [13] for buses and bidimensional grids. Bertossi and Bonuccelli [2] proved the intractability of optimal $L(0,1)$-coloring, giving also optimal solutions for rings and complete binary trees. Finally, as another related case, observe that the classical minimum vertex coloring problem on undirected graphs arises when $\sigma = 2$ and $\delta_1 = 1$. Thus, the minimum vertex coloring problem consists in finding an optimal $L(1)$-coloring.

This paper further investigates the $L(2,1,1)$-coloring problem and starts to investigate also the $L(\delta_1, 1, \ldots, 1)$-coloring problem. Since those problems have been proved to be NP-complete by Griggs and Yeh [10], optimal solutions for special networks are considered. In particular, solutions to the $L(2,1,1)$-coloring problem are exhibited for rings and cellular grids. Finally, solutions to the $L(\delta_1, 1, \ldots, 1)$-coloring problem are also given, when $\sigma \geq 3$, for rings, with $1 \leq \delta_1 \leq \lfloor \frac{\lambda}{2} \rfloor$, and for bidimensional grids, with $1 \leq \delta_1 \leq \lfloor \frac{\sigma-1}{2} \rfloor$. In other words, as it will be clear later on, given any $\delta_1 \geq 1$, the $L(\delta_1, 1, \ldots, 1)$-coloring problem is solved on bidimensional grids and rings for every $\sigma \geq 2\delta_1 + 1$ using as few colors as the $L(1, 1, \ldots, 1)$-coloring problem.

In all cases, optimal solutions are provided by means of efficient channel assignment algorithms. For all networks a channel can be assigned to any vertex in constant time, provided that the relative position of the vertex in the network is locally known. In the case that the vertices do not initially know their own relative positions in the network, channels can be assigned in parallel to the vertices after the execution of simple distributed algorithms for computing the vertex positions, which require optimal time and number of messages.

Finally, it is also discussed when channel separation, as considered in this paper, is better than adding guard frequencies between adjacent channels.

2 Preliminaries

The channel assignment problem on a network N with no channel separation constraint and co-channel reuse distance σ, namely the $L(1,1,\ldots,1)$-coloring problem, can be reduced to a classical coloring problem on an *augmented* graph $G_{N,\sigma}$ obtained as follows. The vertex set of $G_{N,\sigma}$ is the same as the vertex set of N, while an edge $[r,s]$ belongs to the edge set of $G_{N,\sigma}$ iff the distance $d(r,s)$ between the vertices r and s in N satisfies $d(r,s) \leq \sigma - 1$. Now, colors must be assigned to the vertices of $G_{N,\sigma}$ so that every pair of vertices connected by an edge is assigned a couple of different colors and the minimum number of colors is used. Hence, the role of *maximum clique* in $G_{N,\sigma}$ is apparent for deriving lower bounds on the minimum number of channels for the $L(1,1,\ldots,1)$-coloring problem on N. A *clique* K for $G_{N,\sigma}$ is a subset of vertices of $G_{N,\sigma}$ such that for each pair of vertices in K there is an edge. By well-known graph theoretical results, a clique of size k in the augmented graph $G_{N,\sigma}$ implies that at least k different colors are needed to color $G_{N,\sigma}$. In other words, the size of the largest clique in $G_{N,\sigma}$ is a lower bound for the number of channels required to solve the channel assignment problem without channel separation constraint. Clearly, in the presence of both channel separation and co-channel reuse distance constraints, at least as many channels are required as in the presence of the channel separation constraint only. Formally, a lower bound for the $L(1,1,\ldots,1)$-coloring problem is also a lower bound for the $L(\delta_1,1,\ldots,1)$-coloring problem, with $\delta_1 \geq 1$. In particular, lower bounds for the $L(1,1)$- and $L(1,1,1)$-coloring problems hold also for the $L(2,1)$- and $L(2,1,1)$-coloring problems.

Let the *complement graph* $\overline{G} = (V, \overline{E})$ of a graph $G = (V, E)$ be the graph having the same vertex set V as G and having the edge set $\overline{E}$ obtained by swapping edges and non-edges in E. Recall that a *Hamilton path* is a path that traverses each vertex of a graph exactly once.

Lemma 1 [10] *Consider the $L(\delta_1, 1, \ldots, 1)$-coloring problem, with $\delta_1 \geq 2$, on a graph $G = (V, E)$ such that $d(u, v) < \sigma$ for every pair of vertices u and v in V. Then, $\lambda = |V| - 1$ if and only if $\overline{G}$ has a Hamilton path.*

Consider the *star* graph S_ρ which consists of a *center* vertex c with degree ρ, and ρ *ray* vertices of degree 1.

Lemma 2 [10] *Let the center c of S_ρ be already colored. Then, the largest color required for an $L(2,1)$-coloring of S_ρ is at least:*

$$\lambda = \begin{cases} \rho + 1 & \text{if } f(c) = 0 \text{ or } f(c) = \rho + 1, \\ \rho + 2 & \text{if } 0 < f(c) < \rho + 1. \end{cases}$$

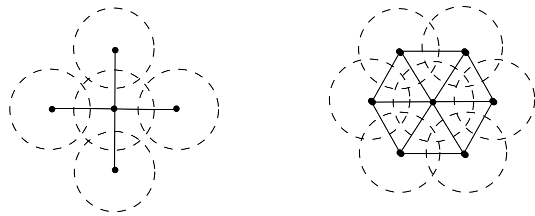
In this paper, several network topologies are examined, namely, bidimensional grids, cellular networks, and rings. Such networks model the regular placement of stations in the Euclidean plane with no obstacles, when the transmission region of each station is a circle of fixed radius centered on the transmitter site. Each vertex of the network represents a station, and an edge corresponds to two stations whose transmission regions intersect. Figure 1 shows how stations can be placed in the plane in such a way that the transmission region adjacencies are modeled by some of the above networks. For the sake of simplicity, the cellular networks will be represented by grids, and optimal channel assignment algorithms will be shown for grids of sufficiently large sizes.

Note that the cellular network is currently the most important to the radio engineer, since stations in such a network not only cover the whole plane, but also present the smallest possible transmitter density. Although the other studied networks may not cover the whole plane, they will be suitable for future generations of wireless access systems when only certain areas, where users with extensive bandwidth requirements are expected to be, will be covered. Indeed, since the required transmitter power increases linearly with the bandwidth, high speed radio access will have a very limited range and dense ubiquitous infrastructures (like cellular networks) will have tremendous costs. As a possible alternative, the "infostation"-concept outlines a sparse infrastructure of information-kiosks, close to which high data rate communication is feasible. This infrastructure can be distributed in a Manhattan fashion inside cities (modeled by a bidimensional grid), or in a loop around a city (modeled by a ring) [8, 20].

The channel assignment algorithms to be presented allow any vertex to self-assign its proper channel in constant time, provided that it knows its relative position within the network. If this is not the case, such relative positions can be computed for all the vertices using simple distributed algorithms requiring optimal time and optimal number of messages. Assume that each vertex of the network only knows its own geographic position (e.g. by means of its I.D. or a local geographic position system (GPS)) and the names of its neighbours (which can be easily obtained by the usual topology-exchange distributed algorithm [18]). The vertices are assumed to be asynchronous and can communicate by exchanging control messages (e.g. via dedicated system signals such as SS7 or MAC protocols such as ALOHA). There is only one kind of control message, which is sent by a vertex to tell its geographic position and its relative position to its neighbours. The computation is started by a single vertex, which is the only vertex which initially knows its relative position. When a vertex receives a control message from a neighbour, it is capable of recognizing whether the sender is a North, South, East, or West neighbour, by comparing its geographic position and that of the sender (the agreement about the actual cardinality points can be established and broadcast by the vertex starting the computation, after knowing the GPS positions of its neighbors). When a vertex receives a control message from a neighbour, if it has not yet computed its position and some conditions are met, then it computes its own relative position and in turn sends a control message, otherwise it neglects the message.

3 Optimal $L(\delta_1, 1, \ldots, 1)$-coloring for Bidimensional Grids

A *bidimensional grid* B of size $r \times c$ has r rows and c columns, indexed respectively from 0 to $r-1$ (from top to bottom) and from 0 to $c-1$ (from left to right), with $r \geq 2$ and $c \geq 2$. A generic



(a) (b)

Figure 1: Adjacencies among transmitting regions modeled by (a) bidimensional grids and (b) cellular networks.

vertex u of B will be denoted by $u = (i, j)$, where i is its row index and j is its column index. All internal vertices, i.e. those not on the borders, have degree 4. In particular, an internal vertex $u = (i, j)$ is adjacent to the vertices $(i-1, j), (i, j+1), (i+1, j)$, and $(i, j-1)$.

Optimal solutions for the $L(\delta_1, \delta_2)$- and $L(2, 1, 1)$-coloring problems on bidimensional grids have been provided by Van Den Heuvel et al. [19]. In this section, the $L(\delta_1, 1, \ldots, 1)$-coloring problem is dealt with.

Lemma 3 *There is an $L(\delta_1, 1, \ldots, 1)$-coloring of a bidimensional grid B of size $r \times c$, with $r \geq \sigma$ and $c \geq \sigma$, only if $\lambda \geq \left\lceil \frac{\sigma^2}{2} \right\rceil - 1$.*

Proof Let first restrict to the $L(1, \ldots, 1)$-coloring problem, with co-channel reuse distance σ. Consider a generic vertex $x = (i, j)$ of B, and its opposite vertex at distance $\sigma - 1$ on the same column, i.e., $y = (i - \sigma + 1, j)$. All the vertices of B at distance $\sigma - 1$ or less from both x and y are mutually at distance $\sigma - 1$ or less. Therefore, in the associated graph $G_{B,\sigma}$, they form a clique, and they must be assigned to different colors. In details, such a clique, denoted as $K_B(x, \sigma)$, is defined as follows: $K_B(x, \sigma) = \left\{ (i - \sigma + 1 + t, j - t), \ldots, (i - \sigma + 1 + t, j + t) \ : \ 0 \leq t \leq \left\lfloor \frac{\sigma - 1}{2} \right\rfloor \right\} \ \cup$

$$\left\{ \left(i - \left\lceil \frac{\sigma-1}{2} \right\rceil + t, j - \left\lceil \frac{\sigma-1}{2} \right\rceil + t\right), \ldots, \left(i - \left\lceil \frac{\sigma-1}{2} \right\rceil + t, j + \left\lceil \frac{\sigma-1}{2} \right\rceil - t\right) \ : \ 1 \leq t \leq \left\lceil \frac{\sigma-1}{2} \right\rceil \right\}$$

Summing up over t, the size of the clique results to be

$$|K_B(x,\sigma)| = \sum_{t=0}^{\lfloor\frac{\sigma-1}{2}\rfloor}(2t+1) + \sum_{t=1}^{\lceil\frac{\sigma-1}{2}\rceil}\left(2\left(\left\lceil\frac{\sigma-1}{2}\right\rceil - t\right)+1\right) = \left\lceil\frac{\sigma^2}{2}\right\rceil.$$

Hence, at least $|K_B(x,\sigma)| = \lceil\frac{\sigma^2}{2}\rceil$ colors are required for the $L(1,\ldots,1)$-coloring problem, and thus $\lambda \geq \lceil\frac{\sigma^2}{2}\rceil - 1$. Therefore, the same lower bound holds also for the $L(\delta_1, 1, \ldots, 1)$-coloring problem. □

In the following, an optimal coloring algorithm for bidimensional grids is exhibited. The Grid-$L(\delta_1, 1, \ldots, 1)$-coloring algorithm works for $\sigma \geq 3$ and $\delta_1 \leq \lfloor\frac{\sigma-1}{2}\rfloor$.

Algorithm Grid-$L(\delta_1, 1, \ldots, 1)$-coloring (B, r, c, σ);

- if σ is odd, then assign to each vertex $u = (i,j)$ the color

$$f(u) = \left(\left(\frac{\sigma-1}{2}+1\right)i + \frac{\sigma-1}{2}j\right) \bmod \left\lceil\frac{\sigma^2}{2}\right\rceil$$

- if σ is even and $\Delta = \lfloor\frac{\sigma-1}{2}\rfloor$ is even, let $i' = i \bmod \sigma$ and $j' = j \bmod \sigma$, then assign to each vertex $u = (i,j)$ the color

$$f(u) = \begin{cases} \Delta(\frac{\sigma}{2}i' + j') \bmod \frac{\sigma^2}{2} & \text{if } 0 \leq i' \leq \frac{\sigma}{2}-1 \text{ and } 0 \leq i' \leq \frac{\sigma}{2}-1, \\ & \text{or } \frac{\sigma}{2} \leq i' \leq \sigma-1 \text{ and } \frac{\sigma}{2} \leq j' \leq \sigma-1, \\ \left(\Delta(\frac{\sigma}{2}i' + j') + 1\right) \bmod \frac{\sigma^2}{2} & \text{if } 0 \leq i' \leq \frac{\sigma}{2}-1 \text{ and } \frac{\sigma}{2} \leq j' \leq \sigma-1, \\ & \text{or } \frac{\sigma}{2} \leq i' \leq \sigma-1 \text{ and } 0 \leq j' \leq \frac{\sigma}{2}-1. \end{cases}$$

- if σ is even and $\Delta = \lfloor\frac{\sigma-1}{2}\rfloor$ is odd, let $i' = i \bmod \sigma$ and $j' = j \bmod \sigma$, then assign to each vertex $u = (i,j)$ the color

$$f(u) = \begin{cases} \Delta(\frac{\sigma}{2}i' + j') \bmod \frac{\sigma^2}{2} & \text{if } 0 \leq i' \leq \frac{\sigma}{2}-1 \text{ and } 0 \leq i' \leq \frac{\sigma}{2}-1, \\ & \text{or } \frac{\sigma}{2} \leq i' \leq \sigma-1 \text{ and } \frac{\sigma}{2} \leq j' \leq \sigma-1, \\ \left(\Delta\left(\frac{\sigma^2}{4} - 1 + \frac{\sigma}{2}i' + j'\right)\right) \bmod \frac{\sigma^2}{2} & \text{if } 0 \leq i' \leq \frac{\sigma}{2}-1 \text{ and } \frac{\sigma}{2} \leq j' \leq \sigma-1, \\ & \text{or } \frac{\sigma}{2} \leq i' \leq \sigma-1 \text{ and } 0 \leq j' \leq \frac{\sigma}{2}-1. \end{cases}$$

Before proving correctness and optimality of the above algorithm, a preliminary result is required.

Lemma 4 *Given an even $\sigma \geq 4$, let $\lambda = \frac{\sigma^2}{2} - 1$, and $\Delta = \lfloor\frac{\sigma-1}{2}\rfloor$. If Δ is even, then $k\Delta \bmod \frac{\sigma^2}{2}$ assumes all the even values in the range $[0, \frac{\sigma^2}{2}-1]$ while k varies within the interval $[0, \frac{\sigma^2}{4}-1]$. If Δ is odd, then $k\Delta \bmod \frac{\sigma^2}{2}$ assumes all the values in the range $[0, \frac{\sigma^2}{2}-1]$ while k varies within the interval $[0, \frac{\sigma^2}{2}-1]$.*

Proof Since σ is even, $\frac{\sigma^2}{2} = 2(\Delta+1)^2$. When Δ is even, $k\Delta \bmod \frac{\sigma^2}{2}$ can be rewritten as $2(\frac{\Delta}{2}k \bmod (\Delta+1)^2)$. Consider any value $x \in [0, (\Delta+1)^2 - 1]$. The congruence $\frac{\Delta}{2}k \bmod (\Delta+1)^2$ assumes the value x when $k \equiv x(\Delta^2 - 3) \bmod (\Delta+1)^2$. In other words, $(\Delta^2-3)\frac{\Delta}{2} \equiv 1 \bmod (\Delta+1)^2$, or equivalently, $\Delta^2 - 3$ is the multiplicative inverse of $\frac{\Delta}{2} \bmod (\Delta+1)^2$. Indeed, let $\frac{\Delta}{2} = t$ and $(\Delta+1)^2 = 4t^2 + 4t + 1$. Hence, $(\Delta^2-3)\frac{\Delta}{2} = t(4t^2-3)$, and then

$$
\begin{aligned}
& t(4t^2-3) \bmod (4t^2+4t+1) \equiv \\
& t(4t^2+1+4t-4t-1-3) \bmod (4t^2+4t+1) \equiv \\
& t(4t^2+1+4t) - t(4t+4) \bmod (4t^2+4t+1) \equiv \\
& (-4t^2-4t) \bmod (4t^2+4t+1) \equiv \\
& 1 \bmod (4t^2+4t+1).
\end{aligned}
$$

Therefore, if Δ is even, $\frac{\Delta}{2}k \bmod (\Delta+1)^2$ assumes all the values in $[0, (\Delta+1)^2 - 1]$, and thus $k\Delta \bmod \frac{\sigma^2}{2}$ assumes all the even values in the range $[0, \frac{\sigma^2}{2} - 1]$ while k varies within the interval $[0, \frac{\sigma^2}{4} - 1]$.

When Δ is odd, $k\Delta \bmod \frac{\sigma^2}{2}$ can be rewritten as $k\Delta \bmod 2(\Delta+1)^2$. Consider any value $x \in [0, 2(\Delta+1)^2 - 1]$. The congruence $k\Delta \bmod 2(\Delta+1)^2$ assumes the value x when $k \equiv x\,(\Delta(\Delta+1)-1) \bmod 2(\Delta+1)^2$. In other words, $\Delta\,(\Delta(\Delta+1)-1) \equiv 1 \bmod 2(\Delta+1)^2)$, or equivalently, $\Delta(\Delta+1)-1$ is the multiplicative inverse of $\Delta \bmod 2(\Delta+1)^2$.

Indeed, recalling that $AB \bmod AC = A(B \bmod C)$, it holds

$$
\begin{aligned}
& (\Delta\,(\Delta(\Delta+1)-1)) \bmod 2(\Delta+1)^2 \equiv \\
& (\Delta^2(\Delta+1)-\Delta) \bmod 2(\Delta+1)^2 \equiv \\
& (\Delta+1)\,(\Delta^2 \bmod 2(\Delta+1)) - \Delta) \bmod 2(\Delta+1)^2 \equiv \\
& ((\Delta+1)\,((\Delta^2-\Delta+\Delta) \bmod 2(\Delta+1)) - \Delta) \bmod 2(\Delta+1)^2 \equiv \\
& ((\Delta+1)\,(((\Delta+1)(\Delta \bmod 2) - \Delta) \bmod 2(\Delta+1)) - \Delta) \bmod 2(\Delta+1)^2 \equiv \\
& ((\Delta+1)1 - \Delta) \bmod 2(\Delta+1)^2 \equiv \\
& 1 \bmod 2(\Delta+1)^2.
\end{aligned}
$$

Therefore, if Δ is odd, then $k\Delta \bmod 2(\Delta+1)^2$ assumes all the values in the range $[0, 2(\Delta+1)^2 - 1]$ while k varies within the interval $[0, 2(\Delta+1)^2 - 1]$. □

Theorem 1 *The Grid-$L(\delta_1, 1, \ldots, 1)$-coloring algorithm is optimal for $\sigma \geq 3$ and $1 \leq \delta_1 \leq \lfloor \frac{\sigma-1}{2} \rfloor$.*

Proof When σ is odd, consider w.l.o.g. the vertices in B colored 0. By construction a vertex $u = (i,j)$ gets the color $f(u) = 0$ if $i\frac{\Delta}{2} + j(\frac{\Delta}{2}+1) \equiv 0 \bmod (\Delta(2\Delta+2)+1)$, where

$\Delta = \frac{\sigma-1}{2}$ and $\Delta(2\Delta+2)+1 = \left\lceil \frac{\sigma^2}{2} \right\rceil$. Solving such a congruence is the same as solving the linear Diophantine equation

$$i\frac{\Delta}{2} + j(\frac{\Delta}{2}+1) = c(\Delta(2\Delta+2)+1),$$

for any integer c. All the solutions of this equation are of the form [9]

$$\begin{cases} j &= c \\ i &= c(2\Delta+1) \end{cases}$$

In other words, the coordinates i and j of vertex u must satisfy $i+j = c(2\Delta+2) = c(\sigma+1)$. Therefore, the distance between two vertices colored 0 is at least $\sigma+1$, which satisfies the co-channel reuse constraint. The separation constraint is easily satisfied by construction. Finally, since the algorithm uses as few colors as in Lemma 3, it is optimal for odd values of $\sigma \geq 3$ and $\delta_1 \leq \Delta$.

When σ is even, the algorithm covers the bidimensional grid B with a tessellation of basic tiles T of size $\sigma \times \sigma$, each consisting of 4 sub-tiles of size $\frac{\sigma}{2} \times \frac{\sigma}{2}$. Precisely, as depicted in Figure 2, the left-upper and the right-lower corners of T are colored by sub-tile S_1, while the left-lower and right-upper corners by sub-tile S_2.

When σ and Δ are even, as shown in Table 1, both S_1 and S_2 are colored row by row, assigning colors separated exactly by Δ to any two consecutive vertices. According to Lemma 4, S_1 starts from 0 and spans all the even values in the range $[0, \left\lceil \frac{\sigma^2}{2} \right\rceil - 1]$, while S_2 starts from 1 and spans all the odd values in the range $[0, \left\lceil \frac{\sigma^2}{2} \right\rceil - 1]$.

When σ is even and Δ is odd, as shown in Table 2, both S_1 and S_2 are again colored row by row, assigning colors separated exactly by Δ to any two consecutive vertices. In this case, according to Lemma 4, all the consecutive multiples of $\Delta \bmod \left\lceil \frac{\sigma^2}{2} \right\rceil$ generate a single sequence of distinct $\left\lceil \frac{\sigma^2}{2} \right\rceil$ values in the range $[0, \left\lceil \frac{\sigma^2}{2} \right\rceil - 1]$, with the first half of the sequence coloring S_1, starting from 0, and the second half of the sequence coloring S_2, starting from $\frac{\sigma^2}{4}\Delta \bmod \left\lceil \frac{\sigma^2}{2} \right\rceil$.

By the construction shown in Figure 2 and the fact that, as proved in Lemma 4, S_1 and S_2 consist of all distinct values, it is easy to see that the same color is reused in two vertices of B which are exactly at distance σ, and therefore the co-channel reuse constraint is verified.

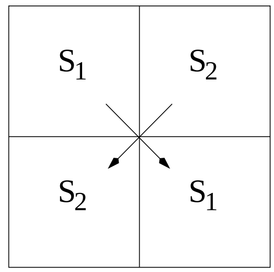


Figure 2: A tile T of size $\sigma \times \sigma$ consisting of 4 sub-tiles each of size $\frac{\sigma}{2} \times \frac{\sigma}{2}$.

In order to check that the separation constraint is also verified, note that inside S_1 and S_2 two consecutive vertices on the same row get two colors separated by Δ, while two consecutive vertices on the same column are separated by $k = \Delta(\frac{\sigma}{2} - 1)$, with $\Delta \leq k \leq \frac{\sigma^2}{2}$, when $\sigma \geq 4$. It remains to show that the separation constraint holds also on the 4 border lines between S_1 and S_2. Note that since σ is even, $\left\lceil \frac{\sigma^2}{2} \right\rceil = \frac{\sigma^2}{2}$ and $\Delta = \frac{\sigma}{2} - 1$. Note also that if Δ is even then $\sigma \geq 6$, while if Δ is odd then $\sigma \geq 4$.

Let Δ be even.

W.l.o.g., the separation between the following 4 pairs of adjacent vertices is evaluated:

1. $u = (\frac{\sigma}{2} - 1, j)$ and $v = (\frac{\sigma}{2}, j)$,
2. $u = (\frac{\sigma}{2} - 1, \frac{\sigma}{2} + j)$ and $v = (\frac{\sigma}{2}, \frac{\sigma}{2} + j)$,
3. $u = (i, \frac{\sigma}{2} - 1)$ and $v = (i, \frac{\sigma}{2})$,
4. $u = (i + \frac{\sigma}{2}, \frac{\sigma}{2} - 1)$ and $v = (i + \frac{\sigma}{2}, \frac{\sigma}{2})$.

Case 1 Consider $u = (\frac{\sigma}{2} - 1, j)$ and $v = (\frac{\sigma}{2}, j)$. Observing Table 1,

$$
\begin{array}{l}
(f(u) - f(v)) \bmod \frac{\sigma^2}{2} \equiv \\
\left(((\frac{\sigma}{2} - 1)\frac{\sigma}{2} + j)\Delta - (j\Delta + 1)\right) \bmod \frac{\sigma^2}{2} \equiv \\
\left(\left(\frac{\sigma}{2} - 1\right)\frac{\sigma}{2}\Delta - 1\right) \bmod \frac{\sigma^2}{2} \equiv \\
\left(\left(\frac{\sigma}{2} - 1\right)^2 \frac{\sigma}{2} - 1\right) \bmod \frac{\sigma^2}{2} \equiv \\
\left(\frac{\sigma}{2}\left((\frac{\sigma^2}{4} - \sigma + 1) \bmod \sigma\right) - 1\right) \bmod \frac{\sigma^2}{2} \equiv \\
\left(\frac{\sigma}{2}\left((\frac{\sigma^2}{4} + 1) \bmod \sigma\right) - 1\right) \bmod \frac{\sigma^2}{2}.
\end{array}
$$

Let $k = (\frac{\sigma^2}{4}+1) \bmod \sigma$. When $k=0$, $(f(u)-f(v)) \bmod \frac{\sigma^2}{2} = \left(\frac{\sigma}{2}k-1\right) \bmod \frac{\sigma^2}{2} = \frac{\sigma^2}{2}-1 > \Delta$, and therefore the separation constraint follows. Whereas, when $1 \leq k \leq \sigma-1$, observing that $\frac{\sigma}{2}k-1 < \frac{\sigma^2}{2}$, it holds $(f(u)-f(v)) \bmod \frac{\sigma^2}{2} = \left(\frac{\sigma}{2}k-1\right) \bmod \frac{\sigma^2}{2} = \frac{\sigma}{2}k-1 \geq \frac{\sigma}{2}-1 = \Delta$, and therefore the separation constraint follows.

Case 2 Consider $u = (\frac{\sigma}{2}-1, \frac{\sigma}{2}+j)$ and $v = (\frac{\sigma}{2}, \frac{\sigma}{2}+j)$. Observing Table 1

$$\begin{array}{l}(f(u)-f(v)) \bmod \frac{\sigma^2}{2} \equiv \\ \left(((\frac{\sigma}{2}-1)\frac{\sigma}{2}+j)\Delta+1-j\Delta\right) \bmod \frac{\sigma^2}{2} \equiv \\ \left(\frac{\sigma}{2}\left((\frac{\sigma^2}{4}-\sigma+1) \bmod \sigma\right)+1\right) \bmod \frac{\sigma^2}{2} \equiv \\ \left(\frac{\sigma}{2}\left((\frac{\sigma}{2}(\frac{\sigma}{2} \bmod 2)+1) \bmod \sigma\right)+1\right) \bmod \frac{\sigma^2}{2}.\end{array}$$

If $\frac{\sigma}{2}$ is even, then $f(u)-f(v) = \frac{\sigma}{2}+1 > \Delta$. If $\frac{\sigma}{2}$ is odd, then $f(u)-f(v) = \frac{\sigma^2}{4}+\frac{\sigma}{2}+1$, since $\frac{\sigma^2}{4}+\frac{\sigma}{2}+1 \leq \frac{\sigma^2}{2}-1$. Again, $f(u)-f(v) > \Delta$ if $\sigma \geq 6$, and the separation constraint is verified.

Case 3 Consider $u = (i, \frac{\sigma}{2}-1)$ and $v = (i, \frac{\sigma}{2})$. Observing Table 1,

$$\begin{array}{l}(f(u)-f(v)) \bmod \frac{\sigma^2}{2} \equiv \\ \left(((i+1)\frac{\sigma}{2}-1)\Delta-(i\frac{\sigma}{2}\Delta+1)\right) \bmod \frac{\sigma^2}{2} \equiv \\ \left((\frac{\sigma}{2}-1)^2-1\right) \bmod \frac{\sigma^2}{2} = \\ \frac{\sigma^2}{4}-\sigma.\end{array}$$

Hence, $f(u)-f(v) \geq \Delta$ when $\sigma \geq 6$.

Case 4 Consider $u = (i+\frac{\sigma}{2}, \frac{\sigma}{2}-1)$ and $v = (i+\frac{\sigma}{2}, \frac{\sigma}{2})$. Observing Table 1,

$$\begin{array}{l}(f(u)-f(v)) \bmod \frac{\sigma^2}{2} \equiv \\ \left(((i+1)\frac{\sigma}{2}-1)\Delta+1-(i\frac{\sigma}{2}\Delta)\right) \bmod \frac{\sigma^2}{2} \equiv \\ \left((\frac{\sigma}{2}-1)\Delta+1\right) \bmod \frac{\sigma^2}{2}.\end{array}$$

Hence, $f(u)-f(v) \geq \Delta$, when $\frac{\sigma}{2}-1 \geq 1$, that is, when $\sigma \geq 4$.

By similar arguments, observing Table 2, one can show that the separation constraint holds also on the 4 border lines between S_1 and S_2 when Δ is odd. □

	0	j	$\frac{\sigma}{2}-1$	$\frac{\sigma}{2}$	$\frac{\sigma}{2}+j$	$\sigma-1$
0	0	$j\Delta$	$\left(\frac{\sigma}{2}-1\right)\Delta$	1	$j\Delta+1$	$\left(\frac{\sigma}{2}-1\right)\Delta+1$
i	$i\frac{\sigma}{2}\Delta$		$\left((i+1)\frac{\sigma}{2}-1\right)\Delta$	$i\frac{\sigma}{2}\Delta+1$		$\left((i+1)\frac{\sigma}{2}-1\right)\Delta+1$
$\frac{\sigma}{2}-1$	$(\frac{\sigma}{2}-1)\frac{\sigma}{2}\Delta$	$(\frac{\sigma}{2}-1)\frac{\sigma}{2}\Delta+j\Delta$	$(\frac{\sigma^2}{4}-1)\Delta$	$(\frac{\sigma}{2}-1)\frac{\sigma}{2}\Delta+1$	$(\frac{\sigma}{2}-1)\frac{\sigma}{2}\Delta+j\Delta+1$	$(\frac{\sigma^2}{4}-1)\Delta+1$
$\frac{\sigma}{2}$	1	$j\Delta+1$	$\left(\frac{\sigma}{2}-1\right)\Delta+1$	0	$j\Delta$	$\left(\frac{\sigma}{2}-1\right)\Delta$
$\frac{\sigma}{2}+i$	$i\frac{\sigma}{2}\Delta+1$		$\left((i+1)\frac{\sigma}{2}-1\right)\Delta+1$	$i\frac{\sigma}{2}\Delta$		$\left((i+1)\frac{\sigma}{2}-1\right)\Delta$
$\sigma-1$	$(\frac{\sigma}{2}-1)\frac{\sigma}{2}\Delta+1$	$(\frac{\sigma}{2}-1)\frac{\sigma}{2}\Delta+j\Delta+1$	$(\frac{\sigma^2}{4}-1)\Delta+1$	$(\frac{\sigma}{2}-1)\frac{\sigma}{2}\Delta$	$(\frac{\sigma}{2}-1)\frac{\sigma}{2}\Delta+j\Delta$	$(\frac{\sigma^2}{4}-1)\Delta$

Table 1: Coloring of tile T when σ is even and Δ is even. All the operations are to be considered mod $\frac{\sigma^2}{2}$.

	0	j	$\frac{\sigma}{2}-1$	$\frac{\sigma}{2}$	$\frac{\sigma}{2}+j$	$\sigma-1$
0	0	$j\Delta$	$\left(\frac{\sigma}{2}-1\right)\Delta$	$\frac{\sigma^2}{4}\Delta$	$(\frac{\sigma^2}{4}+j)\Delta$	$(\frac{\sigma^2}{4}+\frac{\sigma}{2}-1)\Delta$
i	$i\frac{\sigma}{2}\Delta$		$\left((i+1)\frac{\sigma}{2}-1\right)\Delta$	$(\frac{\sigma^2}{4}+i\frac{\sigma}{2})\Delta$		$\left(\frac{\sigma^2}{4}+(i+1)\frac{\sigma}{2}-1\right)\Delta$
$\frac{\sigma}{2}-1$	$(\frac{\sigma}{2}-1)\frac{\sigma}{2}\Delta$	$(\frac{\sigma}{2}-1)\frac{\sigma}{2}\Delta+j\Delta$	$(\frac{\sigma^2}{4}-1)\Delta$	$(\frac{\sigma^2}{2}-\frac{\sigma}{2})\Delta$	$(\frac{\sigma^2}{2}-\frac{\sigma}{2}+j)\Delta$	$(\frac{\sigma^2}{2}-1)\Delta$
$\frac{\sigma}{2}$	$\frac{\sigma^2}{4}\Delta$	$(\frac{\sigma^2}{4}+j)\Delta$	$(\frac{\sigma^2}{4}+\frac{\sigma}{2}-1)\Delta$	0	$j\Delta$	$\left(\frac{\sigma}{2}-1\right)\Delta$
$\frac{\sigma}{2}+i$	$(\frac{\sigma^2}{4}+i\frac{\sigma}{2})\Delta$		$\left(\frac{\sigma^2}{4}+(i+1)\frac{\sigma}{2}-1\right)\Delta$	$i\frac{\sigma}{2}\Delta$		$\left((i+1)\frac{\sigma}{2}-1\right)\Delta$
$\sigma-1$	$(\frac{\sigma^2}{2}-\frac{\sigma}{2})\Delta$	$(\frac{\sigma^2}{2}-\frac{\sigma}{2}+j)\Delta$	$(\frac{\sigma^2}{2}-1)\Delta$	$(\frac{\sigma}{2}-1)\frac{\sigma}{2}\Delta$	$(\frac{\sigma}{2}-1)\frac{\sigma}{2}\Delta+j\Delta$	$(\frac{\sigma^2}{4}-1)\Delta$

Table 2: Coloring of tile T when σ is even and Δ is odd. All the operations are to be considered mod $\frac{\sigma^2}{2}$.

Note that, given any $\delta_1 \geq 1$, the Grid-$L(\delta_1, 1, \ldots, 1)$-coloring algorithm solves the $L(\delta_1, 1, \ldots, 1)$-coloring problem for every $\sigma \geq 2\delta_1 + 1$ using as few colors as the $L(1, 1, \ldots, 1)$-coloring problem. In particular, given $\delta_1 = 2$, the above algorithm solves the $L(2, 1, \ldots, 1)$-coloring problem for every $\sigma \geq 5$. The only values of the reuse distance not covered by the above algorithm when $\delta_1 = 2$ are $\sigma = 3$ and 4. However, in such cases, the $L(2,1)$- and $L(2,1,1)$-coloring problems had been solved by Van Den Heuvel et al. [19]. Therefore, the $L(2, 1, \ldots, 1)$-coloring problem can be optimally solved for any value of σ.

The coloring algorithms presented above allow any vertex v to self-assign its color in $O(1)$ time, provided that v knows its relative position (i, j) within the bidimensional grid. If this is not the case, the relative positions of all the vertices can be computed by a simple distributed algorithm as follows (see the end of Section 2 for the proper assumptions).

The computation is started by the upper-left corner vertex in the network, which is the only vertex knowing its position $(0, 0)$. A control message is structured as $CM(v, g_v, i, j)$, where g_v and (i, j) are the geographic and relative positions of v, respectively. When a vertex u receives

$CM(v, g_v, i, j)$ from a North neighbour v, then u computes its relative position $(i+1, j)$ and sends $CM(u, g_u, i+1, j)$ so as to propagate the computation downwards along the columns of the grid. In the first row, however, if v is a West neighbour of u and $i = 0$, then u computes its position $(0, j+1)$ and sends $CM(u, g_u, 0, j+1)$.

It is easy to see that the overall number of messages required is $O(rc)$ while the total time is $O(r+c)$, assuming that a message reaches its destination in $O(1)$ time. Since there are rc vertices in the grid and the grid diameter is $O(r+c)$, the channel assignment for all the vertices can be performed in a distributed fashion so as to require an optimal time and an optimal number of messages.

4 Optimal $L(2,1,1)$-coloring for Cellular Grids

A *cellular grid* C of size $r \times c$, with $r \geq 2$ and $c \geq 2$, is obtained from a bidimensional grid B of the same size augmenting the set of edges with left-to-right *diagonal* connections. Specifically, each vertex $u = (i, j)$ of C is also connected to the vertices $v = (i-1, j-1)$ and $z = (i+1, j+1)$. Hence, each vertex has degree 6, except for the vertices on the borders.

An optimal solution for the $L(\delta_1, \delta_2)$-coloring problem on cellular grids has been provided by Van Den Heuvel et al. [19]. In the following, an optimal solution for the $L(2,1,1)$-coloring problem is presented.

Lemma 5 *There is an $L(2,1,1)$-coloring of a cellular grid C of size $r \times c$, with $r \geq 4$ and $c \geq 4$, only if $\lambda \geq 11$.*

Proof Given the cellular grid $C = (V, E)$, consider the augmented graph $G_{C,4} = (V, E')$ and the subgraph D of C illustrated in Figure 3. All the 12 vertices of D are mutually at distance 3 or less, and they form a clique in $G_{C,4}$. Hence, they must be assigned to all different colors, and $\lambda \geq 11$. □

Figure 3 shows how to color the subgraph D in such a way that the channel separation constraint is verified for every two adjacent vertices. Moreover, Figure 4 shows a complete

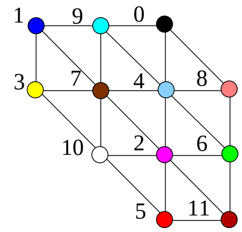


Figure 3: The subgraph D of C whose vertices form a clique in $G_{C,4}$, and an optimal $L(2,1,1)$-coloring for it.

coloring of a cellular grid C obtained by replicating the coloring for the subgraph D. Note that the channel separation constraint is verified not only for the vertices belonging to each copy of D, but also for the vertices belonging to the borders of two contiguous copies of D.

Formally, the coloring of a cellular grid can be described as follows.

Algorithm Cellular-$L(2,1,1)$-coloring (C, r, c);

- If $r \geq 4$ and $c \geq 4$, then assign to each vertex $u = (i,j)$ the color

$$f(u) = \begin{cases} 0 & \text{if } (i+j) \equiv 2 \bmod 6, i \text{ is even, and } j \text{ is even} \\ 1 & \text{if } (i+j) \equiv 0 \bmod 6, i \text{ is even, and } j \text{ is even} \\ 2 & \text{if } (i+j) \equiv 4 \bmod 6, i \text{ is even, and } j \text{ is even} \\ 3 & \text{if } (i+j) \equiv 1 \bmod 6, i \text{ is odd, and } j \text{ is even} \\ 4 & \text{if } (i+j) \equiv 3 \bmod 6, i \text{ is odd, and } j \text{ is even} \\ 5 & \text{if } (i+j) \equiv 5 \bmod 6, i \text{ is odd, and } j \text{ is even} \\ 6 & \text{if } (i+j) \equiv 5 \bmod 6, i \text{ is even, and } j \text{ is odd} \\ 7 & \text{if } (i+j) \equiv 2 \bmod 6, i \text{ is odd, and } j \text{ is odd} \\ 8 & \text{if } (i+j) \equiv 4 \bmod 6, i \text{ is odd, and } j \text{ is odd} \\ 9 & \text{if } (i+j) \equiv 1 \bmod 6, i \text{ is even, and } j \text{ is odd} \\ 10 & \text{if } (i+j) \equiv 3 \bmod 6, i \text{ is even, and } j \text{ is odd} \\ 11 & \text{if } (i+j) \equiv 0 \bmod 6, i \text{ is odd, and } j \text{ is odd} \end{cases}$$

Theorem 2 *The Cellular-$L(2,1,1)$-coloring algorithm is optimal for cellular grids of size $r \times c$, with $r \geq 4$ and $c \geq 4$.*

Proof In order to prove that the channel separation constraint is verified, it is useful to introduce the *Manhattan distance* $m(u,v)$ between any two vertices u and v, where $m(u,v)$ is the length of a shortest path between u and v including only horizontal and vertical edges, thus excluding diagonal edges. Now, any two consecutive colors are considered and it will be proved that such colors cannot be assigned to two adjacent vertices. For example, consider the pair of colors 2 and 3. A vertex $u = (i,j)$ gets the color 2 if and only if both i and j are even, and $i + j \equiv 4 \bmod 6$, while a vertex $v = (h, k)$ is colored 3 if and only if h is odd, k is even, and $h + k \equiv 1 \bmod 6$. The vertices u and v might belong to the same column, but to different rows. In this case, their distance is at least 3. In the case that they do not belong to the same column, they have Manhattan distance $m(u, v) = 3$. Hence, the vertex v which is closest to u and assigned to color 3 is $v = (i + 1, j + 2)$, as illustrated in Figure 4. Keeping track of the diagonal edges, the actual distance $d(u, v)$ is 2, and therefore the channel separation constraint is still verified. An analogous argument can be repeated for any pair of consecutive colors c and $c + 1$, with $0 \leq c \leq 10$.

To show that the co-channel reuse constraint holds, one notes that two vertices $u = (i, j)$ and $v = (h, k)$ get the same color if and only if their Manhattan distance $m(u, v) = 6$, and both $|i - h|$ and $|j - k|$ are even. Due to the diagonal edges, the actual distance $d(u, v)$ is at least 4. Indeed, the actual distance $d(u, v)$ could be 3 when $m(u, v) = 6$, but in this case $|i - h|$ and $|j - k|$ cannot be both even.

The optimality follows from the lower bound shown in Lemma 5. □

Finally, note that, when the vertices do not initially know their relative position within the cellular grid, a distributed algorithm can again be executed. The computation still starts from vertex $(0, 0)$, but it propagates along the "diagonals" of the grid.

5 Optimal $L(2, 1, 1)$- and $L(\delta_1, 1, \ldots, 1)$-coloring for Rings

A *ring* R of size n is a sequence of n vertices, indexed consecutively from 0 to $n - 1$, such that vertex i is connected to both vertices $(i - 1) \bmod n$ and $(i + 1) \bmod n$.

An optimal solution for the $L(2, 1)$-coloring problem on rings has been provided by Griggs and Yeh [10]. In this section, optimal solutions are exhibited for the $L(2, 1, 1)$ and $L(\delta_1, 1, \ldots, 1)$-coloring problems on rings.

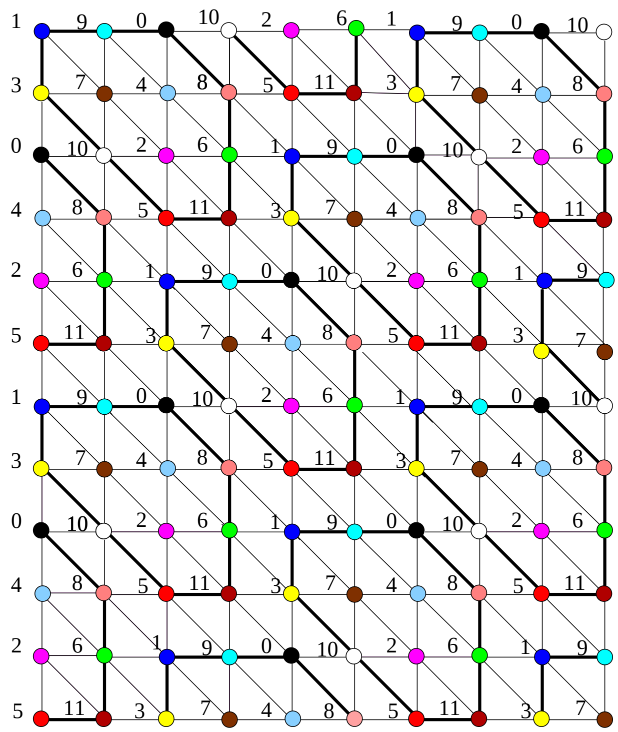


Figure 4: Optimal $L(2,1,1)$-coloring for a cellular grid C.

Lemma 6 *There is an $L(2,1,1)$-coloring for a ring R only if:*

$$\lambda \geq \begin{cases} 2(n-1) & \text{if } n = 1,2,3 \\ 4 & \text{if } n = 4,8, \text{ or } n \geq 12 \\ 3 + \left\lceil \frac{n \bmod 4}{\lfloor \frac{n}{4} \rfloor} \right\rceil & \text{if } 5 \leq n \leq 11 \text{ and } n \neq 8 \end{cases}$$

Proof For $n \leq 3$, R is a clique. Therefore, vertex i must be colored $2i$, for $0 \leq i \leq n-1$. For $n = 4$, by Lemma 1, there is no $L(2,1,1)$-coloring with $\lambda = 3$. Hence, $\lambda \geq 4$. For $n = 6$, 7 and 11, since any optimal $L(2,1,1)$-coloring uses at least as many colors as an optimal $L(1,1,1)$-coloring, the lower bound of λ derives from the lower bound proved in [3]. Finally, for $n = 5$, 8, 9, 10 and $n \geq 12$, the lower bound of λ derives from the lower bound proved in [10] since any optimal $L(2,1,1)$-coloring uses at least as many colors as an optimal $L(2,1)$-coloring. □

Algorithm Ring-$L(2,1,1)$-coloring (R, n);

1. if $n = 1,2,3$ assign to each vertex i the color $f(i) = 2i$;
2. if $n = 6$ assign to each vertex i the color

i	0	1	2	3	4	5
$f(i)$	0	2	4	1	3	5

3. if $n = 7$ assign to each vertex i the color

i	0	1	2	3	4	5	6
$f(i)$	0	2	4	6	1	3	5

4. if $n = 11$ assign to each vertex i the color

i	0	1	2	3	4	5	6	7	8	9	10
$f(i)$	0	2	4	1	3	5	0	2	4	1	3

5. if $n = 4,5,8,9,10$ or $n \geq 12$, let $\theta = 4\left(\lfloor \frac{n}{4} \rfloor - (n \bmod 4)\right)$, and assign to each vertex i of R the color

$$f(i) = \begin{cases} 0 & \text{if } i \equiv 0 \bmod 4 \text{ and } i < \theta, \text{ or } (i-\theta) \equiv 0 \bmod 5 \text{ and } i \geq \theta \\ 1 & \text{if } i \equiv 2 \bmod 4 \text{ and } i < \theta, \text{ or } (i-\theta) \equiv 3 \bmod 5 \text{ and } i \geq \theta \\ 2 & \text{if } (i-\theta) \equiv 1 \bmod 5 \text{ and } i \geq \theta \\ 3 & \text{if } i \equiv 3 \bmod 4 \text{ and } i < \theta, \text{ or } (i-\theta) \equiv 4 \bmod 5 \text{ and } i \geq \theta \\ 4 & \text{if } i \equiv 1 \bmod 4 \text{ and } i < \theta, \text{ or } (i-\theta) \equiv 2 \bmod 5 \text{ and } i \geq \theta \end{cases}$$

Theorem 3 *The Ring-$L(2,1,1)$-coloring algorithm is optimal for rings.*

Proof The first θ vertices are colored repeating $\lfloor \frac{n}{4} \rfloor - (n \bmod 4)$ times the color sequence $0,4,1,3$ of length 4, and the others repeating $n \bmod 4$ times the sequence $0,2,4,1,3$ of length

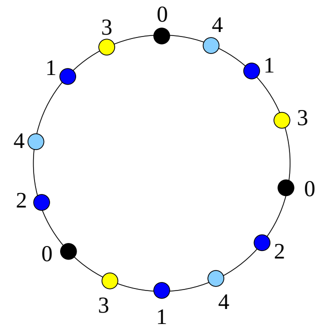


Figure 5: Optimal coloring obtained by the Ring-$L(2,1,1)$-coloring algorithm for a ring of size $n=14$, where $\theta = 4\left(\lfloor\frac{14}{4}\rfloor - (14 \bmod 4)\right) = 4$.

5 (see Figure 5). Since two vertices i and j get the same color only if they are at distance 4 or 5, and two adjacent colors have a gap of at least 2, the co-channel reuse and channel separation constraints hold. Observe that the largest color used when $n = 6,\ 7,\ 11$ is, respectively, 5, 6, and 5 which matches the lower bound given by $3 + \left\lceil \frac{n \bmod 4}{\lfloor \frac{n}{4} \rfloor} \right\rceil$. In all the other cases, $\lambda = 4$. Thus, optimality follows by Lemma 6. $\square$

Lemma 7 *Consider a ring R of size n and let $\sigma \geq 3$ and $\delta_1 \geq 1$. There is an $L(\delta_1, 1, \ldots, 1)$-coloring for R only if*

$$\lambda \geq \begin{cases} (n-1)\delta_1 & \text{if } n \leq 3 \\ n-1 & \text{if } 4 \leq n \leq \sigma \text{ and } 1 \leq \delta_1 \leq \lceil \frac{n}{2} \rceil - 1 \\ \delta_1 + \frac{n}{2} & \text{if } n \text{ is even, } 4 \leq n \leq \sigma, \text{ and } \delta_1 \geq \lceil \frac{n}{2} \rceil \\ 2\delta_1 & \text{if } n \text{ is odd, } 5 \leq n \leq \sigma, \text{ and } \delta_1 \geq \lceil \frac{n}{2} \rceil \\ \sigma - 1 + \lceil \frac{n \bmod \sigma}{\lfloor \frac{n}{\sigma} \rfloor} \rceil & \text{if } n > \sigma \end{cases}$$

Proof When $n \leq 3$, R is a clique, and thus all the vertices must get a color multiple of δ_1. Therefore, $\lambda \geq (n-1)\delta_1$.

When $4 \leq n \leq \sigma$ and $1 \leq \delta_1 \leq \lceil \frac{n}{2} \rceil - 1$, $G_{R,\sigma}$ is a clique. Therefore, all the vertices must get different colors, and thus $\lambda \geq n-1$.

When n is even, $4 \leq n \leq \sigma$ and $\delta_1 \geq \lceil \frac{n}{2} \rceil$, all the vertices must get different colors since $\sigma \geq n$. However n colors are not enough when $\delta_1 \geq \lceil \frac{n}{2} \rceil$, as it is argued below by contradiction. Suppose, indeed, that n colors are enough and let u and v be the two vertices that get colors

$f(u) = n-1$ and $f(v) = n-\delta_1-1$, respectively. Since $n \geq 4$, there is a vertex, say w, connected with v such that $f(w) = t$ and $|n-\delta_1-t-1| \geq \delta_1$. This implies that $n-\delta_1-t-1 \geq \delta_1$, since the case $-(n-\delta_1-t-1) \geq \delta_1$ would require $t \geq n$. As a consequence of $n-\delta_1-t-1 \geq \delta_1$, $\delta_1 \leq \frac{n-t-1}{2}$ results, which contradicts the hypothesis $\delta_1 \geq \frac{n}{2}$. It remains to show how many colors are required. When n is even, the vertices of R can be partitioned in two independent sets S and T of the same size $\frac{n}{2}$ which consist of all the even vertices and all the odd vertices, respectively. To use the minimum number of colors, the vertices of S must get colors $\{0, 1, \ldots, \frac{n}{2}-1\}$. Since any vertex of T is adjacent to a pair of vertices in S, the smallest color that can be used in T is $\delta_1 + 1$. Moreover, since all the vertices of T must get different colors, at least the color $\delta_1 + \frac{n}{2}$ must be used.

When n is odd, $5 \leq n \leq \sigma$ and $\delta_1 \geq \lceil \frac{n}{2} \rceil$, n colors are not enough as proved in the previous case. However, more than $\delta_1 + \frac{n}{2} + 1$ colors are needed. In particular, since n is odd, at least $2\delta_1 + 1$ colors are required. To prove this, suppose by contradiction that $\lambda = 2\delta_1 - 1$ is the largest used color which is assigned w.l.o.g. to vertex 0. Now, vertex 1 can only be colored by t, with $t \leq \delta_1 - 1$, to satisfy the separation constraint. By the same token, vertex 2 can get only a color $f(2)$ not smaller than $t+\delta_1$. Vertex 3, thus, gets a color $f(3) \leq f(2)-\delta_1$. In general, odd vertices get colors within $\{0, \ldots, \delta_1 - 1\}$ while even vertices get colors within $\{\delta_1+1, \ldots, 2\delta_1-1\}$. Thus, vertex $n-1$, which is even and adjacent to vertex 0, cannot be colored without violating the separation constraint. Therefore, a contradiction arises and at least one extra color is necessary. Hence, $\lambda \geq 2\delta_1$.

Finally, when $n > \sigma$, each color may appear at most $t = \left\lfloor \frac{n}{\sigma} \right\rfloor$ times. Therefore, at least $\left\lceil \frac{n}{t} \right\rceil$ colors are needed. Observed that $n = \left\lfloor \frac{n}{\sigma} \right\rfloor \sigma + (n \bmod \sigma)$, it follows that at least $\left\lceil \frac{n}{t} \right\rceil = \sigma + \left\lceil \frac{n \bmod \sigma}{\lfloor \frac{n}{\sigma} \rfloor} \right\rceil$ colors are required. Therefore, $\lambda \geq \sigma - 1 + \lceil \frac{n \bmod \sigma}{\lfloor \frac{n}{\sigma} \rfloor} \rceil$. □

In the following, an optimal $L(\delta_1, 1, \ldots, 1)$-coloring algorithm for rings is exhibited.

Ring-$L(\delta_1, 1, \ldots, 1)$-coloring (R, n, σ);

1. if $n \leq 3$, assign to each vertex i the color $f(i) = i\delta_1$, with $0 \leq i \leq n-1$;

2. if $4 \leq n \leq \sigma$ and $1 \leq \delta_1 \leq \lceil \frac{n}{2} \rceil - 1$, assign to each vertex i the color

$$f(i) = \begin{cases} \frac{i}{2} & \text{if } i \text{ is even} \\ \lceil \frac{n}{2} \rceil + \lfloor \frac{i}{2} \rfloor & \text{if } i \text{ is odd} \end{cases}$$

3. if n is even, $4 \leq n \leq \sigma$, and $\delta_1 \geq \lceil \frac{n}{2} \rceil$, assign to each vertex i the color

$$f(i) = \begin{cases} \frac{i}{2} & \text{if } i \text{ is even} \\ \delta_1 + \lceil \frac{i}{2} \rceil & \text{if } i \text{ is odd} \end{cases}$$

4. if n is odd, $5 \leq n \leq \sigma$, and $\delta_1 \geq \lceil \frac{n}{2} \rceil$, assign to each vertex i the color

$$f(i) = \begin{cases} \delta_1 & \text{if } i = n-1 \\ 2\delta_1 & \text{if } i = n-2 \\ \frac{i}{2} & \text{if } i \text{ is even and } i \neq n-1 \\ \delta_1 + \lceil \frac{i}{2} \rceil & \text{if } i \text{ is odd and } i \neq n-2 \end{cases}$$

5. if $n > \sigma$ and $n \equiv 0 \bmod \sigma$, then:

   - Let $\lambda = \sigma - 1 + \lceil \frac{n \bmod \sigma}{\lfloor \frac{n}{\sigma} \rfloor} \rceil = \sigma - 1$, $\Delta = \lfloor \frac{\lambda}{2} \rfloor$, assign to each vertex i the color

$$f(i) = \begin{cases} i\Delta \bmod \sigma & \text{if (} \sigma \text{ is odd) or (} \sigma \text{ is even and } \Delta \text{ is odd)} \\ i\Delta \bmod \sigma & \text{if } \sigma \text{ is even and } \Delta \text{ is even and } i \equiv t \bmod \sigma, \text{ for } 0 \leq t \leq \frac{\sigma}{2} - 1 \\ (i\Delta + 1) \bmod \sigma & \text{if } \sigma \text{ is even and } \Delta \text{ is even and } i \equiv t \bmod \sigma, \text{ for } \frac{\sigma}{2} \leq t \leq \sigma - 1 \end{cases}$$

6. if $n > \sigma$ and $n \not\equiv 0 \bmod \sigma$, then:

   - Let $\lambda = \sigma - 1 + \lceil \frac{n \bmod \sigma}{\lfloor \frac{n}{\sigma} \rfloor} \rceil$, $\Delta = \lfloor \frac{\lambda}{2} \rfloor$, $\zeta = n \bmod \lambda$, and $\theta = \left\lfloor \frac{n}{\lambda} \right\rfloor - \zeta$

   (a) assign to each vertex i, with $i \geq \theta\lambda$, the color

$$f(i) = \begin{cases} j\Delta \bmod (\lambda+1) & \text{if (} (\lambda+1) \text{ is odd) or (} (\lambda+1) \text{ is even and } \Delta \text{ is odd)} \\ j\Delta \bmod (\lambda+1) & \text{if } (\lambda+1) \text{ is even and } \Delta \text{ is even and } i \equiv t \bmod (\lambda+1), \text{ for } 0 \leq t \leq \frac{\lambda+1}{2} - 1 \\ (j\Delta + 1) \bmod (\lambda+1)) & \text{if } (\lambda+1) \text{ is even and } \Delta \text{ is even and } i \equiv t \bmod (\lambda+1), \text{ for } \frac{\lambda+1}{2} \leq t \leq \lambda \end{cases}$$

   where $j = i - \theta\lambda$;

   (b) assign to each vertex i, with $i \leq \theta\lambda - 1$, the color

$$f(i) = \begin{cases} 0 & \text{if } i \equiv 0 \bmod \lambda \\ (i+1)\Delta \bmod (\lambda+1) & \text{if (} (\lambda+1) \text{ is odd) or (} (\lambda+1) \text{ is even and } \Delta \text{ is odd) and } i \not\equiv 0 \bmod \lambda \\ (i+1)\Delta \bmod (\lambda+1) & \text{if } (\lambda+1) \text{ is even and } \Delta \text{ is even and } i \equiv t \bmod \lambda, \text{ for } 1 \leq t \leq \frac{\lambda+1}{2} - 1 \\ ((i+1)\Delta + 1) \bmod (\lambda+1)) & \text{if } (\lambda+1) \text{ is even and } \Delta \text{ is even and } i \equiv t \bmod \lambda, \text{ for } \frac{\lambda+1}{2} \leq t \leq \lambda \end{cases}$$

It is worth noting that the lower bound on λ given in Lemma 7 is not tight when $\sigma = 4$ and $n \equiv 0 \bmod 4$. Indeed, in this case, Lemma 7 gives $\lambda \geq \sigma - 1 + \lceil \frac{n \bmod \sigma}{\lfloor \frac{n}{\sigma} \rfloor} \rceil = 3$, while Lemma 6 provides $\lambda \geq 4$.

The Ring-$L(\delta_1, 1, \ldots, 1)$-coloring algorithm assumes $\sigma \geq 3$ and works for any value of $\delta_1 \geq 1$, when $n \leq \sigma$, and for $\delta_1 \leq \lfloor \frac{\lambda}{2} \rfloor$, when $n > \sigma$. Note that when $\sigma = 4$ and $n \equiv 0 \bmod 4$, such an algorithm solves the $L(1,1,1)$-coloring problem but not the $L(2,1,1)$-coloring problem.

Before proving the correctness and optimality of the Ring-$L(\delta_1, 1, \ldots, 1)$-coloring algorithm, two preliminary lemmas are required.

Lemma 8 *Given $n > \sigma \geq 3$, let $\lambda = \sigma - 1 + \lceil \frac{n \bmod \sigma}{\lfloor \frac{n}{\sigma} \rfloor} \rceil$, and $\Delta = \lfloor \frac{\lambda}{2} \rfloor$. If ($\lambda + 1$ is odd) or ($\lambda + 1$ is even and Δ is odd), $i\Delta \bmod (\lambda + 1)$ assumes all the values in the range $[0, \lambda]$ while i varies within the interval $[0, \lambda]$.*

Proof When $\lambda + 1$ is odd, λ and Δ can be rewritten as $\lambda = 2t$ and $\Delta = t$. Consider any value $x \in [0, 2t]$. In the following, it is shown that there is a value of $i' = x(2t-1) \bmod (2t+1)$, with $i' \in [0, 2t]$, such that $i't \equiv x \bmod (2t+1)$. Indeed such congruence holds because t and $2t+1$ are coprime, and $2t-1$ is the multiplicative inverse of $t \bmod (2t+1)$, as it can be easily proved observing that $(2t-1)t \equiv ((t-1)(2t+1)+1) \equiv 1 \bmod 2t+1$.

When $\lambda + 1$ is even, λ and Δ can be rewritten as $\lambda = 2t - 1$ and $\Delta = t - 1$. Consider any value $x \in [0, 2t-1]$. Since by hypothesis $\Delta = t - 1$ is odd, it follows that t is even, and Δ and $2t$ are coprime integers. As in the previous case, the congruence $i(t-1) \equiv x \bmod 2t$ holds for $i' = x(t-1) \bmod 2t$, observed that $t - 1$ is the multiplicative inverse of $t - 1 \bmod 2t$. In fact, $(t-1)^2 \equiv t^2 + 1 - 2t \equiv 2t(\frac{t}{2}) - 2t + 1 \equiv 1 \bmod 2t$.

In conclusion, in both cases, for any value $x \in [0, \ldots, \lambda]$, there is a value of $i \in [0, \lambda]$ such that $i\Delta \equiv x \bmod (\lambda + 1)$. □

Lemma 9 *Given $n > \sigma \geq 3$, let $\lambda = \sigma - 1 + \lceil \frac{n \bmod \sigma}{\lfloor \frac{n}{\sigma} \rfloor} \rceil$, and $\Delta = \lfloor \frac{\lambda}{2} \rfloor$. If ($\lambda + 1$ is even and Δ is even), $i\Delta \bmod (\lambda + 1)$ assumes all the even values in the range $[0, \lambda - 1]$ while i varies within the interval $[0, \frac{\lambda+1}{2} - 1]$.*

Proof If both $\lambda + 1$ and Δ are even, then they can be rewritten as $\lambda + 1 = 4t + 2$ and $\Delta = 2t$. The congruence $i2t \bmod (4t+2)$ assumes exactly the same values as the congruence $2(it \bmod (2t+1))$. The congruence $it \bmod (2t+1)$ assumes all the values in $[0, 2t]$, by Lemma 8 since $(2t+1)$ is odd. Therefore the congruence $2(it \bmod (2t+1))$ assumes all the even values in the interval $[0, 4t]$. $\square$

Theorem 4 *Given $\sigma \geq 3$, the Ring-$L(\delta_1, 1, \ldots, 1)$-coloring algorithm is optimal for any value of $\delta_1 \geq 1$, when $n \leq \sigma$, and for $\delta_1 \leq \lfloor \frac{\lambda}{2} \rfloor$, when $n > \sigma$.*

Proof The correctness and optimality is proved following the 6 cases of the algorithm.

Case 1: When $n \leq 3$, R is a clique. Therefore, vertex i is colored $i\delta_1$, for $0 \leq i \leq n-1$, and $\lambda = (n-1)\delta_1$.

Case 2: When $4 \leq n \leq \sigma$ and $1 \leq \delta_1 \leq \lceil \frac{n}{2} \rceil - 1$, $G_{R,\sigma}$ is a clique. By construction, the colors of two consecutive vertices of R are at least separated by $\lceil \frac{n}{2} \rceil - 1 \geq \delta_1$. Finally, each vertex gets a different color and, as it is easy to verify, the largest used color is $n-1$ which is assigned either to vertex $n-1$ when n is even or to vertex $n-2$ when n is odd.

Case 3: When n is even, $4 \leq n \leq \sigma$ and $\delta_1 \geq \lceil \frac{n}{2} \rceil - 1$, the largest color used in the algorithm is $f(n-1) = \delta_1 + \lceil \frac{n-1}{2} \rceil = \delta_1 + \frac{n}{2}$, which matches the lower bound for λ. Moreover, it is easy to see that the separation between two consecutive vertices is

$$f(i+1) - f(i) = \begin{cases} \delta_1 & \text{if } i \text{ is odd,} \\ \delta_1 + 1 & \text{if } i \text{ is even,} \end{cases}$$

while $f(n-1) - f(0) = \delta_1 + \frac{n}{2}$.

Case 4: When n is odd, $5 \leq n \leq \sigma$ and $\delta_1 \geq \lceil \frac{n}{2} \rceil - 1$, the coloring of the vertices $0, \ldots, n-3$ of the algorithm is exactly the same as in Case 3, and therefore the separation constraint holds. For the remaining two vertices, $f(n-2) = 2\delta_1$, and $f(n-1) = \delta_1$. Therefore, $f(n-2) - f(n-1) = f(n-1) - f(0) = \delta_1$, while $f(n-2) - f(n-3) = 2\delta_1 - \frac{n-3}{2} > \delta_1$ because by assumption $\delta_1 \geq \lceil \frac{n}{2} \rceil$. Since the largest color used is $2\delta_1$, the solution is optimal.

Case 5: When $n > \sigma$ and $n \equiv 0 \bmod \sigma$, the algorithm solves the $L(\Delta, 1, \ldots, 1)$-coloring, where $\lambda = \sigma - 1$ and $\Delta = \lfloor \frac{\sigma-1}{2} \rfloor$. Therefore the algorithm solves also all the $L(\delta_1, 1, \ldots, 1)$-colorings with $1 \leq \delta_1 \leq \Delta$. The solution proposed is different depending on the parity of σ and Δ.

In particular, when $n > \sigma$ and $n \equiv 0 \bmod \sigma$, and (σ is odd) or (σ is even and Δ is odd), vertex i of R gets color $f(i) = i\Delta \bmod \sigma$. In practice, R is colored by repeating $\frac{n}{\sigma}$ times the sequence

$B = 0,\ \Delta,\ 2\Delta,\ \ldots,\ i\Delta,\ \ldots,\ (\sigma-1)\Delta$, where all the operations above are done $\bmod\ \sigma$. Thus the separation between two consecutive vertices is $|f(i+1)-f(i)| = |(i+1)\Delta \bmod \sigma - i\Delta \bmod \sigma| \geq ((i+1)\Delta - i\Delta) \bmod \sigma = \Delta$. Moreover, by Lemma 8, B consists of σ different colors, and the same color is reused at distance σ.

In contrast, when $n > \sigma$ and $n \equiv 0 \bmod \sigma$, and (σ is even and Δ is even), by Lemma 9, it is known that the vertices of R $i \equiv t \bmod \sigma$ with $0 \leq t \leq \frac{\sigma}{2} - 1$ get all the even colors in the interval $[0, \sigma - 2]$, whereas the vertices of R $i \equiv t \bmod \sigma$ with $\frac{\sigma}{2} \leq t \leq \sigma - 1$ get all the odd colors in the interval $[1, \sigma - 1]$. In practice, R is colored by repeating $\frac{n}{\sigma}$ times the sequence $B = B_{\text{even}}\ B_{\text{odd}}$, where $B_{\text{even}} = 0,\ \Delta,\ 2\Delta,\ \ldots, i\Delta, \ldots,\ (\frac{\sigma}{2}-1)\Delta$ and $B_{\text{odd}} = 1,\ \Delta+1,\ 2\Delta+1,\ \ldots, i\Delta+1, \ldots,\ (\frac{\sigma}{2}-1)\Delta+1$, where all the operations above are mod σ. To check the separation constraint, consider two consecutive vertices of R. Four cases may arise depending on the position of the vertices in R. Indeed, either both vertices get even (odd) colors or one vertex gets an even (odd) color and the other vertex an odd (even) color.

`both even:` $|f(i+1)-f(i)| = |(i+1)\Delta \bmod \sigma - i\Delta \bmod \sigma| = \Delta$;

`both odd:` $|f(i+1)-f(i)| = |((i+1)\Delta+1) \bmod \sigma - (i\Delta+1) \bmod \sigma| = \Delta$;

`even-odd:` $|(\frac{\sigma}{2}-1)\Delta \bmod \sigma) - 1) \bmod \sigma| = (-\Delta) \bmod \sigma - 1 = \sigma - \Delta - 1 > \Delta$;

`odd-even:` $|((\sigma-1)\Delta+1) \bmod \sigma) - 0| = (-\Delta+1) \bmod \sigma = \sigma - \Delta + 1 > \Delta$.

Finally, by Lemma 9, the same color is reused at distance σ.

Case 6: When $n > \sigma \geq 3$ and $n \not\equiv 0 \bmod \sigma$, the algorithm again solves the $L(\Delta, 1, \ldots, 1)$-coloring, where $\Delta = \lfloor \frac{\lambda}{2} \rfloor$, and therefore it solves also all the $L(\delta_1, 1, \ldots, 1)$-colorings with $1 \leq \delta_1 \leq \Delta$. In this case, as proved in Lemma 7, the largest used color is $\lambda = \sigma - 1 + \lceil \frac{n \bmod \sigma}{\lfloor \frac{n}{\sigma} \rfloor} \rceil$. In practice, the algorithm colors R as follows:

- Build the following two sequences, A and B, of length λ and $\lambda + 1$, respectively, where all the operations are mod $(\lambda+1)$:

$$A = \begin{cases} 0,\ 2\Delta, 3\Delta,\ \ldots, i\Delta, \ldots\ \lambda\Delta & \text{if } \lambda+1 \text{ is odd or } (\lambda+1 \text{ is even and } \Delta \text{ is odd}) \\ 0,\ 2\Delta, 3\Delta,\ \ldots,\ i\Delta, \left(\frac{\lambda+1}{2}-1\right)\Delta, & 1,\ 2\Delta+1, 3\Delta+1,\ \ldots, i\Delta+1, \ldots,\ \left(\frac{\lambda+1}{2}-1\right)\Delta+1 \\ & \text{if } \lambda+1 \text{ is even and } \Delta \text{ is even} \end{cases}$$

$$B = \begin{cases} 0,\ \Delta,\ 2\Delta,\ \ldots, i\Delta, \ldots,\ \lambda\Delta & \text{if } (\lambda+1 \text{ is odd) or } (\lambda+1 \text{ is even and } \Delta \text{ is odd}) \\ 0,\ \Delta,\ 2\Delta,\ \ldots, i\Delta, \ldots,\ \left(\frac{\lambda+1}{2}-1\right)\Delta, & 1,\ \Delta+1,\ 2\Delta+1, \ldots, i\Delta+1,\ \ldots,\ \left(\frac{\lambda+1}{2}-1\right)\Delta+1 \\ & \text{if } \lambda+1 \text{ is even and } \Delta \text{ is even} \end{cases}$$

- Let $\zeta = n \bmod \lambda$, and $\theta = \left\lfloor \frac{n}{\lambda} \right\rfloor - \zeta$; color the vertices of R from 0 up to $\theta\lambda - 1$ by repeating θ times the sequence A, and color the remaining vertices of R by repeating ζ times the sequence B. Note that all the vertices of R are colored because the sequences A and B have length, respectively, λ and $\lambda+1$ and $n = \theta\lambda + \zeta(\lambda+1)$.

The proof that the algorithm verifies the separation and reuse constraints, based on Lemmas 8 and 9, is similar to that of Case 5. □

It is worth noting that the two algorithms proposed in this section solve the $L(2,1,\ldots,1)$-coloring problem for any $\sigma \geq 4$. Indeed, the Ring-$L(2,1,1)$-coloring algorithm works for $\sigma = 4$, while the Ring-$L(\delta_1,1,\ldots,1)$-coloring algorithm always solves the $L(2,1,\ldots,1)$-coloring problem for $\sigma \geq 5$. In general, given any $\delta_1 \geq 1$, the Ring-$L(\delta_1,1,\ldots,1)$-coloring algorithm optimally solves the $L(\delta_1,1,\ldots,1)$-coloring problem for every $\sigma \geq 2\delta_1+1$, independently on the size n of the ring and using as few colors as the $L(1,1,\ldots,1)$-coloring problem. Nevertheless, the $L(\delta_1,1,\ldots,1)$-coloring problem may be solved for values of σ smaller than $2\delta_1+1$ for suitable values of $n < (2\delta_1+1)^2$.

Finally, when the vertices of the ring do not initially know their relative positions, a simple distributed algorithm can again be executed to compute them. The computation is started by any single vertex, which designates itself as vertex 0. Assuming for simplicity that the ring is unidirectional, one round along the ring is needed to compute the vertex relative positions and a second round is required to broadcast n to all the vertices. Both the time and the number of messages are $O(n)$, which are optimal.

Network G	$L(1)$	$L(0,1)$	$L(1,1)$	$L(2,1)$	$L(1,1,1)$	$L(2,1,1)$
bus (path)	2	2	3	5	4	5
ring (cycle)	2 or 3	2 or 3	3 or 4	5	4 or 5	5
complete binary tree	2	3	4	5	6	7
bidimensional grid	2	4	5	7	8	9
cellular grid	3	6	7	9	12	12
References	folklore	[2, 13]	[1, 3, 16]	[4, 6, 10, 19]	[3, 16]	[4, 19], this paper

Table 3: Minimum number $\lambda + 1$ of channels used for a sufficiently large network G, when the reuse distance varies between 2 and 4, and the channel separations are 0, 1 or 2.

6 Conclusion

Tables 3 and 4 summarize the results for optimal $L(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$-coloring, known up to now in the literature. Specifically, such tables indicate, for the most common regular networks, the minimum number of channels required. In all the cases, there are efficient algorithms to assign channels to vertices. The channel assigned to any vertex can be computed locally provided that the relative position of the vertex in the network is known. Such a computation can be performed in constant time for all the networks, except for binary trees which require logarithmic time in the number of vertices. In particular, the present paper dealt with the $L(2,1,1)$ and $L(\delta_1, 1, \ldots, 1)$-coloring problems which consider a channel separation between adjacent vertices. The proposed results extend those already known in the literature for the $L(2,1)$, $L(1,1,1)$, and $L(1,\ldots,1)$-coloring problems. For the sake of completeness, Table 3 reports also the minimum number of colors required by the $L(1)$-coloring problem, which reduces to the classical minimum vertex coloring of a graph.

It is worthy to note that the optimal solutions presented in this paper for the $L(2,1,1)$-coloring problem on rings and cellular grids use as few colors as the $L(1,1,1)$-coloring problem on the same networks. Similarly, the $L(\delta_1, 1, \ldots, 1)$-coloring problem on rings and bidimensional grids has been optimally solved in this paper using as few colors as the $L(1,\ldots,1)$-coloring problem on the same networks. In other words, in the cases above, no extra channels are required when the channel separation constraint is introduced to avoid adjacent frequency interference. In all such

Network G	$L(\delta_1,\delta_2)$	$L(1,\ldots,1)$	$L(\delta_1,1,\ldots,1)$
bus (path)	$2\delta_1+\delta_2$	σ	$\max(\sigma,2\delta_1)$
ring (cycle)	–	$\sigma+\lceil\frac{n \bmod \sigma}{\lfloor\frac{n}{\sigma}\rfloor}\rceil$	$\sigma+\lceil\frac{n \bmod \sigma}{\lfloor\frac{n}{\sigma}\rfloor}\rceil$ if $\delta_1\le\lfloor\frac{\lambda}{2}\rfloor$
complete binary tree	–	$2^{\lfloor\frac{\sigma-1}{2}\rfloor+1}+2^{\lceil\frac{\sigma-1}{2}\rceil}-2$	–
bidimensional grid	$2\delta_1+3\delta_2$	$\lceil\frac{\sigma^2}{2}\rceil$	$\lceil\frac{\sigma^2}{2}\rceil$ if $\delta_1\le\lfloor\frac{\sigma-1}{2}\rfloor$
cellular grid	$\begin{cases}3(\delta_1+\delta_2) & \text{if } \delta_1\ge 2\delta_2\\ 9\delta_2 & \text{if } \frac{3}{2}\delta_2\le\delta_1\le 2\delta_2\\ 4\delta_1+3\delta_2 & \text{if } \delta_1\le\frac{3}{2}\delta_2\end{cases}$	$\lceil\frac{3}{4}\sigma^2\rceil\le\lambda+1\le\sigma^2-\sigma+1$	–
References	[19]	[3, 16]	this paper

Table 4: Minimum number $\lambda+1$ of channels used for a sufficiently large network G for arbitrary reuse distance or channel separations (note that in [19] channels are cyclic, that is, channel λ is adjacent to channel 0).

cases, using channel separation is always better than adding guard frequencies between adjacent channels. Indeed, suppose that the bandwidth of a single channel is β and that the bandwidth of a guard frequency is γ, with $0<\gamma\le\beta$. Consider a channel assignment problem with co-channel reuse distance σ. If the $L(1,\ldots,1)$-coloring problem is optimally solved, say using $\lambda+1$ colors, and then a guard frequency is added between adjacent channels to handle the adjacent frequency interference problem, then the overall bandwidth used is

$$W_{\text{guard}}=(\lambda+1)\beta+\lambda\gamma.$$

In contrast, if channel separation is introduced as described by the $L(\delta_1,1\ldots,1)$-coloring problem, say using $\lambda'+1$ colors, then the total bandwidth used is

$$W_{\text{separation}}=(\lambda'+1)\beta.$$

Clearly, if $\lambda=\lambda'$, then $W_{\text{separation}}<W_{\text{guard}}$, which implies that using channel separation is better than using guard frequency. This happens for the $L(\delta_1,1,\ldots,1)$-coloring problem on rings and bidimensional grids, as well as for the $L(2,1,1)$-coloring problem on rings and cellular grids. In the remaining cases, the channel separation technique may or may not be more appealing than the guard frequency technique, depending on the values of γ. For example, consider the

$L(1,1,1)$- and the $L(2,1,1)$-coloring problems on a bidimensional grid. By the above reasoning one obtains

$$W_{\text{guard}} = 8\beta + 7\gamma,$$

$$W_{\text{separation}} = 9\beta,$$

which implies that using channel separation is better than adding guard frequency when $\gamma \geq \frac{1}{7}\beta$.

From a theoretical point of view, it remains as an open question to solve the general $L(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$-coloring problem, with $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_{\sigma-1}$, and to fill the empty entries in Table 4.

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References

[1] R. Battiti, A.A. Bertossi, and M.A. Bonuccelli, "Assigning Codes in Wireless Networks: Bounds and Scaling Properties", *Wireless Networks*, Vol. 5, 1999, pp. 195-209.

[2] A.A. Bertossi and M.A. Bonuccelli, "Code Assignment for Hidden Terminal Interference Avoidance in Multihop Packet Radio Networks", *IEEE/ACM Transactions on Networking*, Vol. 3, 1995, pp. 441-449.

[3] A.A. Bertossi and M.C. Pinotti, "Mappings for Conflict-Free Access of Paths in Bidimensional Arrays, Circular Lists, and Complete Trees", *Journal of Parallel and Distributed Computing*, Vol. 62, 2002, pp. 1314-1333.

[4] A.A. Bertossi, M.C. Pinotti and R. Tan, "Efficient Use of Radio Spectrum in Wireless Networks with Channel Separation between Close Stations", *DIAL M for Mobility; Int'l ACM Workshop on Discrete Algorithms and Methods for Mobile Computing*, Boston, 2000.

[5] H.L. Bodlaender, T. Kloks, R.B. Tan, and J. van Leeuwen, "Approximation λ-Coloring on Graphs", *STACS*, 2000.

[6] G. J. Chang and D. Kuo, "The $L(2,1)$-Labeling Problem on Graphs", *SIAM Journal on Discrete Mathematics*, Vol. 9, 1996, pp. 309-316.

[7] I. Chlamtac and S.S. Pinter, "Distributed Nodes Organizations Algorithm for Channel Access in a Multihop Dynamic Radio Network", *IEEE Transactions on Computers*, Vol. 36, 1987, pp. 728-737.

[8] D. Goodman, J. Borras, N. Mandayam, and R. Yates, "INFOSTATIONS: A New System Model for Data Messaging Services", *47th IEEE Vehicular Technology Conference*, 1997.

[9] H. Griffin, *Elementary Theory of Numbers*, McGraw Hill, New York, 1954

[10] J. R. Griggs and R.K. Yeh, "Labelling Graphs with a Condition at Distance 2", *SIAM Journal on Discrete Mathematics*, Vol. 5, 1992, pp. 586-595.

[11] W.K. Hale, "Frequency Assignment: Theory and Application", *Proceedings of the IEEE*, Vol. 68, 1980, pp. 1497-1514.

[12] I. Katzela and M. Naghshineh, "Channel Assignment Schemes for Cellular Mobile Telecommunication Systems: A Comprehensive Survey", *IEEE Personal Communications*, June 1996, pp. 10-31.

[13] T. Makansi, "Transmitted Oriented Code Assignment for Multihop Packet Radio", *IEEE Transactions on Communications*, Vol. 35, 1987, pp. 1379-1382.

[14] S.T. McCormick, "Optimal Approximation of Sparse Hessians and its Equivalence to a Graph Coloring Problem", *Mathematical Programming,* Vol. 26, 1983, pp. 153–171.

[15] D. Sakai, "Labeling Chordal Graphs: Distance Two Condition", *SIAM Journal on Discrete Mathematics*, Vol. 7, 1994, pp. 133-140.

[16] A. Sen, T. Roxborough, and S. Medidi, "Upper and Lower Bounds of a Class of Channel Assignment Problems in Cellular Networks", Technical Report, Arizona State University, 1997

[17] D.H. Smith, S. Hurley, and S.U. Thiel, "Improving Heuristics for the Frequency Assignment Problem", *European Journal of Operation Research*, Vol. 107, 1998, pp. 76-86.

[18] A.S. Tanenbaum, *Computer Networks*, Prentice-Hall, Englewood Cliffs, 1989.

[19] J. Van den Heuvel, R. A. Leese, and M.A. Shepherd, "Graph Labelling and Radio Channel Assignment", *Journal of Graph Theory*, Vol. 29, 1998, pp. 263-283.

[20] J. Zander, "Trends and Challenges in Resource Management Future Wireless Networks", *IEEE Wireless Communications & Networks Conference*, 2000.