On time observables in thermal theories.

Nicola Pinamonti [1]
Department of Mathematics, Faculty of Science, University of Trento, & Istituto Nazionale di Alta Matematica "F.Severi", unità locale di Trento & Istituto Nazionale di Fisica Nucleare, Gruppo Collegato di Trento, via Sommarive 14, I-38050 Povo (TN), Italy

Abstract. We study the the existence of an observable representing time in thermal theories at thermodynamical limit. That observable turns out to be described, as usual, by a selfadjoint operator that, with the energy of the system, forms a representation of the Weyl-Heisenberg group. This situation is different from the extent of standard quantum mechanic where, because of the semi boundedness of the energy spectrum and due to Pauli theorem, such selfadjoint time operator does not exists. In the thermal case, because of the presence of the reservoir, the energy is not bounded from below and thus Pauli theorem does not apply. We work in an abstract framework where the thermal properties are translated into modular properties of particular von Neumann algebras. As practical example, the thermal theories we have in mind arise, as in the case of Bisognano Wichmann theorem, considering vacuum sectors of global algebraic quantum theories restricted to some particular bounded region of the global spacetime. In these cases the time operator acquires a particular meaning in terms of some global spacetime properties. Finally, forcing the energy to be positive, the spectral representation of the time operator can be described by a positive-operator-valued measure, in accordance with known quantum mechanical situations.

1 Introduction

In ordinary quantum mechanics observables are described by selfadjoint operators, or equivalently, by their corresponding spectral measures. This description seems to be not completely satisfactory in fact, due to Pauli theorem [1], it is not possible to describe the time coordinate by means of a selfadjoint operator if its conjugate energy is bounded from below. In general relativistic theory the situation is even more complicated, indeed the same problem arises in measuring coordinates of events, the joint spectrum of conjugate momenta being contained in the upper half cone. On the other hand that kind of observables should be present in a quantum theory, in fact more or less every measuring process deal with localization in time (for example when a detector click). It appeared clear that the definition of observable in therms of selfadjoint operators must be generalized. A generalized observable is described in therm of positive operator valued measures (POVM) instead of ordinary spectral measures [2, 3, 4, 5]. With this tool both the time observable in quantum theory and the quantum coordinate of an event in

[1] E-mail: pinamont@science.unitn.it

Minkowski spacetime were defined [3, 6, 7, 8, 9, 10, 11, 12, 13, 14]. This new tool is not a merely mathematical generalization, but sometimes it carries also new physical description. We recall that an observable represented by a POVM on a given Hilbert space $\mathfrak{H}'$ becomes selfadjoint in a bigger Hilbert space $\mathfrak{H}$. The meaning of the states orthogonal to $\mathfrak{H}'$ in $\mathfrak{H}$ is sometime obscure, sometime it was related with the Hilbert space describing the apparatus of measurement [5]. Notice that there is not a unique way to do the embedding $\mathfrak{H}' \to \mathfrak{H}$. On the other hand there are physically reasonable situations in which the energy spectrum is not bounded from below, as for example systems in thermal equilibrium with some reservoir. In the cases a selfadjoint "time" should be found, where negative energy part can be interpreted as the energy of the reservoir gained in a process. Moreover studying these cases it can put in evidence some new physical interpretation of the POVM. We assume that a theory is thermal if it satisfies the Kubo Marting Schwinger (KMS) condition, moreover, since we would like to have a continuous spectrum for the energy, we consider theories at thermodynamic limit. This setup is translated, in a more abstract way, requiring the presence of a von Neumann algebra of observables in "standard" form. In relativistic theories thermal theories arises for example considering the vacuum sector of quantum theories restricted in some particular subspace of the manifold as for example in the Bisognano Wichmann theorem [15] were thermality comes from the choice of a different time evolution. In this paper we study the presence and the properties of a time operator in these cases. To this end we make use of the standard modular theory of Tomita and Takesaki and some remarkable generalizations of Bisognano Wichmann theorem due to Borchers [16]. A consequence of finding the time operator is that some subspace of the manifold we are considering arises as target space of some measuring process, that is for example the case presented as an example in section 3 concerning two dimensional Minkowski spacetime. Remarkably that should be true also in de Sitter spacetime, where a particular vacuum sector appears thermal in static coordinate. Moreover we shall show that restricting to the positive-energy sub Hilbert space produces the well known situation. That is a sort of inverse Neumark theorem. Notice that in this case the initial Hilbert space has a clear physical meaning. In the next section we present the abstract framework in which the theory is developed: Tomita Takesaki modular theory and some Borchers results for von Neumann algebras of observables. Then, in section 3 we present two concrete examples where the time operator arises. Finally, in section 4 some conclusions are given.

2 Algebraic approach

To build up a copy of conjugate observables representing "energy" and "time" in a thermal theory, we seek for two groups of *-automorphisms of a suitable von Neumann algebra representing the observable of the system, generated respectively by "energy" and "time". Moreover in the natural Hilbert space, the two automorphisms must give rise to a representation of the Weyl-Heisenberg group. Since thermality is cast by the KMS condition, the presence of a *-automorphisms generated by the "energy" is assured. It then remains to fix that generated by "time". The results presented here use thermal KMS states on a von Neumann algebra. We could start our considerations also in terms of a C^*-algebra $\mathfrak{A}$ equipped with an algebraic KMS state ω. Then the GNS representation $\pi_\omega(\mathfrak{A})$ of $\mathfrak{A}$ on the Hilbert space $\mathfrak{H}_\omega$ produces a von Neumann algebra $\mathfrak{M} := \pi_\omega(\mathfrak{A})''$ on $\mathfrak{H}_\omega$ with the cyclic vector Ω_ω. For this reason it seems not so restrictive to consider KMS state on von Neumann algebras. The first important point to stress is that for defining a KMS state we need a one-parameter group of automorphisms, usually interpreted as the natural time evolution of the system. Tomita-Takesaki [17] theorem says that, given a von Neumann algebra $\mathfrak{M}$ on the Hilbert space $\mathfrak{H}$, in standard form (i.e. there is a cyclic and separating state Ω in $\mathfrak{H}$), there exists a one-parameter group τ of *-automorphisms of $\mathfrak{M}$, called group of modular automorphisms, such that Ω is a τ-KMS state at inverse temperature 2π. τ is constructed as follows: there is a linear operator S from $\mathfrak{M}\Omega$ to $\mathfrak{M}\Omega$ such that $SA\Omega = A^*\Omega$ for every A in $\mathfrak{M}$. Since S is closeable, the polar decomposition $\overline{S} = J_{\mathfrak{M}}\Delta_{\mathfrak{M}}^{1/2}$ is unique. $\Delta_{\mathfrak{M}}$, called modular operator, is a selfadjoint positive operator, while $J_{\mathfrak{M}}$, representing a linear conjugation, is an anti-unitary operator. The *-automorphisms τ are generated by $D := \log \Delta_{\mathfrak{M}}/(2\pi)$: in $\mathfrak{H}$ the automorphisms τ are represented by the one parameter group of unitary operators $V(t) := \Delta_{\mathfrak{M}}^{it}$. The inverse theorem is also valid. Given an algebraic state ω on a C^*-algebra which is KMS with respect to a $*$-automorphism τ, let $(\mathfrak{H}_\omega, \pi_\omega(\mathfrak{A}), \Omega_\omega)$ be its GNS triple, then the von Neumann algebra $\mathcal{M} := \pi_\omega(\mathfrak{A})''$ is in standard form on $\mathfrak{H}_\omega$ w.r. to the cyclic and separating state Ω_ω. Moreover the $*$-automorphisms τ are represented by unitary operators $V(t) := \Delta_{\mathfrak{M}}^{it/(2\pi)}$ which admit D as generator [18] [19].

2.1. Two dimensional Weyl group as conjugate observables. In this context we are ready to introduce a time operator T that, if exists, should be the conjugate momentum to D. In a more precise formulation, there should be another unitary group of transformation $W(a)$ on $\mathfrak{H}$ such that $W(a)$ and $V(t)$ form a representation of the canonical commutation relations in Weyl form: $W(a)$ and $V(t)$ should be strongly continuous representation of $\mathbb{R}$ and

$$V(t)W(a) = e^{iat}W(a)V(t). \tag{1}$$

It is nothing but a projective representation of the abelian $\mathbb{R}^2$ group:
$U(t,a) := V(t)W(a)$; $U(t,a)U(t',a') = e^{i(at')}U(t+t',a+a')$ called Weyl-Heisenberg group. Working with the Weyl-Heisenberg unitary group instead of using generators avoids problems with domains. Up to unitarities and in a separable Hilbert space, there is only one irreducible representation of Weyl-Heisenberg group by Stone-von Neumann theorem.

2.2. Borchers framework. There are cases were the group of automorphisms generated by time we are looking for arises naturally [20]. Let us recall the following meaningful theorem due to Borchers [16].

Theorem 2.1. *(Borchers) Let $\mathfrak{M}$ be a von Neumann-algebra on $\mathfrak{H}$ a separable Hilbert space and Ω a cyclic and separating vector, if there is a unitary one-parameter group $U(a)$ leaving Ω invariant, and generated by a positive operator then:*
(a) if $U(a)\mathfrak{M}U(-a) \subset \mathfrak{M}$ for every positive a then

$$\text{(B1)} \qquad \Delta_{\mathfrak{M}}^{it}U(a)\Delta_{\mathfrak{M}}^{-it} = U(e^{-2\pi t}a)\forall t, a \in \mathbb{R} \qquad \text{(B2)} \qquad JU(a)J = U(a)^*;$$

(b) if instead $U(a)\mathfrak{M}U(-a) \subset \mathfrak{M}$ for every negative a then

$$\text{(B1)} \qquad \Delta_{\mathfrak{M}}^{it}U(a)\Delta_{\mathfrak{M}}^{-it} = U(e^{2\pi t}a)\forall t, a \in \mathbb{R} \qquad \text{(B2)} \qquad JU(a)J = U(a)^*;$$

Notice that the theorem casts in an abstract form the content of Bisognano Wichmann theorem [15], or of the result of Hislop and Longo [21] on conformal theories. In the case of Minkowski spacetime, as we shall remind later, $U(a)$ arises as the translation along a particular light coordinate.

2.3. Time and energy in Borchers theory. We introduce the following result as a corollary of Borchers theorem 2.1, but with some interesting physical implications we discuss shortly.

Theorem 2.2. *Let $\mathfrak{M}$ be a von Neumann-algebra on a separable Hilbert space $\mathfrak{H}$ and Ω a cyclic and separating vector, if there is a unitary group of transformation $U(a)$ leaving Ω invariant, and generated by a positive operator H and such that $U(a)\mathfrak{M}U(-a) \subset \mathfrak{M}$ for every positive a, then, on a certain closed subspace $\mathfrak{K}$, two strongly continuous one parameter group of transformation $V(t)$, $W(a)$ exist, satisfying the one-dimensional Weyl-Heisenberg commutation relations (1).*

Proof. First of all notice that $V(t) := \Delta_{\mathfrak{M}}^{it/(2\pi)}$ acts as a one-parameter group of unitary transformations on $\mathfrak{H}$, $\Delta_{\mathfrak{M}}$ being the Tomita Takesaki modular operator that exists since Ω is cyclic and separating. To find $W(a)$ we study the properties of H. From (B1) in theorem 2.1 $\Delta_{\mathfrak{M}}^{it} H \Delta_{\mathfrak{M}}^{-it} = e^{-2\pi t}H$, it implies $\sigma(H) = e^{-2\pi t}\sigma(H)$ thus the positive spectrum $\sigma(H)$ of H has at most two disjoint orbits $\{0\}$ and $(0, +\infty)$. If $0 \in \sigma_c(H)$ (the continuous spectrum) we define $\mathfrak{K} := \mathfrak{H}$. If $0 \in \sigma_p(H)$ (the point spectrum of H) the Hilbert space $\mathfrak{H}$ can be decomposed in two orthogonal invariant subspace $\mathfrak{H} := \mathfrak{H}_0 \oplus \mathfrak{K}$ where H on $\mathfrak{H}_0$ has the spectrum $\{0\}$ while on $\mathfrak{K}$ the spectrum is continuous due to separability. Since $\{0\}$ is invariant for the action of $V(t)$, $\mathfrak{K} := \{0\}^{\perp}$ is also invariant for the action $V(t)$. If the spectral decomposition of H on $\mathfrak{K}$ is $H := \int_{\mathbb{R}^+} \lambda dE(\lambda)$, consider the self-adjoint operator $T := \log(H) = \int_{\mathbb{R}^+} \log(\lambda) dE(\lambda)$. As $W(a) := e^{iaT}$, from (B1) descends that $V(t)W(a) := e^{ita}W(a)V(t)$ on $\mathfrak{K}$. They form a projective unitary representation of Weyl-Heisenberg group. □

Remark: (a) We call D "energy" and T "time", in fact D generates the automorphisms τ of the KMS condition for Ω. Moreover $[D, T] := i$ holds in a dense domain $\mathfrak{S} \subset \mathfrak{K}$. The spectrum of both T and D is the whole line $\mathbb{R}$ so that the energy D is not bounded from below. This fact is not surprising because we are describing thermal states.

(b) The Heisenberg uncertainty relations $(\langle D^2\rangle - \langle D\rangle^2)(\langle T^2\rangle - \langle T^2\rangle) \geq 1/4$ holds on $\mathfrak{S}$.

(c) Since the reference state Ω is invariant under the action of $W(a)$, the averaged value of the time operator T on it has no particular meaning. Instead it can be used to compute the time coordinate of the "fluctuation" over the reference state Ω. If $\psi \in \mathfrak{K}$ then $\langle T\rangle_{\psi} := (\psi, T\psi)$, where $\psi \in \mathfrak{K}$ represents some particular fluctuation on Ω.

(d) Borchers hypotheses may descend from the following more geometrical requirement as found by Wiesbrock [22, 23] (see also [24]) who proved that if $\mathfrak{N} \subset \mathfrak{M}$ are two von Neumann algebras for which Ω is cyclic and separating, there exists a unitary group of transformation $U(a)$ on $\mathfrak{H}$, generated by a positive operator H (which is the selfadjoint closure of the essentially selfadjoint operator $\log(\Delta_{\mathfrak{N}}) - \log(\Delta_{\mathfrak{M}})$) satisfying (B1) in theorem 2.1, whose action is invariant for $\mathfrak{M}$ if a is positive.

(e) In practical examples, see below, both $U(a)$ (generated by H) and $V(t)$ arise as geometric groups of translations in particular coordinates. It is remarkable that $W(a)$, that should

represent "energy" translations, is generated by a modification of a time translator $\log(H)$.
(f) Since $\mathfrak{H}_0$ plays the role of "vacuum" with respect to H, it is not surprising that a representation of the Weyl-Heisenberg group do not exist on $\mathfrak{H}_0$. In fact also for a free particle in some Fock spaces there is no way to define meaningful canonical commutation relations on the vacuum subspace, position and impulse form a Weyl-Heisenberg group only in the subspaces orthogonal to the vacuum.

2.4. The interplay with POVM. Suppose to have a von Neumann algebra $\mathfrak{M}$ on a Hilbert space $\mathfrak{H}$ with the hypotheses of theorem 2.2. We can then find a subspace $\mathfrak{K} \subset \mathfrak{H}$ on which acts the two unitary group of transformation $V(t)$ and $W(a)$ satisfying the Weyl-Heisenberg commutation relations and generated respectively by D and T. Being $\mathfrak{K}_+ \subset \mathfrak{K}$ such that D on $\mathfrak{K}_+$ has positive spectrum, if $P : \mathfrak{K} \to \mathfrak{K}_+$ is the orthogonal projector, $T' := PTP$ turns out to be a generalized observable (POVM). If $E : \mathcal{B} \to \mathcal{L}(\mathfrak{K})$ is the projector valued measure (PVM) from Borel sets $\mathcal{B}$ of $\mathbb{R}$ to operators in $\mathfrak{K}$, such that $T := \int_{\mathbb{R}} \log(\lambda) dE(\lambda)$, $F(X) := PE(X)$ defines a POVM from the same Borel sets $\mathcal{B}$ to operators in $\mathfrak{K}_+$ and

$$T' := \int_{\mathbb{R}} \log(\lambda) dF(\lambda).$$

Notice that, because of Pauli theorem, F must be a real POVM in general (in $F(X)F(X) \neq F(X)$), in fact D the conjugate momentum of T' on $\mathfrak{K}_+$ has a spectrum bounded from below. We have here a sort of inverse Neumark theorem. Imposing some constraint in $\mathfrak{K}$ to the energy", T' turns to be a generalized observable.

3 Example of thermal theories with time

In this section we present some concrete examples of time operators T.

3.1. Time operator of quantum particle on the half line. We shall study the second quantization of a one dimensional particle associated with the so called tachyon field. This theory was proposed some years ago by Sewell [25], restricting some Minkowskian free field theory on the Killing horizon. Sometime in literature this is related with lightfront holography [26]. Recently its relation with conformal theory was put forward [27, 28, 29]. For conformal theory on the circle see for example [30, 31, 32, 33] and reference therein. Consider the classical wavefunctions ψ in $\mathcal{S}$: the set of smooth real function vanishing with every derivative at infinity. Equip $\mathcal{S}$ with the symplectic form $\sigma(\psi, \psi') := \int_{\mathbb{R}} \psi \partial_x \psi' - \psi' \partial_x \psi dx$, invariant under change of coordinate x. Once the coordinate x is chosen, every wavefunction can be decomposed in positive and negative frequency parts: $\psi(x) := \psi_+(c) + \overline{\psi_+(x)}$, where the positive frequency part is $\psi_+(x) := \int_0^\infty \frac{e^{-iEx}}{\sqrt{4\pi E}} \tilde{\psi}_+(E)$. The set of ψ_+ turns out to be (dense in) a Hilbert space $\mathfrak{H}_{\mathbb{R}} \sim L^2(\mathbb{R}^+, dE)$ with the scalar product $\langle \psi_+, \psi'_+ \rangle := -i\sigma(\overline{\psi}_+, \psi'_+)$. The Weyl quantum fields $\hat{W}(\psi) := e^{i\hat{\phi}(\psi)}$ on the standard Fock space $\mathfrak{F}(\mathfrak{H}_{\mathbb{R}})$ generates a unitary GNS representation π of the Weyl algebra associated with the pair $(\mathcal{S}, \sigma)$. We consider now the subalgebra $\mathcal{W}(\mathcal{S})_+$ made by concrete Weyl operators $\hat{W}(\psi)$ smeared by wavefunction with support contained in the positive part of the real line. $\mathfrak{M}(\mathbb{R}) := \pi(\mathcal{W}(\mathcal{S})_+)''$ is a von Neumann algebra on $\overline{\mathfrak{F}(\mathfrak{H}_{\mathbb{R}})}$

and the vacuum Ω of $\mathfrak{F}(\mathfrak{H}_{\mathbb{R}})$ is a cyclic and separating state on it [30, 31, 32, 33, 34, 35]. Remarkably, in this case, the one-parameter group of transformations $V(t) := \Delta^{it/(2\pi)}$ (Δ being the modular operator) has a geometrical meaning, it is made by $x-$dilation generated by $D := \int_{\mathbb{R}} x : \partial_x\hat{\phi}(x)\partial_x\hat{\phi}(x) : dx$. The theory is thermal, namely Ω is a KMS state at temperature $1/(2\pi)$ w.r. to $V(t)$. $x-$translations can be represented by the unitary operators $U(a)$ generated by the positive operator $H := \int_{\mathbb{R}} : \partial_x\hat{\phi}(x)\partial_x\hat{\phi}(x) : dx$. Ω is invariant under $U(a)$ and $\mathfrak{M}(\mathbb{R})$ is invariant under the action of $U(a)$ provided a is positive, then the hypothesis of theorem 2.2 are fulfilled. We can build $\mathfrak{K} \subset \overline{\mathfrak{F}(\mathfrak{H}_{\mathbb{R}})}$ and $W(a)$ that with $V(t)$ satisfies the Weyl-Heisenberg commutation relation (1) on $\mathfrak{K}$. $\mathfrak{K}$ is the orthogonal to the vacuum Ω in the Fock space. $H_{|\mathfrak{K}}$ has continuous spectrum only. The time operator T generating $W(a)$ is $\log(H)$: $T := \int \log\lambda dE(\lambda)$ where $H_{|\mathfrak{K}} = \int \lambda dE(\lambda)$. In a more messy language $T := \int_0^\infty \log(E) a_E^\dagger a_E dE$, where $a_E^\dagger$, a_E are the creation and annihilation operator of the state at fixed $H = E$. As noticed in the remarks of the above section, it is intriguing that the time operator can be build using the spectral decomposition H.

3.2. Rindler coordinates in two dimensional Minkowski spacetime. We consider for simplicity only the two dimensional Minkowski spacetime $\mathcal{M}$ but the example we are presenting here can be generalized straightforwardly also in the case of four dimensional Minkowski spacetime or also on some other spacetime having a bifurcate Killing horizon (e.g. de Sitter spacetime). Here we show that the time related with the bust (Rindler time) and also the position, can be represented by selfadjoint operators on some suitable Hilbert space. The metric of $\mathcal{M}$ is $ds^2 = -dt^2 + dx^2$. The Poincaré group, which acts as the isometric transformations, is made by spacetime translations and by the one parameter boost. The concrete free massive or massless local Weyl fields $\hat{W}(f) := e^{i\hat{\phi}(f)}$, smeared by local function on the Fock space $\mathfrak{F}(\mathfrak{H}_{\mathcal{M}})$ with Poincaré-invariant vacuum Ω, generate a GNS unitary representation $\pi_{\mathcal{M}}$ of the abstract Weyl algebra $\mathcal{W}(\mathcal{M})$ associated with the standard symplectic structure on Minkowski[2]. See [15, 19, 18] for details. Consider $\mathcal{W}(\mathcal{R}) \subset \mathcal{W}(\mathcal{M})$ the subalgebra of fields smeared by local function having domain in the right Rindler wedge $\mathcal{R} := \{(t,x), 0 < |t| < x\}$. $\pi_{\mathcal{R}}(W(f)) := \pi_{\mathcal{M}}(W(f))$ is also a concrete representation of $\mathcal{W}(\mathcal{R})$ in $\mathfrak{F}(\mathfrak{H}_{\mathcal{M}})$. $\mathfrak{M}(\mathcal{R}) := \pi_{\mathcal{R}}(\mathcal{W}(\mathcal{R}))''$ is a von Neumann algebra acting on the Hilbert space $\mathfrak{F}(\mathfrak{H}_{\mathcal{M}})$. Ω is a cyclic and separating state for $\mathfrak{M}(\mathcal{R})$ [15], then it has a modular operator Δ and a modular reflection J. Interestingly these two operator have a geometrical meaning, in fact Δ^{it} represents the boosts and J acts on $\mathfrak{M}(\mathcal{R})$ as a geometrical reflection mapping the right wedge $\mathcal{R}$ into the left one. D, the generator of $\Delta^{it/(2\pi)}$, represents the energy of the accelerating observer. $\mathfrak{M}(\mathcal{R})$ is not invariant under the action of time translation, unitarily implemented on $\mathfrak{F}(\mathfrak{H}_{\mathcal{M}})$ by $U_H(a) := e^{iaH}$, for every $a \neq 0$, then the Borchers theorem 2.1 does not apply for U_H and consequently also the theorem 2.2 does not permit to find a time operator. We can circumvent the problem noticing that there are two one parameter group of transformation corresponding to displacements in Minkowksi light coordinates, respectively $x^+ = (t+x)/2$ and $x^- = (t-x)/2$. They are unitarily implementable on $\mathfrak{F}(\mathfrak{H}_{\mathcal{M}})$. We call the corresponding unitary operators $U_+(a)$ and $U_-(a)$, generated respectively by H_+ and

[2] The local smearing is realized by mapping local function f to wavefunction ψ satisfying the massive Klein Gordon equation by means of the standard causal propagator.

H_-. $U_+(a)\mathfrak{M}(\mathcal{R})U_+(a)^\dagger \subset \mathfrak{M}(\mathcal{R})$ for $a > 0$ and $U_-(b)\mathfrak{M}(\mathcal{R})U_-(b)^\dagger \subset \mathfrak{M}(\mathcal{R})$ for $b < 0$. Both groups keep invariant the vacuum Ω and are generated by positive operators. By theorem 2.2 it is possible to find two operators $T_+ = \log(H_+)$ and $T_- = -\log(H_-)$. Both T_+, D and T_-, D satisfy the Weyl-Heisenberg commutation relation (1) on a subspace $\mathfrak{K} \in \mathfrak{F}(\mathfrak{H}_{\mathcal{M}})$, the orthogonal of the vacuum. If $U_+(a)$ and $U_-(b)$ commute, as for example for massless fields, it is possible to combine both in the following way: $T = (T_+ + T_-)/2$ and $X = (T_+ - T_-)/2$. Notice that $[T, D] = i$ while X commute with D. T represents the Rindler time while X is the Rindler position. The explicit construction of T_+ and T_- descends straightforwardly as in the above example from the spectral representations of H_+ and H_-. Notice that Rindler points arises as elements of the joint spectrum of T and R on a subset of $\mathfrak{F}(\mathfrak{H}_{\mathcal{M}})$.

4 Comments

We have built a time observable in thermal theories at thermodynamical limit equipped with another one-parameter group of symmetry which exists under the hypothesis of Borchers theorem 2.1. This observable is described by a selfadjoint operator whose associates one-parameter group of unitary transformation satisfy the Weyl-Heisenberg commutation relations with unitary time translations. This is not the usual situation in ordinary quantum mechanic where a selfadjoint operator describing the time displacement cannot exists. Moreover we have studied the restriction of that operators in subspaces where the energy is bounded from below. We have shown that in this way time observables turns to be re-constructed by a POVM as in the ordinary quantum mechanic. With these tools we have analyzed two practical situations one related with conformal theories and the related with description of Rindler observables in Minkowski vacuum sector. We have shown that in the latter case the Rindler time arises, surprisingly, as a function of the generators of Minkowksi lightlike translations: $T = \log H_+ - \log H_-$. As an interesting consequence, the points in the Rindler wedge arises as elements of the joint spectrum of two observables in Minkowski vacuum.

Appendix

A Explicit form of T on the one particle Hilbert space

We study the restriction of the Weyl-Heisenberg groups studied in section 3.1 on the one particle Hilbert space $\mathfrak{H} = L^2(\mathbb{R}^+, dE)$ where $W(a)$ and $V(t)$ have this form:

$$W(a)\psi(E) := E^{ia}\psi(E), \qquad V(t)\psi(E) := e^{\frac{t}{2}}\psi(Ee^t).$$

In $\mathfrak{H}$ it is easy to show that they are unitary and that $V(t)^\dagger = V(-t)$, $V(t)^\dagger V(t) = 1$ and that $W(a)^\dagger = W(-a)$, $W(a)^\dagger W(a) = 1$, and $W(a)V(t) := e^{ist}V(t)W(a)$. These are two strongly continuous unitary representation of $\mathbb{R}$ in $\mathfrak{H}$ which form a representation of the Weyl-Heisenberg commutation relations on $\mathfrak{H}$. $W(a) = e^{iaT}$,$V(t) = e^{itD}$. To show the explicit form of T and D

we unitarily map $L^2(\mathbb{R}^+, dE)$ onto $L^2(\mathbb{R}, dq)$ where $q = \log E$ and for every ψ_E in $L^2(\mathbb{R}+, dE)$ $\psi_q(q) := \sqrt{e^q}\psi_E(e^q)$ is in $L^2(\mathbb{R}, dq)$. On a suitable common dense domain this T and D takes the following standard form

$$T = q, \qquad D = i\frac{d}{dq}.$$

A.1. Time in position representation. For completeness we would like to present T and D in position representation where $\psi_+(x) := \int_0^\infty \frac{e^{-iEx}}{\sqrt{4\pi E}}\tilde{\psi}(E)dE$ for every $\tilde{\psi}$ in $\mathfrak{H}$. The action of $V(t)$ on $\psi_+(x)$ is $V(t)\psi_+(x) = \psi_+(e^t x)$ corresponding to a dilation generated by $D = ix\frac{d}{dx}$. On the other hand the action of W on $\psi_+(x)$ is more complicated. It is a bit simpler considering the action of W on wavefunction ($\psi(x) := \psi(x)_+ +$ c.c.) instead, where

$$W(a)\psi(x) := \psi(x) * \left(\frac{|x|^{ia}}{\Gamma(ia)} + \frac{|x|^{-ia}}{\Gamma(-ia)}\right)\frac{\pi}{\sinh(\pi a)}\frac{e^{\frac{\pi}{2}a\sigma(x)}}{x}$$

where $*$ is for a convolution, $\sigma(x)$ is the distribution representing the signum of x. The action of T turns to be:

$$T\psi_+(x) := \psi_+(x) * i\left(\frac{\log(|x|) + \gamma}{x}\right)$$

where the second term in the convolution needs to be intended with the prescription of principal value, and γ is the logarithmic derivative of the gamma function evaluated in i: $\gamma = \Gamma(1)'/\Gamma(1)$.

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References

[1] W. Pauli *"Die algemainen Prinzipen der Wellenmechanik"* Handbuch der Physik, edited by S.Fl ugge, Vol. V/1, p.60, Springer Verlag, Berlin.

[2] M. A. Naimark, *Spectral functions of a symmetric operator* Izv. Akad. Nauk. SSSR Ser. Mat. **4** (1940).Ina

[3] G. Ludwig, *Foundations of quantum mechanics..* Texts and Monographs in Physics. Springer-Verlag, New York, (1983).

[4] P. Busch, P. J. Lahti and P. Mittelstaedt, *"The Quantum Theory of Measurement,"* Lecture Notes in Phys., Springer Verlag, Berlin (1991).

[5] P. Busch, M. Grabowski and P. J. Lahti, *"Operational Quantum Physics,"* Lecture Notes in Phys., Springer Verlag, Berlin (1995).

[6] Y. Aharonov, J. Oppenheim, S. Popescu, B. Reznik and W. G. Unruh, *"Measurement of Time-of-Arrival in Quantum Mechanics,"* Phys. Rev. A **57**, 4130 (1998).

[7] W. G. Unruh and R. M. Wald, *"Time And The Interpretation Of Canonical Quantum Gravity,"* Phys. Rev. D **40**, 2598 (1989).

[8] R. Giannitrapani, *"Positive-Operator-Valued Time Observable in Quantum Mechanics,"* Int. Journ. Theor. Phys. **36**, 1575 (1997).

[9] R. Giannitrapani, *"Quantum Coordinates of an Event in Local Quantum Physics,"* J. Math. Phys. **39**, 5180 (1998).

[10] M. Toller, *"Localization of Events in Space-Time,"* Phys. Rev. A **59**, 960 (1999).

[11] M. Toller, *"Variance of the quantum coordinates of an event,"* Int. J. Theor. Phys. **38**, 2015 (1999).

[12] R. Brunetti and K. Fredenhagen, *"When does a detector click?,"* arXiv:quant-ph/0103144 (2001).

[13] R. Brunetti and K. Fredenhagen,
"Remarks on time-energy uncertainty relations," Rev. Math. Phys. **14**, 897 (2002).

[14] M. Toller, *"Events in a Non-Commutative Space-Time,"* Phys. Rev. D **70**, 024006 (2004).

[15] J. J. Bisognano and E. H. Wichmann, *"On The Duality Condition For A Hermitian Scalar Field,"* J. Math. Phys. **16**, 985 (1975).

[16] H. J. Borchers, *"The CPT theorem in two-dimensional theories of local observables,"* Commun. Math. Phys. **143**, 315 (1992).

[17] Takesaki, M. Tomita, *"Theory of Modular Hilbert-Algebras and its Application"* Lecture Notes in Mathematics Vol, 128. Springer Verlag, Berlin (1970).

[18] R. Haag *"Local quantum physics: fields, particles, algebras"*, Springer Verlag, Berlin, (1992).

[19] O. Bratteli, D. W. Robinson, *"Operator Algebras and Quantum Statistical Mechanics I"* Second Edition, Springer Verlag, Berlin, (1987).

[20] R. Brunetti, D. Guido and R. Longo, *"Modular localization and Wigner particles,"* Rev. Math. Phys. **14**, 759 (2002).

[21] P. D. Hislop and R. Longo, *"Modular Structure Of The Local Algebras Associated With The Free Massless Scalar Field Theory,"* Commun. Math. Phys. **84**, 71 (1982).

[22] H. W. Wiesbrock, *"Half-Sided Modular Inclusions of von Neumann-Algebras,"* Commun. Math. Phys. **157**, 83 (1993) [Erratum-ibid. **184**, 683 (1997)].

[23] H. W. Wiesbrock, *"Symmetries and modular intersections of von Neumann algebras,"* Lett. Math. Phys. **39**, 203 (1997).

[24] H. Araki and L. Zsido, *"Extension of the structure theorem of Borchers and its application to half-sided modular inclusions,"* Rev. Math. Phys. **17**, 491 (2005).

[25] G.L. Sewell, *"Quantum fields on manifolds: PCT and gravitationally induced thermal states,"* Ann. Phys. (NY) **141** 201 (1982).

[26] B. Schroer, *"Lightfront holography and area density of entropy associated with localization on wedge-horizons,"* Int. J. Mod. Phys. A **18**, 1671 (2003).

[27] D. Guido, R. Longo, J. E. Roberts and R. Verch, *"Charged sectors, spin and statistics in quantum field theory on curved spacetimes,"* Rev. Math. Phys. **13**, 125 (2001).

[28] V. Moretti, N. Pinamonti, *"Holography and SL(2,R) symmetry in 2D Rindler spacetime,"* J. Math. Phys. **45** 230 (2004). *"Quantum Virasoro algebra with central charge $c = 1$ on the horizon of a $2D-$Rindler spacetime"* J. Math. Phys. **45** 257 (2004).

[29] V. Moretti, N. Pinamonti, *"Bose Einstein condensate and Spontaneous Breaking of Conformal Symmetry on Killing Horizons,"* J. Math. Phys. **46** 062303 (2005).

[30] D. Buchholz, G. Mack, I. Todorov, *"The Current Algebra On The Circle As A Germ Of Local Field Theories"*, Nucl. Phys. Proc. Suppl. **5**, 20 (1988).
D. Buchholz, H.S. Mirbach, *"Haag duality in conformal quantum field theory"*, Rev. Math. Phys. **2**, 105 (1990).

[31] F. Gabbiani, J. Fröhlich, *"Operator algebras and conformal field theory"*, Commun. Math. Phys. **155**, 569 (1993).

[32] D. Guido, R. Longo, *"The Conformal spin and statistics theorem"*, Commun. Math. Phys. **181**, 11 (1996).

[33] S. Carpi, *"On The Representation Theory of Virasoro Nets"*, Commun. Math. Phys. **244**, 261 (2003).

[34] C. Dappiaggi, V. Moretti and N. Pinamonti, *"Rigorous steps towards holography in asymptotically flat spacetimes,"* arXiv:gr-qc/0506069 (2005).

[35] V. Moretti, *"Uniqueness theorem for BMS-invariant states of scalar QFT on the null boundary of asymptotically flat spacetimes and bulk-boundary observable algebra correspondence,"* arXiv:gr-qc/0512049 (2005).