Uniqueness theorem for BMS-invariant states of scalar QFT on the null boundary of asymptotically flat spacetimes and bulk-boundary observable algebra correspondence

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Abstract. Features of scalar QFT defined on the causal boundary $\Im^+$ of an asymptotically flat at null infinity spacetime and based on the BMS-invariant Weyl algebra $\mathcal{W}(\Im^+)$ are discussed.
(a) (i) It is noticed that the natural BMS invariant pure quasifree state λ on $\mathcal{W}(\Im^+)$, recently introduced by Dappiaggi, Moretti an Pinamonti, enjoys positivity of the self-adjoint generator of u-translations with respect to *every* Bondi coordinate frame $(u, \zeta, \overline{\zeta})$ on $\Im^+$, ($u \in \mathbb{R}$ being the affine parameter of the complete null geodesics forming $\Im^+$ and $\zeta, \overline{\zeta}$ complex coordinates on the transverse 2-sphere). This fact may be interpreted as a remnant of spectral condition inherited from QFT in Minkowski spacetime (and it is the spectral condition for free fields when the bulk is the very Minkowski space). (ii) It is also proved that cluster property under u-displacements is valid for every (not necessarily quasifree) pure state on $\mathcal{W}(\Im^+)$ which is invariant under u displacements. (iii) It is established that positivity of the self-adjoint generator of u-translations in *a fixed* Bondi frame individuates the BMS-invariant state λ uniquely (without requiring BMS invariance) in the class of pure algebraic quasifree states on $\mathcal{W}(\Im^+)$: there is exactly one algebraic pure quasifree state which is invariant under u-displacements (of a fixed Bondi frame) and has positive self-adjoint generator of u-displacements. It coincides with the GNS-invariant state λ. (iv) Finally it is showed that in the folium of a pure u-displacement invariant state (like λ but not necessarily quasifree) on $\mathcal{W}(\Im^+)$ the state itself is the only state invariant under u-displacement.
(b) It is proved that all the theory can rested for spacetimes asymptotically flat at null infinity which also admit future time completion i^+ (and fulfills some other requirements related with global hyperbolicity). In this case a $*$-isomorphism exists automatically which identifies the (Weyl) algebra of observables of a linear fields propagating in the bulk spacetime with a sub algebra of $\mathcal{W}(\Im^+)$. Moreover the preferred BMS-invariant state λ on $\mathcal{W}(\Im^+)$ induces a preferred state on the field algebra in the bulk spacetime.

1   Introduction.

1.1. *Summary the relevant results established in [DMP05] and some extensions.*   Throughout $\mathbb{R}^+ := [0, +\infty)$, $\mathbb{N} := \{0, 1, 2, \ldots\}$ and $C^\infty(A), C^\infty(A; \mathbb{C})$ denote respectively the real and complex linear space of smooth respectively real-valued, complex-valued functions on the manifold

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A. If $\chi : M \to N$ is a diffeomorphism, χ^* is the natural extension to tensor bundles (counter-, co-variant and mixed) from M to N (Appendix C in [Wa84]). Concerning spacetimes we adopt general definitions of [Wa84] (especially cap.8 therein). The metric has signature $- + \cdots +$. We use terminology, notation and definitions given in [DMP05]. An **asymptotically flat spacetime** is a four-dimensional smooth spacetime which is a *vacuum spacetime asymptotically flat at (future $\Im^+$ and past $\Im^-$) null and and spatial (i^0) infinity* as defined in [Wa84] (see definition B.1 in appendix B). Actually it is not required that it satisfies vacuum Einstein equations everywhere, it is sufficient that it does in a neighborhood of $\Im^+ \cup \Im^- \cup \{i^0\}$ or, more weakly, "approaching" the boundary $\Im^+ \cup \Im^- \cup \{i^0\}$ as discussed on p.278 of [Wa84] and the past null infinity is not involved in all our results.

In [DMP05] we have considered a Weyl algebra constructed on the future null boundary of a asymptotically flat spacetime (M, g). By definition (see appendix B) (M, g) can be identified with a submanifold, with boundary $\partial M = \Im^+ \cup \Im^- \cup \{i^0\}$, of an unphysical spacetime $(\tilde{M}, \tilde{g})$ with $\tilde{g} = \Omega^2 g$ (where, for sake of simplicity, we omitted to write explicitly the embedding map). $\Omega : \tilde{M} \to [0, +\infty)$ is smooth in $\tilde{M} \setminus \{i^0\}$ (at i^0 it is at least C^2 and satisfies $\Omega(x) = 0$ in $\overline{M}$ if and only if $x \in \Im^+ \cup \Im^- \cup \{i^0\}$. $\Im^+ = (\partial J^+(i^0)) \setminus \{i^0\}$ is a 3-dimensional submanifold of $\tilde{M}$ diffeomorphic to $\mathbb{R} \times \mathbb{S}^2$. $\Im^+$ is the union of integral lines of the, nonvanishing on $\Im^+$, null field $n^\mu := \tilde{g}^{\mu\nu}\nabla_\nu\Omega$ and is equipped with the degenerate metric $\tilde{h}$ induced by $\tilde{g}$. As far as the only structure on $\Im^+$ is concerned, changes of the unphysical spacetime $(\tilde{M}, \tilde{g})$ associated with a fixed asymptotically flat spacetime (M, g), are completely encompassed by **gauge transformations** $\Omega \to \omega\Omega$ valid in a neighborhood of $\Im^+$ (not including i^0 in general), with ω smooth and strictly positive. Under these gauge transformations the triple $(\Im^+, \tilde{h}, n)$ transforms as

$$\Im^+ \to \Im^+ , \quad \tilde{h} \to \omega^2\tilde{h} , \quad n \to \omega^{-1}n . \tag{1}$$

If C is the class of the triples $(\Im^+, \tilde{h}, n)$ transforming as in (1) for a fixed asymptotically flat vacuum spacetime, there is no general physical principle to single out a preferred element in C. On the other hand C is *universal* for all asymptotically flat spacetimes [Wa84]: If C_1 and C_2 are the classes of triples associated respectively to (M_1, g_2) and (M_2, g_2) there is a diffeomorphism $\gamma : \Im_1^+ \to \Im_2^+$ such that for suitable $(\Im_1^+, \tilde{h}_1, n_1) \in C_1$ and $(\Im_2^+, \tilde{h}_2, n_2) \in C_2$,

$$\gamma(\Im_1^+) = \Im_2^+ , \quad \gamma^*\tilde{h}_1 = \tilde{h}_2 , \quad \gamma^*n_1 = n_2 .$$

With an appropriate choice of ω, explicitly required to exist in the very definition of asymptotically flat spacetime (condition 5(b) in def. B.1), the tangent vector n turns out to be that of complete geodesics. ω is completely fixed by requiring that, in addition, the nondegenerate metric on the transverse section of $\Im^+$ is, constantly along geodesics, the standard metric of $\mathbb{S}^2$ in $\mathbb{R}^3$. We indicate by ω_B and $(\Im^+, \tilde{h}_B, n_B)$ that value of ω and the associated triple respectively. For $\omega = \omega_B$, a **Bondi frame** on $\Im^+$ is a global coordinate system $(u, \zeta, \bar{\zeta})$ on $\Im^+$, where $u \in \mathbb{R}$ is an affine parameter of the complete null $\tilde{g}$-geodesics whose union is $\Im^+$ and $\zeta, \bar{\zeta} \in \mathbb{S}^2 \equiv \mathbb{C} \cup \{\infty\}$ being complex coordinates on the section of $\Im^+$ transverse to the geodesics: $\zeta = e^{i\phi}\cot(\theta/2)$ with θ, ϕ usual spherical coordinates of $\mathbb{R}^3$. The metric on the transverse section of $\Im^+$ now reads $2(1 + \zeta\bar{\zeta})^{-2}(d\zeta \otimes d\bar{\zeta} + d\bar{\zeta} \otimes d\zeta)$.

A diffeomorphisms $\chi : \Im^+ \to \chi$ belongs to **BMS group** G_{BMS} if $\chi^* \tilde{h}$ and $\chi^* n$ differ from $\tilde{h}$ and n at most by a rescaling (1). These diffeomorphisms represent "asymptotic isometries" of M in the precise sense discussed in [Wa84] and highlighted in Proposition 2.1 in [DMP05]. Henceforth, whenever it is not explicitly stated otherwise, *we consider as admissible realizations of the unphysical metric on $\Im^+$ only those metrics $\tilde{h}$ which can be reached via transformations in G_{BMS} from a metric whose associated triple is $(\Im^+, \tilde{h}_B, n_B)$.*

In coordinates of a fixed Bondi frame $(u, \zeta, \overline{\zeta})$, the group G_{BMS} is realized as semidirect group $SO(3,1){\uparrow} \ltimes C^\infty(\mathbb{S}^2)$, where $(\Lambda, f) \in SO(3,1){\uparrow} \times C^\infty(\mathbb{S}^2)$ acts as

$$u \quad \to \quad u' := K_\Lambda(\zeta, \overline{\zeta})(u + f(\zeta, \overline{\zeta})) \,, \tag{2}$$

$$\zeta \quad \to \quad \zeta' := \Lambda\zeta := \frac{a_\Lambda \zeta + b_\Lambda}{c_\Lambda \zeta + d_\Lambda} \,, \quad \overline{\zeta} \to \overline{\zeta}' := \Lambda\overline{\zeta} := \frac{\overline{a_\Lambda}\overline{\zeta} + \overline{b_\Lambda}}{\overline{c_\Lambda}\overline{\zeta} + \overline{d_\Lambda}} \,. \tag{3}$$

K_Λ is the smooth function on $\mathbb{S}^2$

$$K_\Lambda(\zeta, \overline{\zeta}) := \frac{(1 + \zeta\overline{\zeta})}{(a_\Lambda \zeta + b_\Lambda)(\overline{a_\Lambda}\overline{\zeta} + \overline{b_\Lambda}) + (c_\Lambda \zeta + d_\Lambda)(\overline{c_\Lambda}\overline{\zeta} + \overline{d_\Lambda})} \quad \text{and} \quad \begin{bmatrix} a_\Lambda & b_\Lambda \\ c_\Lambda & d_\Lambda \end{bmatrix} = \Pi^{-1}(\Lambda) \,. \tag{4}$$

Above Π is the well-known surjective covering homomorphism $SL(2, \mathbb{C}) \to SO(3,1){\uparrow}$ (see [DMP05] for further details). Two Bondi frames are connected each other through the transformations (2),(3) with $\Lambda \in SU(2)$. Conversely, any coordinate frame $(u', \zeta', \overline{\zeta}')$ on $\Im^+$ connected to a Bondi frame by means of an arbitrary BMS transformation (2),(3) is *physically equivalent* to the latter from the point of view of General Relativity, but it is not necessarily a Bondi frame in turn. A global reference frame $(u', \zeta', \overline{\zeta}')$ on $\Im^+$ related with a Bondi frame $(u, \zeta, \overline{\zeta})$ by means of a BMS transformation (2)-(3) will be called **admissible frame**.

Remark 1.1. The notion of Bondi frame is useful but *conventional*. Any physical object must be invariant under the whole BMS group, i.e. under asymptotic symmetries (which include proper symmetries) of M, and not only under the subgroup of G_{BMS} connecting Bondi frames.

We recall the reader that an asymptotically flat spacetime (M, g) is said to be **strongly asymptotically predictable** [Wa84] if there is an open set $\tilde{V} \subset \tilde{M}$ with $\overline{M \cap J^-(\Im^+)} \subset \tilde{V}$ (the closure being referred to $\tilde{M}$) such that $(\tilde{V}, \tilde{g})$ is globally hyperbolic. Notice that also $M_{\tilde{V}} := \tilde{V} \cap M$ is globally hyperbolic under these hypotheses and a spacelike smooth Cauchy surface $\tilde{S}$ for $(\tilde{V}, \tilde{g})$ through i^0 individuates a smooth Cauchy surface S for $(M_{\tilde{V}}, g)$ when restricting to $M_{\tilde{V}}$. In particular Minkowski spacetime $M = \mathbb{M}^4$ is strongly asymptotically predictable with $M_{\tilde{V}} = \mathbb{M}^4$. If ϕ is smooth with compactly supported Cauchy data and solves the massless **conformally-coupled Klein-Gordon equations** in $M_{\tilde{V}}$

$$P\phi = 0 \,, \quad \text{where } P := -g^{\mu\nu}\nabla_\mu\nabla_\nu + \tfrac{1}{6}R \,, \tag{5}$$

the limit ψ of $(\omega_B \Omega)^{-1}\phi$ toward $\Im^+$ is smooth (Proposition 2.3 in [DMP05]). The action of asymptotic isometries on ϕ in the bulk corresponds to an action of G_{BMS} on ψ, in a fixed Bondi frame, (Proposition 2.3 in [DMP05])

$$\left(A_{(\Lambda,f)}\psi\right)(u', \zeta', \overline{\zeta}') := K_\Lambda(\zeta, \overline{\zeta})^{-1}\psi(u, \zeta, \overline{\zeta}) \,. \tag{6}$$

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All that may suggest to think the rescaled boundary values ψ as wavefunctions on $\Im^+$ and define a QFT based on a suitable symplectic space containing these wavefunctions where the BMS group acts as in (6) and it is the symmetry group of the theory. In fact, in [DMP05] we introduced a simple notion of QFT on $\Im^+$ based on a certain Weyl algebra of observables associated with a symplectic space $(\mathcal{S}(\Im^+), \sigma)$ with

$$\mathcal{S}(\Im^+) := \left\{ \psi \in C^\infty(\Im^+) \ \middle| \ \sup_{(\zeta,\overline{\zeta}\in\mathbb{S}^2)} |u|^{-k}|\partial_u^k\partial_\zeta^m\partial_{\overline{\zeta}}^n\psi| \to 0 \,, \text{as } |u| \to +\infty \,, \forall k,m,n \in \mathbb{N} \right\} . \quad (7)$$

Here we *enlarge* $\mathcal{S}(\Im^+)$ (the reason will be clear later) up to the space $\mathsf{S}(\Im^+) \supset \mathcal{S}(\Im^+)$ with

$$\mathsf{S}(\Im^+) := \left\{ \psi \in C^\infty(\Im^+) \ \middle| \ \psi \text{ and } \partial_u\psi \text{ belong to } L^2(\mathbb{R}\times\mathbb{S}^2, du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta})) \right\} \quad (8)$$

$\epsilon_{\mathbb{S}^2}$ is (see below) the standard volume form of the unit 2-sphere $\mathbb{S}^2 = \mathbb{C}\cup\{\infty\}$. Both spaces are invariant under the action (6) of BMS group so that the choice of a Bondi frame in the definitions is immaterial. The nondegenerate symplectic form σ is the following. If $\psi_1, \psi_2 \in \mathcal{S}(\Im^+)$ or, more generally $\psi_1, \psi_2 \in \mathsf{S}(\Im^+)$,

$$\sigma(\psi_1,\psi_2) := \int_{\mathbb{R}\times\mathbb{S}^2} \left(\psi_2\frac{\partial\psi_1}{\partial u} - \psi_1\frac{\partial\psi_2}{\partial u} \right) du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}) \,, \quad \epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}) := \frac{2d\zeta\wedge d\overline{\zeta}}{i(1+\zeta\overline{\zeta})^2} \,, \quad (9)$$

The Weyl algebra $\mathcal{W}(\Im^+)$ is that associated with the pair $(\mathsf{S}(\Im^+), \sigma)$ (see appendix C). The generators of that Weyl algebras are denoted by $W(\psi)$, $\psi \in \mathsf{S}(\Im^+)$. By definition they do not vanish and satisfy *Weyl relations* (or CCR)

$$(W1) \qquad W(-\psi) = W(\psi)^* \,, \qquad\qquad (W2) \qquad W(\psi)W(\psi') = e^{i\sigma(\psi,\psi')/2}W(\psi+\psi') \,.$$

$\mathcal{W}(\Im^+)$ is uniquely determined, up to (isometric $*$-algebra) isomorphisms by the requirement that it is a C^* algebra generated by non null elements $W(\psi)$ fulfilling (W1) and (W2) (see Appendix C). The formal interpretation of generators is $W(\psi) = e^{i\sigma(\psi,\Psi)}$, $\sigma(\psi,\Psi)$ denotes the usual *symplectically smeared field operator* (see appendix C).

Remark 1.2. Naturalness of the symplectic space $(\mathsf{S}(\Im^+), \sigma)$ and the associated Weyl algebra is consequence of the following three facts. (i) σ *is invariant under the action (6) of BMS group* as proved in Theorem 2.1 [DMP05], the enlargement of $\mathcal{S}(\Im^+)$ does not affect the proof. (ii) Under suitable hypotheses the Weyl algebra of linear QFT in the bulk identify with a sub algebra of $W(\Im^+)$. Let us illustrate this point. Let (M,g) be an asymptotically flat spacetime, strongly asymptotically predictable with respect to $\tilde{V} \subset \tilde{M}$. Define $\mathcal{S}_P(M_{\tilde{V}})$ to be the real linear space of real smooth solutions ϕ in $M_{\tilde{V}}$ of Klein-Gordon equation (5) which have compact support on Cauchy surfaces in $M_{\tilde{V}}$ and define the Cauchy-surface invariant symplectic form defined as

$$\sigma_{M_{\tilde{V}}}(\phi_1,\phi_2) := \int_\Sigma (\phi_1\partial_{n_\Sigma}\phi_2 - \phi_2\partial_{n_\Sigma}\phi_1) \, d\mu_\Sigma^{(g)} \,, \quad \text{for } \phi_1,\phi_2 \in \mathcal{S}_P(M_{\tilde{V}}), \quad (10)$$

$\Sigma \subset M_{\tilde{V}}$ being a smooth spacelike Cauchy surface with unit, future directed, normal vector n_Σ and measure $\mu_\Sigma^{(g)}$ induced by g. In this context $\mathcal{W}_P(M_{\tilde{V}})$ denotes the Weyl algebra of the quantum field ϕ in the bulk associated with the symplectic space $(\mathcal{S}(M_{\tilde{V}}), \sigma_{M_{\tilde{V}}})$ with Weyl generators

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$W_{M_{\tilde{V}}}(\phi)$, $\phi \in \mathcal{S}(M_{\tilde{V}})$. Proposition 4.1 in [DMP05] reads

Proposition 1.1.   *Let (M, g) be an asymptotically flat spacetime, strongly asymptotically predictable with respect to $\tilde{V} \subset \tilde{M}$. Assume that both conditions below hold true for the projection map $\Gamma_{M_{\tilde{V}}} : \mathcal{S}_P(M_{\tilde{V}}) \ni \phi \mapsto \left((\omega_B \Omega)^{-1} \phi \right) \restriction_{\mathfrak{I}^+} :$*
(a) $\Gamma_{M_{\tilde{V}}}(\mathcal{S}_P(M_{\tilde{V}})) \subset \mathcal{S}(\mathfrak{I}^+)$,
(b) *symplectic forms are preserved by $\Gamma_{M_{\tilde{V}}}$, that is, for all $\phi_1, \phi_2 \in \mathcal{S}(M_{\tilde{V}})$,*

$$\sigma_{M_{\tilde{V}}}(\phi_1, \phi_2) = \sigma(\Gamma_{M_{\tilde{V}}} \phi_1, \Gamma_{M_{\tilde{V}}} \phi_2) \,, \tag{11}$$

Then $\mathcal{W}_P(M_{\tilde{V}})$ can be identified with a sub C^*-algebra of $\mathcal{W}(\mathfrak{I}^+)$ by means of a C^*-algebra isomorphism $\imath$ uniquely determined by the requirement

$$\imath(W_{M_{\tilde{V}}}(\phi)) = W(\Gamma_{M_{\tilde{V}}} \phi) \,, \quad \text{for all } \phi \in \mathcal{S}_P(M_{\tilde{V}}) \,, \tag{12}$$

By Proposition 4.1 in [DMP05] the conditions (a) and (b) are fulfilled at least when (M, g) is the four-dimensional Minkowski spacetime (in this case $M_{\tilde{V}} = M$) : $\mathcal{S}(\mathfrak{I}^+)$ (and thus $\mathsf{S}(\mathfrak{I}^+)$) includes the limit ψ to $\mathfrak{I}^+$ of the rescaled solutions $(\omega_B \Omega)^{-1} \phi$ of (5) in $M_{\tilde{V}}$ and $\sigma(\psi_1, \psi_2)$ coincides with the limit to $\mathfrak{I}^+$ of the bulk symplectic form for smooth solutions ϕ.
However, it is worth noticing that $(\mathsf{S}(\mathfrak{I}^+), \sigma)$ does not depend on the particular (asymptotically flat and strongly asymptotically predictable) spacetime M whose $\mathfrak{I}^+$ is the future causal boundary.

A preferred quasifree pure BMS-invariant state λ on $\mathcal{W}(\mathfrak{I}^+)$ has been introduced in [DMP05]. The extent is not affected by the enlargement of $\mathcal{S}(\mathfrak{I}^+)$ to $\mathsf{S}(\mathfrak{I}^+)$. Fix a Bondi frame $(u, \zeta, \overline{\zeta})$. For $\psi \in \mathsf{S}(\mathfrak{I}^+)$ define its **positive-frequency part** ψ_+ (with respect to u) as follows:

$$\psi_+(u, \zeta, \overline{\zeta}) := \int_{\mathbb{R}} e^{-iEu} \frac{\widetilde{\psi_+}(E, \zeta, \overline{\zeta})}{\sqrt{4\pi E}} dE \,, \quad \frac{\widetilde{\psi_+}(E, \zeta, \overline{\zeta})}{\sqrt{2E}} := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{+iEu} \psi(u, \zeta, \overline{\zeta}) du \,, \tag{13}$$

with $\widetilde{\psi_+}(E, \zeta, \overline{\zeta}) := 0$ for $E \notin \mathbb{R}^+$. With our enlargement of $\mathsf{S}(\mathfrak{I}^+)$ the Fourier transforms in (13) must be understood as the Fourier-Plancherel transforms (see Appendix D). From proposition D.1, the right-hand side of (9) can be computed also for positive frequency parties ψ_{1+}, ψ_{2+} when $\psi_1, \psi_2 \in \mathsf{S}(\mathfrak{I}^+)$, provided the derivatives involved in (9) be interpreted in distributional sense. A Hermitian scalar product arises in the complexified space of positive frequency parts:

$$\langle \psi_{1+}, \psi_{2+} \rangle := -i\sigma(\overline{\psi_{1+}}, \psi_{2+}) \,. \tag{14}$$

Since $\mathsf{S}(\mathfrak{I}^+) \supset \mathcal{S}(\mathfrak{I}^+)$, Theorem 2.2 in [DMP05] implies that the Hilbert completion $\mathcal{H}$ of the complexified space of positive frequency parts is isometrically isomorphic to $L^2(\mathbb{R}^+ \times \mathbb{S}^2, dE \otimes \epsilon_{\mathbb{S}^2})$ (no matter the enlargement of $\mathcal{S}(\mathfrak{I}^+)$). In particular

$$\langle \psi_{1+}, \psi_{2+} \rangle = \int_{\mathbb{R}^+ \times \mathbb{S}^2} \overline{\widetilde{\psi_{1+}}(E, \zeta, \overline{\zeta})} \, \widetilde{\psi_{2+}}(E, \zeta, \overline{\zeta}) \, dE \otimes \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}) \,, \quad \text{for every pair } \psi_1, \psi_2 \in \mathsf{S}(\mathfrak{I}^+). \tag{15}$$

Since $\mathsf{S}(\Im^+) \supset \mathsf{8}(\Im^+)$, Theorem 2.2 in [DMP05] implies also that the $\mathbb{R}$-linear map K : $\mathsf{S}(\Im^+) \ni \psi \mapsto \psi_+ \in \mathcal{H}$ has dense range. Since, by (13) and (14) one also has $\sigma(\psi_1, \psi_2) = -2Im\langle K\psi_1, K\psi_2\rangle$, we conclude (see proposition C.2 in Appendix C) that there is a unique pure quasifree regular state λ which satisfies,

$$\lambda(W(\psi)) = e^{-\mu_\lambda(\psi,\psi)/2}\,, \quad \text{for all } \psi \in \mathsf{S}(\Im^+) \text{ and where } \mu_\lambda(\psi_1, \psi_2) := Re\langle \psi_{1+}, \psi_{2+}\rangle \quad (16)$$

and the GNS triple $(\mathfrak{H}, \Pi, \Upsilon)$ is made of the Fock space $\mathfrak{H}$ with cyclic vector given by the vacuum Υ and one-particle space $\mathcal{H}$. The representation Π is completely determined by the identity, valid for every $\psi \in \mathsf{S}(\Im^+)$, $\Pi(W(\psi)) := e^{i\overline{\Psi(\psi)}}$ where, following notation as in [DMP05] we write $\Psi(\psi)$ in place of $\sigma(\psi, \Psi)$ for the sake of simplicity. Similarly, regarding the GNS triple of λ and field operators from now on we omit pedices $_\lambda$.

Let us pass to focus on BMS invariance of the theory and the state λ. We recall the reader that a state ω on a C^*-algebra $\mathcal{A}$ is **invariant** under a faithful representation β of a group G made of of $*$-automorphisms if $\omega(\beta_g(a)) = \omega(a)$ for every $g \in G$ and every $a \in \mathcal{A}$. Invariance of ω under β implies that β is *unitarily implementable* in the GNS representation $(\mathfrak{H}_\omega, \Pi_\omega, \Upsilon_\omega)$ of ω and there is a unique unitary representation $U : G \ni g \mapsto U_g$ acting on $\mathfrak{H}_\omega$ *leaving fixed the cyclic vector* [Ar99], that is

$$U_g\Pi_\omega(a)U_g^\dagger = \Pi_\omega(\beta_g(a)) \quad \text{and} \quad U_g\Upsilon_\omega = \Upsilon_\omega\,, \quad \text{for all pairs } g \in G,\, a \in \mathcal{A}. \quad (17)$$

The remaining unitary representations $\{V_g\}_{g\in\mathcal{G}}$ of $\mathcal{G}$ which implement the group on $\mathfrak{H}_\omega$ may transform Υ up to a phase e^{ia_g} only. They therefore differ from U for that phase only, it being $V_g\Pi_\omega(a)\Upsilon_\omega = e^{ia_g}\Pi_\omega(\beta_g(a))\Upsilon_\omega = e^{ia_g}U_g\Pi_\omega(a)\Upsilon_\omega$. When G is a topological/Lie group there is no guarantee, in general, for strong continuity of U. Strong continuity would imply, via Stone theorem, the existence of self-adjoint generators of the transformations of the group which, very often, have some physical interest.

A group $G \ni g$ acting on a symplectic space S by means of transformations B_g preserving the symplectic form σ, induces a representation β of G made of $*$-automorphisms on the Weyl algebra $\mathcal{W}_{\mathsf{S},\sigma}$ (see theorem 5.2.8 in [BR022]): it is uniquely determined by $\beta_g(W(\psi)) := W\left(B_{g^{-1}}\psi\right)$ for every $g \in G$ and $\psi \in \mathsf{S}$. We call β the **representation canonically induced by G**.

Concerning $\mathcal{W}(\Im^+)$, σ is invariant under the action (6) of G_{BMS} and thus the representation α canonically induced by G_{BMS} (6) on $\mathcal{W}(\Im^+)$ is uniquely determined by the requirement

$$\alpha_g(W(\psi)) = W\left(A_{g^{-1}}\psi\right)\,, \quad \text{for every } g \in G_{BMS} \text{ and } \psi \in \mathsf{S}(\Im^+). \quad (18)$$

α turns out to be faithful. With the extended definition of symplectic space we have the following theorem which embodies theorems 2.3 and 2.4 in [DMP05].

Theorem 1.1. *The state λ on $\mathcal{W}(\Im^+)$ (16) with GNS triple $(\mathfrak{H}, \Pi, \Upsilon)$ is invariant under the representation α of G_{BMS} (6) so that λ is independent from the choice of the Bondi frame on $\Im^+$ used in (16). Furthermore the following holds.*
(a) *The unique unitary representation $G_{BMS} \ni g \mapsto U_g$ representing α leaving fixed Υ, is the*

standard tensorialization of the representation $U^{(1)} = U\!\restriction_{\mathcal{H}}$ on the one-particle space $\mathcal{H}$ defined in the Bondi frame on $\Im^+$ used to define λ by

$$\left(U^{(1)}_{(\Lambda,f)}\varphi\right)(E,\zeta,\overline{\zeta}) = \frac{e^{iEK_\Lambda(\Lambda^{-1}(\zeta,\overline{\zeta}))f(\Lambda^{-1}(\zeta,\overline{\zeta}))}}{\sqrt{K_\Lambda(\Lambda^{-1}(\zeta,\overline{\zeta}))}}\varphi\left(EK_\Lambda\left(\Lambda^{-1}(\zeta,\overline{\zeta})\right),\Lambda^{-1}(\zeta,\overline{\zeta})\right), \qquad (19)$$

for every $\varphi \in L^2(\mathbb{R}^+ \times \mathbb{S}^2; dE \otimes \epsilon_{\mathbb{S}^2})$ and $G_{BMS} \ni g \equiv (\Lambda, f)$.
(b) *U is strongly continuous when equipping G_{BMS} with the nuclear topology (see [DMP05]).*

Sketch of proof. By direct inspection, referring to (13), from (19) one sees that (i) $U^{(1)}$ is unitary and, if $\psi \in \mathsf{S}(\Im^+)$, $\psi_+^{(g)}(u,\zeta,\overline{\zeta}) := \int_{\mathbb{R}^+} e^{-iEu}(U^{(1)}_g\widetilde{\psi_+})(E,\zeta,\overline{\zeta})\frac{dE}{\sqrt{4\pi E}}$ is well defined and satisfies (ii) $\psi_+^{(g)} + \overline{\psi_+^{(g)}} = A_g(\psi)$. Let U_g be the tensorialization to the whole Fock space of $U^{(1)}_g$ satisfying $U_g\Upsilon := \Upsilon$. Using $\Pi(W(\psi)) = \exp\overline{i\sigma(\psi,\Psi)}$ (see proposition C.2 in appendix C), from (ii) arises $U_g\Pi(W(\psi))U_g^\dagger = \Pi\left(W\left(A_{g^{-1}}\psi\right)\right)$. This proves (a) as well as the invariance of λ under α because $U_g\Upsilon := \Upsilon$ by constriction. The proof of (b) is exactly that of Theorem 2.4 in [DMP05]. $\square$

Remark 1.3. It has been proved in Section 3 of [DMP05] that, adopting a suitable Wigner's-like representation analysis, (Theorem 3.2 in [DMP05]) the representation $U\!\restriction_{\mathcal{H}}$ is that proper of a *massless* particle with respect to the known *BMS notion of mass* [MC72-75]. (The proof is completely independent on the enlargement of $\mathsf{S}(\Im^+)$ adopted here.) This is particularly relevant because this result suggests that also in the absence of Poincaré symmetry the "geometric notion of mass" which appears in Klein-Gordon equation could have a Schwinger - group theory interpretation in relation to BMS group for asymptotically flat spacetimes.

1.2. *Contents of this paper.* In this paper we primarily focus on one of the final issues raised at the end of [DMP05]. *How is the BMS-invariant state λ unique?* In fact, after some preparatory results given in section 2 section 3 presents an answer to that question based on some peculiarities of the state λ which are examined in the following section. In the practice, first we notice that λ enjoys positivity of the self-adjoint generator of u-translations with respect to *every* admissible frame $(u,\zeta,\overline{\zeta})$ on $\Im^+$. This fact may be interpreted as a remnant of spectral condition inherited from QFT in Minkowski spacetime. Moreover we find that every pure state on $\mathcal{W}(\Im^+)$ invariant under u-displacements with respect to a fixed admissible frame satisfies cluster property with respect to these displacements. Afterwords, in section 3 taking the cluster property into account, we show that the validity of positivity for the self-adjoint generator of u-translations in *a fixed* admissible frame individuates the BMS-invariant state λ uniquely (without requiring BMS invariance). As a second result, we show that in the folium of a pure u-displacement invariant state (like λ but not necessarily quasifree) on $\mathcal{W}(\Im^+)$ the state itself is the only u-displacement invariant state. The proof of the first uniqueness result is essentially obtained by reducing to the hypotheses of a know uniqueness theorem due to Kay [Ka79].

The second issue considered in section 4 concerns the validity of proposition 1.1 which assures that the Weyl algebra of a linear QFT in the bulk is isometrically mapped onto a sub algebra of

$\mathcal{W}(\mathfrak{I}^+)$. We know that the hypotheses of proposition 1.1 are fulfilled for Minkowski spacetime or, similarly, for a strongly asymptotically predictable, asymptotically flat spacetime which coincides with Minkowski spacetime after a Cauchy surface arbitrarily far in the future. The issue is important because the existence of the isometric $*$-homomorphism permits to induce a preferred state in the bulk by the symmetric state λ. We expect that the preferred state is invariant under any asymptotic symmetry (including proper symmetries) of the bulk by construction. We prove in section 4 which the isometric $*$-homomorphism of proposition 1.1 exists whenever it is possible to complete $\mathfrak{I}^+$ by adding the asymptotic future point i^+ in the sense of Friedrich [Fri86-88].

The last section contains some final comments and open questions. The appendices contain proofs of some propositions and recall general definitions and results about quasifree states on Weyl algebras.

2   Some properties of λ, $\mathcal{W}(\mathfrak{I}^+)$ and states on $\mathcal{W}(\mathfrak{I}^+)$.

2.1. *Positivity, u-displacement cluster property for Weyl-generator.* There are two interesting properties of λ which were not mentioned in [DMP05] these are stated in proposition 2.1. Some introductory notions are necessary. For each admissible frame $\mathcal{F} \equiv (u, \zeta, \overline{\zeta})$ there is a one-parameter subgroup $\{\mathcal{T}_t^{(\mathcal{F})}\}_{t \in \mathbb{R}}$ of G_{BMS} defining (active) u-displacements: $\mathcal{T}^{(\mathcal{F})} := \mathcal{T}_t^{(\mathcal{F})}$: $(u, \zeta, \overline{\zeta}) \mapsto (u + t, \zeta, \overline{\zeta})$. In turn, the restriction of α (18) to $\mathcal{T}^{(\mathcal{F})}$ is a $*$-automorphism representation, $\alpha^{(\mathcal{F})} := \{\alpha_t^{(\mathcal{F})}\}_{t \in \mathbb{R}}$ of $\mathcal{T}^{(\mathcal{F})}$. If ω is an $\alpha^{(\mathcal{F})}$-invariant state on $\mathcal{W}(\mathfrak{I}^+)$, it satisfies $\alpha^{(\mathcal{F})}$-**cluster property for Weyl generators** if

$$\lim_{t \to +\infty} \omega \left(W(\psi)\, \alpha_t^{(\mathcal{F})}\big(W(\psi')\big) \right) = \omega\left(W(\psi)\right) \omega\left(W(\psi')\right), \quad \text{for all } \psi, \psi' \in \mathsf{S}(\mathfrak{I}^+), \tag{20}$$

Proposition 2.1.   *The G_{BMS}-invariant state λ on $\mathcal{W}(\mathfrak{I}^+)$ defined in (16) enjoys the following properties with respect to the one-parameter group $\alpha^{(\mathcal{F})}$ of every admissible frame $\mathcal{F}$.*
(a) *The generator $H^{(\mathcal{F})}$ of the unitary group $\{e^{-itH^{(\mathcal{F})}}\}_{t \in \mathbb{R}}$ implementing $\alpha^{(\mathcal{F})}$ leaving fixed the cyclic vector is nonnegative.*
(b) *λ satisfies $\alpha^{(\mathcal{F})}$-cluster property for Weyl generators (20).*

Proof. It is sufficient to prove the thesis for a fixed Bondi frame $\mathcal{F}$. It generalizes to every other admissible frame $\mathcal{F}'$ using the following facts: (i) λ is G_{BMS} invariant, (ii) there is $g \in G_{BMS}$ such that $\alpha_t^{(\mathcal{F}')} = \alpha_g \alpha_t^{(\mathcal{F})} \alpha_{g^{-1}}$ for every $t \in \mathbb{R}$, (iii) α is unitarily implementable leaving fixed the cyclic vector, (iv) unitary equivalences preserve the spectrum of operators.
(a) Construct the state λ referring to the Bondi frame $\mathcal{F}$. In the one-particle space $\mathcal{H} \equiv L^2(\mathbb{R}^+ \times \mathbb{S}^2)$ of the GNS representation of λ, the self-adjoint operator H, such that $(H\phi)(E, \zeta, \overline{\zeta}) := E\phi(E, \zeta, \overline{\zeta})$ defined in the domain of the square-integrable functions ϕ such that $\mathbb{R}^+ \ni E \mapsto E\phi(E, \zeta, \overline{\zeta})$ is square integrable, has spectrum $\sigma(H) := [0, +\infty)$. By construction, if $H^\otimes$ denotes unique the standard tensorialization of H extended to the Fock space $\mathfrak{H}$ with the constraint $H^{(\mathcal{F})}\Upsilon = 0$, $H^{(\mathcal{F})}$ is non negative by construction, moreover one has $e^{-itH^{(\mathcal{F})}}\Upsilon = \Upsilon$ as well as,

from (13)

$$e^{-itH^{(\mathcal{F})}} \Pi\left(W(\psi)\right) e^{itH^{(\mathcal{F})}} = \Pi\left(W(A_{\mathcal{T}^{(\mathcal{F})}_{-t}}\psi)\right) = \Pi\left(\alpha_{\mathcal{T}^{(\mathcal{F})}_t}\left(W(\psi)\right)\right) = \Pi\left(\alpha_t^{(\mathcal{F})}\left(W(\psi)\right)\right) \ .$$

So that $e^{-itH^{(\mathcal{F})}}$ implements $\alpha_t^{(\mathcal{F})}$ and has nonnegative generator.

(b) Take $\psi, \psi' \in \mathsf{S}(\mathfrak{S}^+)$. If $\psi'_t(u, \zeta, \overline{\zeta}) := \psi'(u+t, \zeta, \overline{\zeta})$, using Weyl relations, (49), invariance of λ under $\alpha^{(\mathcal{F})}$ and (ii) in lemma C.1, one has

$$\lambda\left(W(\psi)\, \alpha_t^{(\mathcal{F})}\big(W(\psi')\big)\right) = e^{-\langle K\psi, K\psi'_t\rangle}\lambda\left(W(\psi)\right)\lambda\left(W(\psi')\right) \ . \tag{21}$$

By (14) and Fubini-Tonelli theorem: $\langle K\psi, K\psi'_t\rangle = \int_{\mathbb{R}^+} dE e^{-itE} \int_{\mathbb{S}^2} \overline{\widetilde{\psi_+}(E, \zeta, \overline{\zeta})}\widetilde{\psi'_+}(E, \zeta, \overline{\zeta})\epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta})$ and the internal integral defines a $L^1(\mathbb{R}^+, dE)$ function of E. Riemann-Lebsgue lemma implies that $\langle K\psi, K\psi'_t\rangle$ vanishes as $t \to +\infty$ so that (20) holds true from (21). $\square$

Remark 2.1. Consider QFT in Minkowski spacetime $\mathbb{M}^4$ built up Minkowski vacuum $\Upsilon_{\mathbb{M}^4}$ and QFT with Weyl algebra $\mathcal{W}(\mathbb{M}^4)$ on $\mathfrak{S}^+$ referred to $\lambda \equiv \Upsilon$. If a Bondi frame $(u, \zeta, \overline{\zeta})$ (which can be fixed to be a Bondi frame) on $\mathfrak{S}^+$ is associated with a Minkowski reference frame in the bulk, u displacements are in one-to-one correspondence with time translations respect to the Minkowski frame. More precisely, by Theorems 4.1 and 4.2 there is the unitary equivalence $\mathcal{U}$ which unitarily implements, in the respective GNS Hilbert spaces, the $*$-isomorphisms $\imath : \mathcal{W}(\mathbb{M}^4) \to \mathcal{W}(\mathfrak{S}^+)$ arising from proposition 1.1 (Proposition 4.1 in [DMP05]), mapping $\Upsilon_{\mathbb{M}^4}$ into Υ. Under the unitary equivalence $\mathcal{U}$ the self-adjoint generator of time displacements of the Weyl algebra in the bulk is transformed to the self-adjoint generator of u-displacements for the Weyl algebra on $\mathcal{W}(\mathfrak{S}^+)$ and hence the spectra of those operators are identical. Finally, as discussed in [DMP05], changing Minkowski frame by means of a orthochronous Poincaré transformation is equivalent to passing to another admissible frame (in general not a Bondi frame) by means of a suitable transformation (2)-(3). These changes preserve the interplay of time displacements and u-displacements. We conclude that positivity of u-generator for QFT on $\mathfrak{S}^+$ refereed to λ, valid for *every* admissible frame on $\mathfrak{S}^+$ is nothing but the *spectral condition* of QFT in Minkowski spacetime referred to Minkowski vacuum for the free theory in $\mathbb{M}^4$. In Minkowski QFT the spectral condition is a stability requirement: it guarantees that, under small (external) perturbations, the system does not collapse to lower and lower energy states. In this way, we are naturally lead to consider positivity of u-displacement generator with respect to every admissible frame on $\mathfrak{S}^+$ as a natural candidate for replacing the spectral condition in QFT on $\mathfrak{S}^+$ *also when $\mathfrak{S}^+$ itself is not thought as the null boundary of Minkowski spacetime.*

2.2. *Asymptotic properties, extension of cluster property.* The proof of proposition 2.1 yields, as a byproduct, a general property of $(\mathcal{W}(\mathfrak{S}^+), \sigma)$, i.e. *asymptotic commutativity*. The proof of the following proposition is in Appendix A.

Proposition 2.2. *For every admissible frame $\mathcal{F}$ the following facts are valid.*
(a) $\alpha^{(\mathcal{F})}$-**asymptotic commutativity** *holds:*

$$\lim_{t \to +\infty} \left[\alpha_t^{(\mathcal{F})}(a), b\right] = 0, \quad \text{for all } a, b \in \mathcal{W}(\mathfrak{I}^+). \tag{22}$$

(b) *Let ω be a pure (not necessarily quasifree) state $\mathcal{W}(\mathfrak{I}^+)$ with GNS representation $(\mathfrak{H}_\omega, \Pi_\omega, \Upsilon_\omega)$ and assume that there exist a unitary group $U^{(\mathcal{F})}$ implementing $\alpha^{(\mathcal{F})}$ on $\mathfrak{H}_\omega$. Then ω satisfies $\alpha^{(\mathcal{F})}$-**weak asymptotic commutativity**:*

$$\text{w-}\lim_{t \to +\infty} \left[U_t^{(\mathcal{F})} A U_t^{(\mathcal{F})\dagger}, B\right] = 0, \quad \text{for all pairs } A \in \Pi_\omega(\mathcal{W}(\mathfrak{I}^+)), \, B \in \mathfrak{B}(\mathfrak{H}_\omega) \tag{23}$$

where $\mathfrak{B}(\mathfrak{H}_\omega)$ is the space of bounded operators on $\mathfrak{H}_\omega$, and w-lim denotes the weak operatorial topology limit.

To conclude this technical subsection we give a final proposition which extends $\alpha^{(\mathcal{F})}$-cluster property to the whole Weyl algebra giving also another related property. If $\mathcal{F}$ is a Bondi frame on $\mathfrak{I}^+$, we say that a state ω (not necessarily quasifree) on $\mathcal{W}(\mathfrak{I}^+)$ satisfies $\alpha^{(\mathcal{F})}$-**cluster property** (in the full-$\mathcal{W}(\mathfrak{I}^+)$ version) if

$$\lim_{t \to +\infty} \omega(a\, \alpha_t^{(\mathcal{F})}(b)) = \omega(a)\omega(b), \quad \text{for all } a, b \in \mathcal{W}(\mathfrak{I}^+). \tag{24}$$

Proposition 2.3. *Let $\mathcal{F}$ be an admissible frame on $\mathfrak{I}^+$ and ω a pure (not necessarily quasifree) state on $\mathcal{W}(\mathfrak{I}^+)$ with GNS triple $(\mathfrak{H}_\omega, \Pi_\omega, \Upsilon_\omega)$. If ω is $\alpha^{(\mathcal{F})}$-invariant the following holds.*
(a) *ω satisfies $\alpha^{(\mathcal{F})}$-cluster property.*
(b) *If $A \in \Pi_\omega(\mathcal{W}(\mathfrak{I}^+))$ and $U^{(\mathcal{F})}$ is a unitary group implementing $\alpha^{(\mathcal{F})}$ on $\mathfrak{H}_\omega$, one has*

$$\text{w-}\lim_{t \to +\infty} U_t^{(\mathcal{F})} A U_t^{(\mathcal{F})\dagger} = \langle \Upsilon_\omega, A \Upsilon_\omega \rangle I. \tag{25}$$

Proof (a) is an immediate consequence of (b) when writing the statement (a) in the GNS space $\mathcal{H}_\omega$ using GNS theorem, with $A = \Pi_\omega(a)$ and $B = \Pi_\omega(b)$. To prove (a), take $B \in \Pi_\omega(\mathcal{W}(\mathfrak{I}^+))$ and $\Phi \in \mathfrak{H}_\omega$. If $A_t = U_t^{(\mathcal{F})} A U_t^{(\mathcal{F})\dagger}$ and $P_0 = |\Upsilon_\omega\rangle\langle\Upsilon_\omega|$ we have

$$\langle \Phi, BA_t\Upsilon_\omega \rangle = \langle B^\dagger\Phi, [A_t, P_0]\Upsilon_\omega \rangle + \langle B^\dagger\Phi, P_0 A_t\Upsilon_\omega \rangle = \langle B^\dagger\Phi, [A_t, P_0]\Upsilon_\omega \rangle + \langle \Phi, B\Upsilon_\omega \rangle\langle \Upsilon_\omega, A_t\Upsilon_\omega \rangle.$$

The second term on the right-hand side is noting but $\langle \Phi, B\Upsilon_\omega \rangle\langle \Upsilon_\omega, A\Upsilon_\omega \rangle$ because ω is $\alpha^{(\mathcal{F})}$ invariant, whereas the first term vanishes by weak asymptotic commutativity. By asymptotic commutativity we also get $\lim_{t \to +\infty}\langle \Phi, A_t B\Upsilon_\omega \rangle = \lim_{t \to +\infty}\langle \Phi, BA_t\Upsilon_\omega \rangle = \langle \Phi, B\Upsilon_\omega \rangle\langle \Upsilon_\omega, A\Upsilon_\omega \rangle$. Since $\{B\Upsilon_\omega\}$ is dense in $\mathfrak{H}_\omega$, for every $\Psi \in \mathfrak{H}_\omega$ and $\epsilon > 0$, there is $B_\epsilon \in \Pi_\omega(\mathcal{W}(\mathfrak{I}^+))$ with $\|B_\epsilon\Upsilon_\omega - \Psi\| < \epsilon$. Therefore, if $\langle A \rangle := \langle \Upsilon_\omega, A\Upsilon_\omega \rangle$, it results that $|\langle \Phi, A_t\Psi \rangle - \langle A \rangle\langle \Phi, \Psi \rangle|$ is bounded by $|\langle \Phi, A_t B_\epsilon\Upsilon_\omega \rangle - \langle A \rangle\langle \Phi, B_\epsilon\Upsilon_\omega \rangle| + |\langle \Phi, A_t(\Psi - B_\epsilon\Upsilon_\omega) \rangle - \langle A \rangle\langle \Phi, (\Psi - B_\epsilon\Upsilon_\omega) \rangle|$. The first term tends to 0 as $t \to +\infty$, whereas the second is bounded by $\epsilon\|\Phi\| \, \|A_t\| + |\langle A \rangle| = \epsilon\|\Phi\| \|A\| + |\langle A \rangle|\|$. Finally, with the procedure based on standard properties of lim sup, lim inf used in the proof of proposition , one obtains $\lim_{t \to +\infty}\langle \Phi, A_t\Psi \rangle = \langle \Upsilon_\omega, A\Upsilon_\omega \rangle\langle \Phi, \Psi \rangle$. $\square$

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3 The uniqueness theorem.

3.1. *The uniqueness theorem.* Making profitable use of cluster invariance, we are able to establish that λ is the unique quasifree pure state on $\mathcal{W}(\Im^+)$ which is $\alpha^{(\mathcal{F})}$ invariant for an, arbitrarily chosen, admissible frame $\mathcal{F}$ and it has nonnegative self-adjoint generator for the unitary implementation of $\alpha^{(\mathcal{F})}$. No requirement about the full BMS invariance is necessary. Moreover, dropping the quasifree hypotheses, we show that in the folium[2] of a pure $\alpha^{(\mathcal{F})}$-invariant state on $\mathcal{W}(\Im^+)$ the state itself is the only $\alpha^{(\mathcal{F})}$-invariant state. Below "BMS-invariant" for a state means "invariant under the $*$-automorphism representation α (18) of G_{BMS}".

Theorem 3.1. *Consider an arbitrary admissible frame $\mathcal{F}$ on $\Im^+$.*
(a) *The BMS-invariant state λ defined in (16) is the unique pure quasifree state on $\mathcal{W}(\Im^+)$ satisfying both:*
   (i) it is invariant under $\alpha^{(\mathcal{F})}$,
   (ii) the unitary group which implements $\alpha^{(\mathcal{F})}$ leaving fixed the cyclic GNS vector is strongly continuous with nonnegative generator.
(b) *Let ω be a pure (not necessarily quasifree) state on $\mathcal{W}(\Im^+)$ which is BMS-invariant or, more weakly, $\alpha^{(\mathcal{F})}$-invariant. ω is the unique state on $\mathcal{W}(\Im^+)$ satisfying both:*
   (i) it is invariant under $\alpha^{(\mathcal{F})}$,
   (ii) it belongs to the folium of ω.

Remarks. **(1)** The condition (ii) in (a) is equivalent to the requirement that there is a strongly-continuous unitary group $\{e^{-itH^{(\mathcal{F})}}\}_{t\in\mathbb{R}}$ implementing $\alpha^{(\mathcal{F})}$, such that $\inf\sigma(H^{(\mathcal{F})}) \geq \langle\Upsilon, H^{(\mathcal{F})}\Upsilon\rangle$, Υ being the cyclic GNS vector.
(2) From a general result in the appendix C, strong continuity for the unitary group implementing $\alpha^{(\mathcal{F})}$ leaving the cyclic vector unchanged for a state ω is equivalent to continuity at 0 of $\mathbb{R} \ni t \mapsto \omega\left(W(\psi)\,\alpha_t^{(\mathcal{F})}(W(\psi'))\right)$ for all $\psi, \psi' \in \mathsf{S}(\Im^+)$.

Proof of theorem 3.1. (a) Consider a state ω invariant under a one-parameter group of $*$-automorphisms $\alpha^{(\mathcal{F})}$, supposing that $\mathcal{F}$ is a Bondi frame, and let us indicate by $\{U_t^{(\mathcal{F})}\}_{t\in\mathbb{R}}$ the unique unitary group which implements $\alpha^{(\mathcal{F})}$ leaving the GNS cyclic vector Υ_ω fixed. From now on we represent wavefunctions in coordinates $(u, \zeta, \overline{\zeta})$ of $\mathcal{F}$. Since ω is quasifree on has $\Pi_\omega(W(\psi)) = e^{i\Psi_\omega(\psi)}$ and thus, in particular, for every $x \in \mathbb{R}$, $U_t^{(\mathcal{F})}e^{i\Psi_\omega(x\psi)}U_{-t}^{(\mathcal{F})}\Upsilon_\omega = e^{i\Psi_\omega(x\psi_t)}\Upsilon_\omega$, where $\psi_t(u, \zeta, \overline{\zeta}) := \psi(u+t, \zeta, \overline{\zeta})$. Using the fact that $U_{-t}^{(\mathcal{F})}\Upsilon_\omega = \Upsilon_\omega$ and applying Stone theorem, it results $U_t^{(\mathcal{F})}a^\dagger(K_\omega\psi)\Upsilon_\omega = a^\dagger(K_\omega\psi_t)\Upsilon_\omega$. In other words the one-particle space $\mathcal{H}_\omega$ is invariant under $U_t^{(\mathcal{F})}$ and its restriction to $\mathcal{H}_\omega$, $V^{(\mathcal{F})} := U^{(\mathcal{F})}\!\restriction_{\mathcal{H}_\omega}$ is unitary as well. Tensorialization of $V^{(\mathcal{F})}$, assuming also invariance of Υ_ω, produces a unitary representation of $\alpha^{(\mathcal{F})}$ which leaves Υ_ω invariant and thus it must coincide with $U^{(\mathcal{F})}$. As a consequence we can restrict

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[2]The *folium* of an algebraic state ω is the convex body of the states which are representable by means of either vector or density matrices in the GNS Hilbert space of ω.

our discussion to the one-particle space $\mathcal{H}_\omega$. The fact that the $U^{(\mathcal{F})}$ is strongly continuous with positive self-adjoint generator implies that $V^{(\mathcal{F})}$ is strongly continuous with positive self-adjoint generator H_ω. Notice also that, if $\psi_t(u, \zeta, \overline{\zeta}) := \psi_t(u+t, \zeta, \overline{\zeta})$, by construction, $V_t^{(\mathcal{F})} K_\omega \psi = K_\omega \psi_t$ for every $t \in \mathbb{R}$ and $\psi \in \mathsf{S}$.

Now consider the triple $(K_\omega, \mathcal{H}_\omega, V^{(\mathcal{F})})$ associated with ω (where $K_\omega : \mathsf{S}(\Im^+) \to \mathcal{H}_\omega$ is the function in lemma C.1 and proposition C.2) and the analog for λ, $(K, \mathcal{H}, V^{(\mathcal{F})})$. We want to reduce to use the following remarkable result due to Kay [Ka79].

Lemma 3.1. *Let $\mathcal{W}_{\mathsf{S}, \sigma}$ be a Weyl algebra equipped with a one-parameter group of $*$-automorphisms $\beta = \{\beta_t\}_{t \in \mathbb{R}}$ canonically induced by a one-parameter group of transformations $B = \{B_t\}_{t \in \mathbb{R}}$ of S preserving the symplectic form σ. Suppose that, for $k = 1, 2$, there are triples $(K_k, \mathcal{H}_k, V_k)$ where: $\mathcal{H}_k$ are complex Hilbert spaces, $K_k : \mathsf{S} \to \mathcal{H}_k$ and $V_k = \{V_{k\,t}\}_{t \in \mathbb{R}}$ are strongly continuous one-parameters groups of unitary operators on $\mathcal{H}_k$. Suppose that the following holds as well.*
(a) K_k are $\mathbb{R}$-linear with dense range and $\sigma(\psi, \psi') = -2Im\langle K_k \psi, K_k \psi' \rangle_{\mathcal{H}}$, with $\psi, \psi' \in \mathsf{S}$.
(b) $V_{k\,t} K_k \psi = K_k B_t \psi$ for every $t \in \mathbb{R}$ and $\psi \in \mathsf{S}$.
(c) The self-adjoint generators H_k of V_k have nonnegative spectrum.
(d) $\overline{Ran H_k} = \mathcal{H}_k$.
With these hypotheses there is a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ with $U K_1 = K_2$.

Notice that (2) of proposition C.2 implies that, under the hypotheses of lemma 3.1, the pure quasifree states ω_1 and ω_2, respectively individuated by $(K_1, \mathcal{H}_1)$ and $(K_2, \mathcal{H}_2)$, must coincide. Turning back to the proof of theorem 3.1 , the triples $(K_\omega, \mathcal{H}_\omega, V^{(\mathcal{F})})$ and $(K, \mathcal{H}, V^{(\mathcal{F})})$ satisfies hypotheses (a) by lemma C.1 and (d) in proposition C.2. (b) and (c) hold true by construction/hypotheses for ω and by proposition 2.1 for ω. To conclude the proof of theorem 3.1 it is now sufficient to establish the validity of (d), i.e. that $\overline{Ran H_\omega} = \mathcal{H}_\omega$ and the analog for the generator H of $V^{(\mathcal{F})}$. Since H_ω, H are self-adjoint, it is equivalent to prove that $Ker H_\omega = \{0\}$ and $Ker H = \{0\}$. It is trivially true for the generator H (see the proof of proposition 2.1). Let us prove that $Ker H_\omega = \{0\}$ from cluster property, which is valid for ω due to (a) of proposition 2.3. Dealing with as in (b) in the proof of proposition 2.1 one obtains

$$\omega\left(W(\psi)\,\alpha_t^{(\mathcal{F})}\big(W(\psi')\big)\right) = e^{-\langle K_\omega \psi, K_\omega \psi_t' \rangle} \omega\left(W(\psi)\right) \omega\left(W(\psi')\right) . \tag{26}$$

Since $\omega\left(W(\psi)\right) = e^{-\mu_\omega(\psi, \psi)/2} \neq 0$ for every $\psi \in \mathsf{S}(\Im^+)$, (26) together with cluster property, imply that $e^{-\langle K_\omega \psi, K_\omega \psi_t' \rangle} = 1$ as $t \to +\infty$. In other words for every $\epsilon > 0$ there is $T_\epsilon \in \mathbb{R}$ with

$$\left\langle K_\omega \psi, K_\omega \psi_t' \right\rangle \in \bigcup_{n \in \mathbb{Z}} B_\epsilon(2\pi in) , \quad \text{if } t > T_\epsilon ,$$

where $B_\delta(\zeta) := \{z \in \mathbb{C} \mid |z - \zeta| < \delta\}$. However, the map $(T_\epsilon, +\infty) \ni t \mapsto \left\langle K_\omega \psi, K_\omega \psi_t' \right\rangle = \left\langle K_\omega \psi, e^{-itH_\omega} K_\omega \psi' \right\rangle$ is continuous with connected domain and thus it must have connected range. Hence, if ϵ is small enough, the range is contained in a single ball $B_\epsilon(2\pi in_{\psi, \psi'})$. In turn, it implies

$$\lim_{t \to +\infty} \frac{1}{2\pi i} \left\langle K_\omega \psi, e^{-itH_\omega} K_\omega \psi' \right\rangle = n_{\psi, \psi'} \in \mathbb{Z} , \quad \text{for all } \psi, \psi' \in \mathsf{S}(\Im^+) .$$

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Linearity in ψ implies that $n_{\alpha\psi,\psi'} = \alpha n_{\psi,\psi'} \in \mathbb{Z}$ for every $\alpha \in \mathbb{R}$. Since $n_{\psi,\psi'} \in \mathbb{Z}$, it it is possible only if $n_{\psi,\psi'} = 0$ for all $\psi, \psi' \in \mathsf{S}(\mathfrak{I}^+)$ and hence: $\langle K_\omega\psi, e^{-itH_\omega}K_\omega\psi' \rangle \to 0$ for all $\psi, \psi' \in \mathsf{S}(\mathfrak{I}^+)$ if $t \to +\infty$. The result extends to the whole space $\mathcal{H}_\omega$. Indeed, if $\phi \in \mathcal{H}_\omega$,

$$\left| \langle \phi, e^{-itH_\omega}K_\omega\psi' \rangle \right| \leq \left| \langle K_\omega\psi, e^{-itH_\omega}K_\omega\psi' \rangle \right| + \left| \langle (\phi - K_\omega\psi), e^{-itH_\omega}K_\omega\psi' \rangle \right| ,$$

now, using $||e^{-itH_\omega}|| = 1$,

$$0 \leq \left| \langle \phi, e^{-itH_\omega}K_\omega\psi' \rangle \right| \leq \left| \langle K_\omega\psi, e^{-itH_\omega}K_\omega\psi' \rangle \right| + ||\phi - K_\omega\psi|| ||K_\omega\psi'|| .$$

As a consequence, for every sequence $\{t_n\}$ with $t_n \to +\infty$ as $n \to +\infty$ and for every $\psi \in \mathsf{S}(\mathfrak{I}^+)$,

$$0 \leq \liminf_{n\to+\infty} \left| \langle K_\omega\psi, e^{-it_nH_\omega}K_\omega\psi' \rangle \right| \leq \limsup_{n\to+\infty} \left| \langle K_\omega\psi, e^{-it_nH_\omega}K_\omega\psi' \rangle \right| \leq ||\phi - K_\omega\psi|| ||K_\omega\psi'|| ,$$

As $\overline{RanK_\omega} = \mathcal{H}_\omega$, we can take $K_\omega\psi \to \phi$ in order to conclude that, for every $\phi \in \mathcal{H}_\omega$ and every $\psi' \in \mathsf{S}(\mathfrak{I}^+)$: $\langle \phi, e^{-itH_\omega}K_\omega\psi' \rangle \to 0$ as $t \to +\infty$. Making use of the identity $\langle \phi, e^{-itH_\omega}K_\omega\psi' \rangle = \langle e^{itH_\omega}\phi, K_\omega\psi' \rangle$ and employing the same procedure, the result extends to the right entry of the scalar product too. Summing up, cluster property yields

$$\lim_{t\to+\infty} \langle \phi, e^{-itH_\omega}\phi' \rangle = 0 , \quad \text{for all } \phi, \phi' \in \mathcal{H}_\omega. \tag{27}$$

It is now obvious that, if there were $\phi_0 \in KerH_\omega \backslash \{0\}$ one would find $\langle \phi_0, e^{-itH_\omega}\phi_0 \rangle = \langle \phi_0, \phi_0 \rangle \neq 0$ so that (27) and cluster property, valid by proposition 2.3, would be violated. Therefore $KerH_\omega = \{0\}$.

Finally, we pass to consider the case where $\mathfrak{F}$ in the hypotheses is not a Bondi frame. Let $\mathfrak{F}_0$ be a Bondi frame. There is $g \in G_{BMS}$ such that, for every $t \in \mathbb{R}$, $\alpha_t^{(\mathfrak{F})} = \alpha_g \alpha_t^{(\mathfrak{F}_0)} \alpha_{g^{-1}}$. The state ω' such that $\omega'(a) := \omega(\alpha_g(a))$ is invariant under $\alpha^{(\mathfrak{F}_0)}$ by construction. By direct inspection one sees that the GNS triple of ω' is $(\mathfrak{H}_{\omega'}, \Pi_{\omega'}, \Upsilon_{\omega'}) = (\mathfrak{H}_\omega, \Pi_\omega \circ \alpha_g, \Upsilon_\omega)$. As a consequence, if $\{U_t\}_{t\in\mathbb{R}}$ implements $\alpha^{(\mathfrak{F})}$ for ω leaving Υ_ω invariant, it also implements $\alpha^{(\mathfrak{F}_0)}$ for ω' leaving fixed $\Upsilon_{\omega'} = \Upsilon_\omega$. Since, by hypotheses $\{U_t\}_{t\in\mathbb{R}}$ is strongly continuous with positive generator and $\mathfrak{F}_0$ is a Bondi frame, we can apply the result proved above for Bondi frames obtaining that $\omega' = \lambda$. That is $\omega \circ \alpha_g = \lambda$. Since λ is BMS invariant, we have that $\omega = \lambda \circ \alpha_{g^{-1}} = \lambda$.

(b). Let $(\mathfrak{H}_\omega, \Pi_\omega, \Upsilon_\omega)$ be the GNS triple of a state ω as in the hypotheses. A generic element in the folium of ω is a positive trace-class operator $\rho : \mathfrak{H} \to \mathfrak{H}$ with $tr\rho = 1$ and has spectral decomposition $\rho = \sum_{i\in I} p_i |\Psi_i\rangle\langle\Psi_i|$, where $p_i \geq 0$ and $\sum_i p_i = 1$. If $\rho \neq \lambda$ (i.e. $\rho \neq |\Upsilon\rangle\langle\Upsilon|$) and ρ is $\alpha^{(\mathfrak{F})}$ invariant, the operator $P_0^\perp \rho P_0^\perp / tr(\rho P_0^\perp)$ ($P_0^\perp$ denoting the orthogonal projector normal to Υ_ω) is another well-defined $\alpha^{(\mathfrak{F})}$-invariant state in the folium of ω. Therefore, without loss of generality we assume that each Ψ_i in $\rho = \sum_{i\in I} p_i |\Psi_i\rangle\langle\Psi_i|$ satisfies $\langle \Upsilon_\omega, \Psi_i \rangle = 0$ and we prove that every p_i must vanish whenever ρ is invariant under $\alpha^{(\mathfrak{F})}$. Take $A = \Pi_\omega(a)$ with $a \in \mathcal{W}(\mathfrak{I}^+)$ and let $A_t := \Pi_\omega(\alpha_t^{(\mathfrak{F})}(a))$. Since both ω and ρ are $\alpha^{(\mathfrak{F})}$ invariant, one has:

$$tr\left(\rho |A\Upsilon_\omega\rangle\langle A\Upsilon_\omega| \right) = tr\left(\rho |A_t\Upsilon_\omega\rangle\langle A_t\Upsilon_\omega| \right) = \sum_{i\in I} p_i |\langle \Psi_i, A_t\Upsilon_\omega \rangle|^2 = \lim_{t\to+\infty} \sum_{i\in I} p_i |\langle \Psi_i, A_t\Upsilon_\omega \rangle|^2$$

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$$= \lim_{t \to +\infty} \sum_{i \in I} p_i |\langle \Psi_i, P_0 A_t \Upsilon_\omega \rangle + \langle \Psi_i, [A_t, P_0] \Upsilon_\omega \rangle|^2 = \lim_{t \to +\infty} \sum_{i \in I} p_i |\langle \Psi_i, [A_t, P_0] \Upsilon_\omega \rangle|^2 = 0$$

In the last step we used $\langle \Psi_i, [A_t, P_0] \Upsilon_\omega \rangle \to 0$ as $t \to +\infty$ due to weak asymptotic commutativity of the state ω and we have interchanged the symbols of series and limit using Lebesgue dominated convergence for the measure which counts the points of I, using the t-uniform bound $|p_i |\langle \Psi_i, [A_t, P_0] \Upsilon_\omega \rangle|^2| \leq p_i 2||a||$ and noticing that $\sum_{i \in I} 2||a|| p_i = 2||a|| < +\infty$ by hypotheses. Since $\{A \Upsilon_\omega\}$ is dense in $\mathfrak{H}$ and $0 = tr(\rho |A \Upsilon_\omega\rangle\langle A \Upsilon_\omega|) = \sum_{i \in I} p_i |\langle \Psi_i, A \Upsilon_\omega \rangle|^2$, using a procedure based on Lebesgue's theorem again, one finds that $|\langle \Psi_i, \Psi_i \rangle| = 0$ and thus $\Psi_i = 0$ for every $i \in I$ as wanted. (b) has been proved. $\square$

4 Algebraic interplay bulk - $\Im^+$ in the presence of i^+ and induction of preferred states.

Proposition 1.1 assures that the Weyl algebra of a linear QFT in the bulk is isometrically mapped onto a sub algebra of $\mathcal{W}(\Im^+)$ provided some hypotheses are fulfilled. We know that the hypotheses of proposition 1.1 are fulfilled for Minkowski spacetime or, similarly, for a strongly asymptotically predictable, asymptotically flat spacetime which coincides with Minkowski spacetime after a Cauchy surface arbitrarily far in the future. However the proof of the validity of these hypotheses given in [DMP05] exploited the fact that the *causal (Lichnerowicz') propagator* of the massless Klein-Gordon operator is strictly supported on the surface of the lightcone. It is known that, in general curved spacetimes the support includes a "tail" supported inside the lightcone (this is equivalent to the failure of Huygens principle barring for "plane-wave spacetimes") [Gü88, Fr75]. In the following we show that actually the relevant hypotheses of proposition 1.1 and its thesis hold true also for another class of spacetimes which are flat at null infinity but not necessarily at space infinity and admit future time completion i^+ (Minkowski spacetime is one of them). The existence of such spacetimes in the class of vacuum solutions of Einstein equations was studied by Friedrich [Fri86-88] (actually his approach concerned spacetimes with past time completion i^- but the re-adaptation to our case is immediate). Recasting the definition in [Fri86-88] in a language more useful for our goals, we have:

Definition 4.1. *A time-oriented four-dimensional smooth spacetime (M, g) which solves vacuum Einstein equations is called* **asymptotically flat spacetime with future time infinity** i^+ *if there is a smooth spacetime $(\tilde{M}, \tilde{g})$ with a preferred point i^+, a diffeomorphism $\psi : M \to \psi(M) \subset \tilde{M}$ and a map $\Omega : \psi(M) \to [0, +\infty)$ so that $\tilde{g} = \Omega^2 \psi^* g$ and the following facts hold. (We omit to write explicitly ψ and ψ^* in the following).*
(1) $J^-(i^+)$ *is closed and $M = J^-(i^+) \setminus \partial J^-(i^+)$ (topology and causal sets being referred to $(\tilde{M}, \tilde{g})$). Thus $M = I^-(i^+)$, i^+ is in the future and time-like related with all the points of M and $\partial M = \Im^+ \cup \{i^+\}$ where $\Im^+ := \partial J_-(i^+) \setminus \{i^+\}$ is the* **future null infinity**.
(2) *M is strongly causal.*
(3) *Ω can be extended to a smooth function on $\tilde{M}$.*

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(4) $\Omega\lceil_{\partial J_-(i^+)} = 0$ *but* $d\Omega(x) \neq 0$ *for* $x \in \Im^+$ *and* $d\Omega(i^+) = 0$ *but* $\tilde{\nabla}_\mu \tilde{\nabla}_\nu \Omega(i^+) = -2\tilde{g}_{\mu\nu}(i^+)$.
(5) *For a strictly positive smooth function* ω *defined in a neighborhood of* $\Im^+$ *which satisfies* $\tilde{\nabla}_\mu(\omega^4 n^\mu) = 0$ *on* $\Im^+$, *the integral curves of* $\omega^{-1}n$ *are complete on* $\Im^+ \cup \Im^-$.

Remark 4.1.   **(1)** In [Fri86-88], interchanging i^+ with i^- the spacetimes defined above were called *vacuum spacetime with complete null cone at past infinity.*
(2) As in the case of asymptotic flat spacetime at null and space infinity, the requirement that (M, g) satisfies Einstein vacuum equations can be relaxed to the requirement that it does in a neighborhood of $\Im^+$ as far as one is interested in the geometric structure of $\Im^+$.
(3) The conditions (4) and (5) were stated into a very different, but equivalent, form in [Fri86-88]. In particular the last condition in (3) is the non degenerateness of the Hessian $\tilde{\nabla}_\mu \tilde{\nabla}_\nu \Omega(i^+)$, which implies $\tilde{\nabla}_\mu \tilde{\nabla}_\nu \Omega(i^+) = c g_{\mu\nu}(i^+)$ for some $c < 0$ ($c > 0$ in [Fri86-88] due to the use of signature $+---$.) We fixed the value of the constant c in analogy with def.B.1, since global rescaling of Ω are irrelevant. With our choices the null vector $\tilde{\nabla}^\mu \Omega$ is future directed along $\Im^+$.

In comparison with the analog definition B.1 for asymptotic flat spacetime at null and space infinity we notice that, replacing i^0 with i^+, the property (5)(a) is now automatically fulfilled since $\tilde{g}$, differently from i^0, is now smooth at i^+. Therefore the topology of $\Im^+$ is once again $\mathbb{R} \times \mathbb{S}^2$. Thus dealing with as for the analogous analysis performed in [Wa84] for asymptotically flat spacetime at null and space infinity, the parts in condition (4) and (5) referring to $\Im^+$, together with the fact that (M, g) satisfies vacuum Einstein equations (in a neighborhood of $\Im^+$ at least) assure that $\Im^+$ is a smooth null 3-surface made of the union of complete null geodesics with respect to the metric $\omega\tilde{g}$ and that these geodesics are the integral curves of $\omega^{-1}n$. The gauge transformations (1) and the BMS group have exactly the same meaning as in the case of asymptotically flat spacetime at null and space infinity. One can introduce the preferred gauge ω_B, Bondi frames and admissible frames once again. Therefore BMS-invariant Wayl QFT based on $(\mathsf{S}(\Im^+), \sigma)$ (with the preferred BMS invariant state λ) can be recast as it stands for asymptotically flat spacetime with future time infinity too.
We come to the main result of this section. Let (M, g) be a *globally hyperbolic* asymptotically flat spacetime with future time infinity spacetime, Define $\mathsf{S}_P(M)$ to be the real linear space of real smooth solutions ϕ in M of Klein-Gordon equation (5) which have compact support on Cauchy surfaces in M and define the Cauchy-surface invariant symplectic form defined as

$$\sigma_M(\phi_1, \phi_2) := \int_\Sigma (\phi_1 \partial_{n_\Sigma} \phi_2 - \phi_2 \partial_{n_\Sigma} \phi_1) \, d\mu_\Sigma^{(g)}, \quad \text{for } \phi_1, \phi_2 \in \mathsf{S}_P(M), \tag{28}$$

$\Sigma \subset M$ being a smooth spacelike Cauchy surface with unit, future directed, normal vector n_Σ and measure $\mu_\Sigma^{(g)}$ induced by g. In this context $\mathcal{W}_P(M)$ denotes the Weyl algebra of the quantum field ϕ in the bulk associated with the symplectic space $(\mathsf{S}(M), \sigma_M)$ with Weyl generators $W_M(\phi)$, $\phi \in \mathsf{S}(M)$.

Theorem 4.1.   *Let* (M, g) *be an asymptotically flat spacetime with future time infinity. Suppose that in the associated unphysical spacetime* $(\tilde{M}, \tilde{g})$ *there is a open set* $V \subset \tilde{M}$ *with* $\overline{M} \subset V$ *and* $(V, \tilde{g})$ *globally hyperbolic. Then the following facts hold.*

(a) (M, g) *is globally hyperbolic.*
(b) *the projection map* $\Gamma_M : \mathcal{S}_P(M) \ni \phi \mapsto \left((\omega_B\Omega)^{-1}\phi\right) \upharpoonright_{\mathfrak{S}^+}$ *is well-defined and satisfies*
$\Gamma_M(\mathcal{S}_P(M)) \subset \mathsf{S}(\mathfrak{S}^+)$.
(c) *The symplectic forms are preserved by* Γ_M, *that is, for all* $\phi_1, \phi_2 \in \mathcal{S}_P(M)$,

$$\sigma_M(\phi_1, \phi_2) = \sigma(\Gamma_M\phi_1, \Gamma_M\phi_2) \,. \tag{29}$$

(d) $\mathcal{W}_P(M)$ *can be identified with a sub C^*-algebra of* $\mathcal{W}(\mathfrak{S}^+)$ *by means of a C^*-algebra isomorphism $\imath$ uniquely determined by the requirement*

$$\imath(W_M(\phi)) = W(\Gamma_M\phi) \,, \quad \text{for all } \phi \in \mathcal{S}_P(M) \,. \tag{30}$$

and thus, in particular, the state λ on $\mathcal{W}(\mathfrak{S}^+)$ induces a quasifree state λ_M on the field algebra of the bulk $\mathcal{W}_P(M)$ by means of

$$\lambda_M(a) := \lambda(\imath(a)) \,, \quad \text{for every } a \in \mathcal{W}_P(M) \,. \tag{31}$$

Proof. Without loss of generality we assume $V = \tilde{M}$. We need a preliminary result given by the following lemma.

Lemma 4.1. *Consider a set $K \subset M$. With obvious notations one has $J^-(K; M) = J^-(K; \tilde{M})$ and $J^+(K; M) = J^+(K; \tilde{M}) \cap M$.*

Proof. To prove the identities, notice that every $\tilde{M}$-causal curve completely contained in M is a M-causal curve by construction therefore $J^-(K; M) \subset J^-(K; \tilde{M})$ and $J^+(K; M) \subset J^+(K; \tilde{M}) \cap M$. To go on, in the former case suppose that there is $s \in J^-(K; \tilde{M})$ with $s \notin J^-(K; M)$. s must belong to an (at least continuous) $\tilde{M}$-causal past-directed curve $\gamma : [0, 1] \to \tilde{M}$ from $q \in K$ to s including points not contained in M. Since $\gamma(0) = q \in M$, the point $x = \gamma(t_x)$ such that $t_x = \sup\{t \in [0, 1] \,|\, \gamma(u) \in M \,, \text{for } u \in [0, t]\}$ must belong to $\partial M = J^-(i^+; \tilde{M}) \setminus I^-(i^+; \tilde{M})$. Notice also that, by construction $(q \in M = I^-(i^+; \tilde{M}))$ there must be a past-directed $\tilde{M}$-timelike line γ' from i^+ to q. If $x \in J^-(i^+; \tilde{M}) \setminus I^-(i^+; \tilde{M})$ any (continuous) causal curves from i^+ to x must be a portion of a smooth null a geodesic (Corollary to Theorem 8.1.2 in [Wa84]). In the considered case however, the continuous causal curve obtained by joining γ' and γ up to x is a continuous causal curve and it is not a portion of a null geodesic by construction. We conclude that s cannot exist and $J^-(K; M) = J^-(K; \tilde{M})$. In the latter case, suppose that $s \in M$ satisfies $s \in J^+(K; \tilde{M})$ but $s \notin J^+(K; M)$. There must be at least one past-directed $\tilde{M}$-causal curve from s to $p \in K$ containing points in $\tilde{M} \setminus M$. In particular as before there is a past-directed causal curve γ from $s \in I^-(i^+; \tilde{M})$ to $x \in \partial J^-(i^+; \tilde{M})$ and, in turn, there is a timelike past-directed curve from i^+ to s. By construction, the past-directed causal curve obtained by joining γ' and γ fails to be a null geodesics so that s cannot exists and thus $J^+(K; M) = J^+(K; \tilde{M}) \cap M$. $\square$

Proof of (a). Since $\tilde{M}$ is globally hyperbolic $J^+(p; \tilde{M}) \cap J^-(q; \tilde{M})$ is compact (Theorem 8.3.10 [Wa84]). But one also has by lemma 4.1 $J^+(p; \tilde{M}) \cap J^-(q; \tilde{M}) = J^+(p; \tilde{M}) \cap J^-(q; M) = (J^+(p; \tilde{M}) \cap M) \cap J^-(q; M)) = J^+(p; M) \cap J^-(q; M)$ which, in turn is compact as well. This is

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enough to establish that M is globally hyperbolic, it being strongly causal (see Remark at the end of Cap. 8 of [Wa84]).

Proof of (b). Now we pass to consider causal (Lichnerowicz) propagators $E := D_- - D_+$ [Di96], A and R being respectively the advanced and retarded fundamental solutions associated with Klein-Gordon operator P in a globally hyperbolic spacetime N. $D_\pm : C_0^\infty(N) \to C^\infty(N)$ are uniquely defined by the requirements that (i) they have the indicated domain and range, (ii) for every $f \in C_0^\infty(N)$, one has $P(D_\pm f) = f$ with (iii) $D_+ f, D_- f$ respectively supported in $J^+(supp f)$ and $J^-(supp f)$. Now exploit the fact that (see Appendix D of [Wa84]) in M – where $\Omega > 0$ is smooth – the following identity is fulfilled

$$\Omega^{-3}(-g^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{1}{6}R)\phi = (-\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu + \frac{1}{6}\tilde{R})\Omega^{-1}\phi$$

and that, by lemma 4.1, $J^-(supp f; M) = J^-(supp f; \tilde{M})$ and $J^+(supp f; M) = J^+(supp f; \tilde{M}) \cap M$. In this way one easily gets that, if $f \in C_0^\infty(M)$ and with obvious notation,

$$\Omega(x)^{-1}(Ef)(x) = \tilde{E}(\Omega^{-3}f)(x), \quad \text{for every } x \in M. \tag{32}$$

□

The proof of item (b) is obtained by collecting together the following three lemmata and taking into account that the standard measure of $\mathbb{S}^2$ used in the definition of $\mathcal{S}(\mathfrak{S}^+)$ is finite.

Lemma 4.2. $\Gamma_M\phi$ *is well defined and is a smooth function on $\mathfrak{S}^+$ for every $\phi \in \mathcal{S}_P(M)$.*
Proof. Consider a smooth solution ϕ in M of the equation $P\phi = 0$ (5) with compactly supported Cauchy data, i.e. $\phi \in \mathcal{S}_P(M)$. Then As (M,g) is globally hyperbolic [Wa94], there is $C_0^\infty(M)$ with $\phi = Ef$. Since $\Omega^{-3}f \in C_0^\infty(M) \subset C_0^\infty(\tilde{M})$ we may also consider the solution $\tilde{\phi} := \tilde{E}(\Omega^{-3}f)$ which is smooth and well defined in the whole globally hyperbolic spacetime $(\tilde{M}, \tilde{g})$ and in particular on $\mathfrak{S}^+ \cup \{i^+\}$. Due to (32) one has $\tilde{\phi}(x) = \Omega^{-1}(x)\phi(x)$ if $x \in M$. This proves $\Omega^{-1}\phi$ extends to a smooth function on $\tilde{M}$ and in particular to $\mathfrak{S}^+$. Since ω_B is smooth and strictly positive in a neighborhood of $\mathfrak{S}^+$, the analog holds considering $(\omega_B\Omega)^{-1}\phi$. □

Lemma 4.3. *Referring to a Bondi frame $(u, \zeta, \overline{\zeta})$ on $\mathfrak{S}^+$ and representing $supp(\Gamma_M\phi)$ in those coordinates, if $\phi \in \mathcal{S}_P(M)$ there is $Q_\phi \in \mathbb{R}$ with $supp(\Gamma_M\phi) \subset [Q_\phi, +\infty) \times \mathbb{S}^2$.*
Proof. Consider a Bondi frame $(u, \zeta, \overline{\zeta})$ on $\mathfrak{S}^+$, *with u future oriented*, and ϕ and f as above. $\omega\Gamma_M\phi = (\tilde{E}\Omega^{-3}f) \upharpoonright_{\mathfrak{S}}^+ = (\tilde{D}_-\Omega^{-3}f) \upharpoonright_{\mathfrak{S}}^+ -(\tilde{D}_+\Omega^{-3}f) \upharpoonright_{\mathfrak{S}}^+$ However $(\tilde{D}_-\Omega^{-3}f) \upharpoonright_{\mathfrak{S}}^+ = 0$ because $J^-(supp(\Omega^{-3}f); \tilde{M}) = J^-(supp(\Omega^{-3}f); M) \subset M = I^-(i^+; \tilde{M})$.
Hence $supp(\Gamma_M\phi) = supp(\omega^{-1}\tilde{D}_-\Omega^{-3}f) \cap \mathfrak{S}^+ = supp(\tilde{D}_-f) \cap \mathfrak{S}^+$ (in fact $\Omega^{-3}f$ and f have equal support and $\omega > 0$ on $\mathfrak{S}^+$). Since $supp f$ is compact, there is a Cauchy surface Σ for $\tilde{M}$ in the past of $supp f$ and in the past of i^+ and $supp(\Gamma_M\phi) = supp(\tilde{D}_-f) \cap \mathfrak{S}^+$ by consequence. Since Σ and $\mathfrak{S}^+ \cup \{i^+\} = \partial J^-(i^+; \tilde{M})$ are closed, $S = \partial J^-(i^+; \tilde{M}) \cap \Sigma = \mathfrak{S}^+ \cap \Sigma$ is such. The coordinate function $u : S \to \mathbb{R}$ is smooth and in particular continuous so that it is bounded

below on S by some real Q. The same uniform bound holds for the u coordinate of the points in $supp(\Gamma_M \phi) = supp(\tilde{D}_- f) \cap \Im^+$, since u is future oriented and those points are in the future of S. □

Lemma 4.4. *Referring to a Bondi frame $(u, \zeta, \overline{\zeta})$ on $\Im^+$, if $\phi \in S_P(M)$, for $p = 0, 1$, there is $u_0 \in \mathbb{R}$ sufficiently large and $C_p, M_p > 0$, such that, if $u > u_0$ and for every $(\zeta, \overline{\zeta}) \in \mathbb{S}^2$*

$$\left| \partial_u^p (\Gamma_M \phi)(u, \zeta, \overline{\zeta}) \right| \leq \frac{M_p}{|C_p u - 1|} .$$

Proof. Since $\tilde{M}$ is globally hyperbolic, it is strongly causal. Consider a sufficiently small open neighborhood U of i^+ which is the image of exponential map centered at i^+ and consider $(U, \tilde{g})$ as a spacetime. Strongly causality for $\tilde{M}$ implies that $I^-(i^+; U) = I^-(i^+; \tilde{M}) \cap U$. Therefore $U \cap (\Im^+ \cup \{i^+\}) = U \cap \partial I^-(i^+; \tilde{M}) = \partial^{(U)}(I^-(i^+; \tilde{M}) \cap U) = \partial^{(U)} I^-(i^+; U)$ (the topological boundary $\partial^{(U)}$ being referred to the topology of U). The structure of $\partial^{(U)} I^-(i^+; U)$ is well known (Theorem 8.1.12 in [Wa84]): it is made by the past-directed null geodesics through i^+. Now consider Riemannian normal coordinates centered at i^+: $x \equiv (x^0, \mathbf{x})$ with $\mathbf{x} := (x^1, x^2, x^3)$ and defined in U above. From now on $||\mathbf{x}|| := \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$ and $|x| := \sqrt{(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2}$. In these coordinates $\Im^+ \cup \{i^+\}$ is the conical set $-x^0 = ||\mathbf{x}||$ and any geodesic through i^+ is a straight line $x^\mu(t) = c^\mu t$ for $t \in (-\epsilon, \epsilon)$ and $c^\mu \in \mathbb{R}$ constants. From now on we describe the portion of $\Im^+ \cup \{i^+\}$ in U by means of coordinates $(x^1, x^2, x^3) \in V$ where $V \subset \mathbb{R}^3$ is open and bounded. *We are explicitly assuming that $V \ni (0,0,0)$ corresponding to the tip of the cone i^+ where a conical singularity arises ($\Im^+ \cup \{i^+\}$ is not a submanifold of $\tilde{M}$ whereas $\Im^+$ is).* By direct inspection one sees that:
if $f : \tilde{M} \to \mathbb{R}$ is smooth, its restriction to $\Im^+$ represented in function of $\mathbf{x}$, $f \restriction_{\Im^+} (\mathbf{x}) := f(-||\mathbf{x}||, \mathbf{x})$, and $\partial_{x^i} f \restriction_{\Im^+} (\mathbf{x})$ are smooth and bounded on $V \setminus \{(0,0,0)\}$ for $i = 1, 2, 3$. Boundedness generally fails for higher derivatives due to singularity of $||\mathbf{x}||$ a the origin.
Now consider an integral curve of $\tilde{\nabla}^\mu \Omega$, that is a solution of

$$\frac{dx^\alpha}{d\lambda} = \tilde{g}^{\alpha\beta} \partial_\beta \Omega(x(\lambda)) = -8x^\alpha + O_2^\alpha(x) .$$

where we have used the conditions $d\Omega(i^+) = 0$ and $\partial_\alpha \partial_\beta \Omega(i^+) = -2\tilde{g}_{\alpha\beta}(i^+)$ and the functions O_2^α satisfy $O_2^\alpha(x)/|x| \to 0$ as $|x| \to 0$. As a consequence of standard theorems on dynamical systems, $x = (0,0,0,0)$ is a stable stationary point (the map $x \mapsto |x|^2$ being a Liapunov function for i^+) and thus, for every $\epsilon > 0$ there is $\delta > 0$ such that such that the integral lines satisfy $|x(\lambda)| < \epsilon$ if $|x(0)| < \delta$. Multiplying both members of the differential equation for x^α, summing over α, and dividing for $|x(\lambda)|^2$ the result, one finally gets:

$$\frac{d \ln |x(\lambda)|^2}{d\lambda} = -16 + \frac{O_2(x(\lambda))}{|x(\lambda)|}$$

with $O_2(x)$ enjoying the same behaviour as O_2^α about $x = (0,0,0,0)$. Thus $|O_2(x(\lambda))|/|x(\lambda)|$ can be bounded from above by any arbitrarily small real 2η by taking the above-mentioned $\delta = 2\delta_\eta$

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small enough. With this estimation one gets, taking initial conditions with $|x(0)| < \delta_\eta$:

$$|x(\lambda)| \leq 2\delta_\eta e^{-(8-\eta)\lambda} \tag{33}$$

$\tilde{\nabla}^\mu \Omega$ is a null vector tangent to $\Im^+$ (it can be seen by eq.(36) by multiplying both sides by Ω and considering the limit as $\Omega = 0$ i.e on $\Im^+$). Therefore integral lines with initial condition on $\Im^+$ belong to $\Im^+$ entirely. In this case (33) produces, taking initial conditions with $||\mathbf{x}(0)|| < \delta_\eta$,

$$||\mathbf{x}(\lambda)|| \leq ||\mathbf{x}(0)||e^{-(8-\eta)\lambda} = \delta_\eta e^{-(8-\eta)\lambda} . \tag{34}$$

We come to the main issue. Let us consider a smooth function $\psi : \tilde{M} \to \mathbb{R}$, in particular the solution of Klein-Gordon equation in $(\tilde{M}, \tilde{g})$ (which extend $\Omega^{-1}\phi$, ϕ being an associated solution in (M, g)) considered. We want to evaluate the behaviour of $\omega_B^{-1}\psi\restriction_{\Im}^+$ in a neighborhood of i^+ To this end we consider one of the above integral lines and the function (to be evaluated as $\lambda \to +\infty$)

$$\omega_B(\mathbf{x}(\lambda))^{-1}\psi\restriction_{\Im^+}(\mathbf{x}(\lambda))$$

Barring re-arrangements in the cross section of $\Im^+$, ω_B is defined, along the considered integral lines, by the equation (1.1.18) of [Wa84]

$$\frac{d\omega_B(\mathbf{x}(\lambda))}{d\lambda} = -\frac{1}{2}\frac{\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\Omega}{\Omega}\restriction_{\mathbf{x}(\lambda)}$$

So that we have to study the behaviour at $\lambda \to +\infty$ of

$$\omega_B(\mathbf{x}(\lambda)) = \omega_B(\mathbf{x}(0))e^{-\frac{1}{2}\int_0^\lambda \frac{\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\Omega}{\Omega}\restriction_{\mathbf{x}(\lambda')}d\lambda'} \tag{35}$$

The integrand is only apparently singular ($\Omega = 0$ on $\Im^+$!) and it must be evaluated using vacuum Einstein equations for g $R_{\mu\nu} = 0$ valid at least in a neighborhood of $\Im^+$ and employing the conformal relation between Ricci tensor of g and that of $\tilde{g}$

$$\Omega R_{\mu\nu} = \Omega\tilde{R}_{\mu\nu} + 2\tilde{\nabla}_\mu\tilde{\nabla}_\nu\Omega + \tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}(\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\Omega - 3\Omega^{-1}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}_\beta\Omega) \tag{36}$$

Eq. (11.1.16) of [Wa84]. For $\Omega = 0$ (i.e. on $\Im^+$) one finds

$$\frac{\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\Omega}{\Omega}\restriction_{\Im^+} = \tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\Omega\restriction_{\Im}^+ \tag{37}$$

The right hand side tends to -8 as the argument approach i^+ because of the condition on i^+, $\tilde{\nabla}_\mu\tilde{\nabla}_\nu\Omega(i^+) = -2g_{\mu\nu}(i^+)$. Using this result in (35) and (33) we conclude that, for every $\epsilon > 0$ we can choose a sufficiently small ball B_{δ_ϵ} about $\mathbf{x} = (0,0,0)$ containing all integral curves starting at $t = 0$ inside this ball and such that, on these curves, for $\lambda \geq 0$, $|8 + \frac{\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\Omega}{\Omega}| \leq \epsilon$ so that

$$\omega_B(\mathbf{x}(0))e^{\lambda(4-\epsilon)} \leq \quad \omega(\mathbf{x}(\lambda)) \quad \leq \omega_B(\mathbf{x}(0))e^{\lambda(4+\epsilon)} , \tag{38}$$

$$(4-\epsilon)\omega_B(\mathbf{x}(0))e^{\lambda(4-\epsilon)} \leq \quad \frac{d\omega(\mathbf{x}(\lambda))}{d\lambda} \quad \leq (4+\epsilon)\omega_B(\mathbf{x}(0))e^{\lambda(4+\epsilon)} \tag{39}$$

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Let $\psi : \tilde{M} \to \mathbb{R}$ the solution of Klein-Gordon equation in $(\tilde{M}, \tilde{g})$ (which extend $\Omega^{-1}\phi$, ϕ being an associated solution in (M, g)), if $|\psi \restriction_{\Im}^+ (\mathbf{x})|, |\partial_{x^i} \psi \restriction_{\Im}^+ (\mathbf{x})| \leq M_\psi < +\infty$ in the considered ball B_{δ_ϵ} about $\mathbf{x} = (0, 0, 0)$ (and such a M does exist as discussed at the beginning) and $|\tilde{g}^{\alpha\beta}\partial_\beta\Omega(x(\lambda))| \leq N < +\infty$ in B_δ, (38) and (39) entail, for every $\lambda \geq 0$:

$$\left|\omega(\mathbf{x}(\lambda))^{-1}\psi\restriction_{\Im}^+ (\mathbf{x}(\lambda))\right| \leq \frac{M_\psi}{\omega_B(\mathbf{x}(0))}e^{-\lambda(4+\epsilon)} , \tag{40}$$

$$\left|\frac{d}{d\lambda}\left(\omega(\mathbf{x}(\lambda))^{-1}\psi\restriction_{\Im}^+ (\mathbf{x}(\lambda))\right)\right| \leq \frac{M_\psi(4+\epsilon+3N)}{\omega_B(\mathbf{x}(0))}e^{-\lambda(4-3\epsilon)} . \tag{41}$$

To conclude we extend similar estimations to the case where the parameter of the integral curves is the coordinate u of a Bondi frame. In this case the vector field to integrate is $\omega_B^{-1}(\mathbf{x}(\lambda))\tilde{g}^{\alpha\beta}\partial_\beta\Omega(x(\lambda))$ so that, along every curve $du/d\lambda = \omega_B(\mathbf{x}(\lambda))$. As a consequence, integrating that equation making use of the estimation (38), one has in particular

$$e^{-\lambda(4+\epsilon)} \leq \frac{\omega_B(\mathbf{x}(0))}{(4+\epsilon)u - \omega_B(\mathbf{x}(0))(3+\epsilon)} . \tag{42}$$

As a consequence

$$\left|\frac{\psi\restriction_{\Im}^+ (\mathbf{x}(u))}{\omega(\mathbf{x}(u))}\right| \leq \frac{M_\psi}{(4+\epsilon)u - \omega_B(\mathbf{x}_0)(3+\epsilon)} \tag{43}$$

Moreover

$$\left|\frac{d}{du}\left(\omega(\mathbf{x}(u))^{-1}\psi\restriction_{\Im}^+ (\mathbf{x}(u))\right)\right| = \frac{1}{\omega_B(\mathbf{x}(u))}\left|\frac{d}{d\lambda}\left(\omega(\mathbf{x}(\lambda))^{-1}\psi\restriction_{\Im}^+ (\mathbf{x}(\lambda))\right)\right| \leq \frac{M_\psi(4+\epsilon+3N)}{\omega_B(\mathbf{x}(u))\omega_B(\mathbf{x}_0)}e^{-\lambda(4-3\epsilon)}$$

so that, by (38)

$$\left|\frac{d}{du}\left(\omega(\mathbf{x}(u))^{-1}\psi\restriction_{\Im}^+ (\mathbf{x}(u))\right)\right| \leq \frac{M_\psi(4+\epsilon+3N)}{\omega_B(\mathbf{x}_0)^2}e^{-\lambda(8-4\epsilon)} \leq \frac{M_\psi(4+\epsilon+3N)}{\omega_B(\mathbf{x}_0)^2}e^{-\lambda(4+\epsilon)} .$$

Using (42) one finally achieves

$$\left|\frac{d}{du}\frac{\psi\restriction_{\Im}^+ (\mathbf{x}(u))}{\omega(\mathbf{x}(u))}\right| \leq \frac{M_\psi(4+\epsilon+3N)}{\omega_B(\mathbf{x}_0)[(4+\epsilon)u - \omega_B(\mathbf{x}_0)(3+\epsilon)]} . \tag{44}$$

Consider a ball B_r centered in $\mathbf{x} = (0, 0, 0)$ with radius $r < \delta_\epsilon$, so that all the estimation above are valid for the considered integral curves provided $\mathbf{x}(\lambda = 0) \in \partial B_r$. Using a the Bondi frame $(u, \zeta, \overline{\zeta})$, the coordinates $(\zeta, \overline{\zeta})$ simply parametrize a class of the integral curves $\mathbf{x} = \mathbf{x}(u, \zeta, \overline{\zeta})$. $\mathbf{x}_0(\zeta, \overline{\zeta})$ is the point along the curve which belongs to ∂B_r. In global coordinates $(u, \zeta, \overline{\zeta})$ on $\Im^+$ the sphere ∂B_r is represented as some compact surface with equation $u = b(\zeta, \overline{\zeta})$. For $u \geq b(\zeta, \overline{\zeta})$ the integral line $\mathbf{x} = \mathbf{x}(u, \zeta, \overline{\zeta})$ is completely contained in B_r and thus (43) and (44) are valid. Since ω_B is smooth and strictly positive, it attains minimum $A > 0$ and maximum $B > 0$ on

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the compact smooth manifold B_r. As a consequence, inside B_r, i.e. for $u > B$ and uniformly in $\zeta, \overline{\zeta} \in \mathbb{S}^2$:

$$\left| \frac{\psi\restriction_{\Im}^+ (u, \zeta, \overline{\zeta})}{\omega_B(u, \zeta, \overline{\zeta})} \right| \leq \frac{M_\psi}{(4 + \epsilon)u - B(3 + \epsilon)} \tag{45}$$

$$\left| \frac{\partial}{\partial u} \frac{\psi\restriction_{\Im}^+ (u, \zeta, \overline{\zeta})}{\omega_B(u, \zeta, \overline{\zeta})} \right| \leq \frac{M_\psi(4 + \epsilon + 3N)}{A[(4 + \epsilon)u - B(3 + \epsilon)]} \tag{46}$$

These relations leads immediately to the thesis. $\square$

Collecting together lemmata 4.2, 4.3 and 4.4, one sees immediately that, if $\phi \in \mathcal{S}_P(M)$, $\Gamma_M \phi$ is smooth and belongs to $L^2(\mathbb{R} \times \mathbb{S}^2, du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}))$ together with its u-derivative because they have support included in a set $\{(u, \zeta, \overline{\zeta}) \in \mathbb{R} \times \mathbb{S}^2 \mid u > Q\}$ for some $Q < +\infty$, decay sufficiently fast as $u \to +\infty$ and, finally, $\mathbb{S}^2$ has finite (factor) measure. In other words $\Gamma_M \phi \in \mathsf{S}(\Im^+)$. This ends the proof of (b). $\square$

Proof of (c). Consider $\phi_1, \phi_2 \in \mathcal{S}_P(M)$ and a smooth spacelike Cauchy surface $\Sigma \subset M$. If $K \subset S$ is compact and includes Cauchy data of ϕ_1 and Φ_2, consider an open neighborhood $O \subset \Sigma$ $O \supset K$, $V := \overline{J^+(\Sigma; \tilde{M})} \cap M$(che closure being referred to $\tilde{M}$) includes the support of ϕ_1 and Φ_2 in the region between Σ and $\Im^+ \cup \{i^+\}$. One can arrange in order that the portion of ∂V which does not intesect Σ and $\Im^+ \cup \{i^+\}$ is smooth. Notice that ϕ_i and ϕ_i/Ω vanish smoothly on that portion of the boundary. By direct inspection one finds that, if σ_M is defined as in (11)

$$\sigma_M(\phi_1, \phi_2) = \int_\Sigma (\psi_1 \partial_{\tilde{n}_\Sigma} \psi_2 - \psi_2 \partial_{\tilde{n}_\Sigma} \psi_1) \, d\mu_\Sigma^{(\tilde{g})}$$

where, now, everything is referred to the unphysical metric $\tilde{g} = \Omega^2 g$ and $\psi_i := \Omega^{-1} \phi_i$. These fields are well defined solution of Klein-Gordon equation on $\tilde{M}$ and the right-hand side of the identity above coincides with the integral over Σ of the 3-form locally represented by

$$\chi_{\phi_1, \phi_2} := -\frac{1}{6} \sqrt{|\tilde{g}|} \tilde{g}^{\beta\alpha} (\psi_1 \partial_\beta \psi_2 - \psi_2 \partial_\beta \psi_1) \, \epsilon_{\alpha\beta\mu\nu} dx^\beta \wedge dx^\mu \wedge dx^\nu \, ,$$

($\epsilon_{\alpha\beta\mu\nu}$ is the sign of the permutation $\alpha\beta\mu\nu$ of 1234 or $\epsilon_{\alpha\beta\mu\nu} = 0$ if there are repeated numbers in $\alpha\beta\mu\nu$.) We can use the divergence theorem for the form ω with respect to the region V. As is well-known the fact that ψ_i satisfies Klein-Gordon equation implies immedediatly that the divergence of ω vanishes. Since the boundary terms which are not evaluated on Σ and $\Im^+ \cup \{i^+\}$ do not give contribution, the theorem of divergence reduces to the statement

$$\sigma_M(\phi_1, \phi_2) = \int_{\Im^+} \chi_{\phi_1, \phi_2} \tag{47}$$

we have omitted i^+ since it has negligible measure (as is known an isolated conical singularity at the tip of a cone is too weak to create troubles with integration of smooth forms) and we assume

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orientation on $\Im^+$ compatible with time orientation. It is known [Wa84] that $\Im^+, \Omega, u, \theta, \phi$ form a coordinate system in a full neighborhood of $\Im^+$, θ, ϕ being standard coordinated on $\mathbb{S}^2$ and that coordinate frame reduces to a Bondi frame on $\Im^+$ for $\Omega = 0$ with $\zeta = e^{i\phi} \cot(\theta/2)$. In these coordinates ((11.1.22) in [Wa84], noticing that the metric therein is our $\omega_B^2 \tilde{g}$)

$$\tilde{g}\!\restriction_{\Im}^+ = \frac{-d\Omega \otimes du - du \otimes d\Omega + d\theta \otimes d\theta + \sin^2\theta \, d\phi \otimes d\phi}{\omega_B^2} \, .$$

As coordinates u, θ, ϕ being adapted to $\Im^+$:

$$\int_{\Im^+} \chi_{\phi_1, \phi_2} = - \int_{\mathbb{R} \times \mathbb{S}^2} \sqrt{|\tilde{g}\!\restriction_{\Im^+}|} \ \tilde{g}\!\restriction_{\Im^+}^{\Omega u} \ (\psi_1 \partial_u \psi_2 - \psi_2 \partial_u \psi_1) \, du \wedge d\theta \wedge d\phi \, .$$

Expliciting computations one has

$$\int_{\Im^+} \chi_{\phi_1, \phi_2} = - \int_{\mathbb{R} \times \mathbb{S}^2} \omega_B^{-2} \ (\psi_1 \partial_u \psi_2 - \psi_2 \partial_u \psi_1) \, du \wedge d\theta \wedge d\phi \, .$$

That is, since $\omega_B^{-2} (\psi_1 \partial_u \psi_2 - \psi_2 \partial_u \psi_1) = \omega_B^{-1} \psi_1 \partial_u (\omega_B^{-1} \psi_2) - \omega_B^{-1} \psi_2 \partial_u (\omega_B^{-1} \psi_1)$, and passing to Bondi coordinates,

$$\sigma_M(\phi_1, \phi_2) = \int_{\mathbb{R} \times \mathbb{S}^2} \left[\omega_B^{-1} \psi_1 \partial_u (\omega_B^{-1} \psi_2) - \omega_B^{-1} \psi_2 \partial_u (\omega_B^{-1} \psi_1) \right] du \wedge \frac{d\zeta \wedge d\bar{\zeta}}{i(1 + \zeta\bar{\zeta})^2} = \sigma \left(\frac{\psi_1}{\omega_B}, \frac{\psi_2}{\omega_B} \right) \, .$$

By the very definition of Γ_M, that is just the result we wanted to establish. $\square$

Proof of (d). If $\mathcal{W}(\Im^+)_M$ is the C^*-algebra of $\mathcal{W}(\Im^+)$ generated by generators $W(\Gamma_M \phi)$ for every $\phi \in \mathcal{S}_P(M)$, preservation of symplectic forms by the linear map Γ_M implies immediately (theorem 5.2.8 in [BR022]) that there is a unique (isometric) $*$-algebra isomorphism $\imath$ from $\mathcal{W}_P(M)$ to $\mathcal{W}(\Im^+)$ satisfying (30). The statement concerning the induction of the state λ_M is straightforward. In particular, the fact that the state is quasifree follows immediately from the expression (16) for λ. It implies that λ_M is the quasifree state associated with the scalar product $\mu_M(\phi, \phi') := \mu_\lambda(\Gamma_M \phi, \Gamma_M \phi')$. Preservation of symplectic forms assures that μ_M fulfills (48) with respect to σ_M. $\square$

The proof of the theorem is concluded. $\square$

Remark 4.2. To conclude, we notice that Minkowski spacetime $(\mathbb{M}^4, \eta)$ fulfills both the definitions of asymptotic flat at null and space infinity and asymptotically flat spacetime with future time infinity if $(\tilde{M}, \tilde{g})$ is Einstein closed universe (see [Wa84, DMP05]) in particular $\Im^+$ is the same submanifold of $(\tilde{M}, \tilde{g})$ in both cases. Since Einstein closed universe is globally hyperbolic theorem 4.1 is valid in this case. However the thesis of the theorem is true anyway because of the independent proof given in that case in (a) of Theorem 4.1 [DMP05]. We also know by (b) of Theorem 4.1 in [DMP05] that, in the considered case, the state λ_M induced by λ is nothing but Minkowski vacuum.

5 Discussion and open issues.

A crucial role in proving the uniqueness theorem in played by the fact that the C^* algebra of observables is a Weyl algebra: this fact is essential in obtaining both cluster property for every state which is invariant under u-displacements and asymptotic commutativity, used in establishing the uniqueness theorem. The use of a Weyl algebra to describe quantum observables in standard QFT in a globally hyperbolic spacetime is appropriate as far as the theory deals with linear – i.e. "free" – fields. This is because nonlinear field equations – i.e. the presence of "interaction" – do not preserve the standard symplectic form of field solutions if varying Cauchy surface. However dealing with QFT on $\Im^+$ the extend is different since there is no time evolution – one stays "at the end of time" when interactions of the bulk, if any, have been switched off – and a Weyl algebra may be still appropriate especially if one try to use some "S matrix" formalism (involving QFT on $\Im^-$) in order to describe bulk phenomena in terms of features of QFT on the boundary of the spacetime. We remark that if adopting this point of view the out coming S-matrix theory enjoys a larger symmetry group than the usual Poincaré group provided by BMS group.

Concerning the last statement of theorem 4.1 an important issue deserving further investigation is the validity of Hadamard property [KW91, Ra96] for the state λ_M. In case this property is fulfilled, it make sense to implement a perturbative procedure to stady the back reaction on the metric using the stress-energy tensor operator [Mo03] averaged on λ_M. If Hadamard property fails the gravitational stability of the state is dubious. A first scrutiny seems to shows that, at least near $\Im^+$ the singular support of the two-point function associated to ω_M is included in the set of couple of points connected by means of a null geodesic tis is a first clue for the validity of Hadamard behaviour. Another property of λ_M which is, most probably fulfilled, is its simmetry with respect every proper isometry group of M if any. This is because λ is invariant under BMS group which includes (asymptotic) symmtries. A general open problem, which seems to be quite difficult for several technical reasons, is the extension of the results presented here an in [DMP05] to the case of a massive field. Al these issues will be investigated elsewhere.

A Proof of proposition 2.2

In the following $\mathcal{W}_0$ is the $*$-algebra of finite linear combinations of all $W(\psi)$, $\psi \in \mathsf{S}(\Im^+)$.
(a) First assume that $\mathcal{F}$ is a Bondi frame and the coordinates of that Bondi frame to describe wavefunctions on $\Im^+$. Using Weyl commutation relations one has, if $\psi'_t(u, \zeta, \overline{\zeta}) := \psi'(u + t, \zeta, \overline{\zeta})$

$$||[\alpha_t^{(\mathcal{F})}\left(W(\psi')\right), W(\psi)]|| \leq |\sin \sigma(\psi'_t, \psi)| \, ||W(\psi'_t + \psi)|| = |\sin(2Im\langle K\psi, K\psi'_t\rangle)| \, ||W(\psi'_t + \psi)||,$$

where $K : \mathsf{S}(\Im^+) \to \mathcal{H}$ is that associated with the state λ. The left-hand side vanishes as $t \to +\infty$ because $||W(\phi)|| = 1$ for every $\phi \in \mathsf{S}(\Im^+)$ and moreover, we have seen in the proof of

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proposition 2.1 that $\langle K\psi, K\psi'_t \rangle \to 0$ as $t \to +\infty$. If $\mathcal{F}$ is not a Bondi frame, there is $g \in G_{BMS}$ such that, for every $t \in \mathbb{R}$, $\alpha_t^{\mathcal{F}} = \alpha_g \alpha_t^{\mathcal{F}_0} \alpha_{g^{-1}}$ where $\mathcal{F}_0$ is a Bondi frame. Using the fact that α_g is a isometric $*$-automorphism which transforms a Wayl generators into a Wayl generator and the result above, one gets $||[\alpha_t^{(\mathcal{F})}(W(\psi')), W(\psi)]|| \to 0$ as $t \to +\infty$ again. The result extends to $\mathcal{W}_0$ by linearity. To conclude it is sufficient to extend the result to $\overline{\mathcal{W}_0} = \mathcal{W}(\Im^+)$. For every $\epsilon > 0$ and fixed $a, b \in \mathcal{W}(\Im^+)$ there are $a_\epsilon, b_\epsilon \in \mathcal{W}_0$ with $||a - a_\epsilon|| < \epsilon$, $||b - b_\epsilon|| < \epsilon$. Consider any sequence $t_n \to +\infty$. Since $\alpha^{(\mathcal{F})}$ is isometric $0 \leq \liminf_n |[\alpha_{t_n}^{(\mathcal{F})}(a), b]| \leq \limsup_n |[\alpha_{t_n}^{(\mathcal{F})}(a), b]|$ and

$\limsup_n |[\alpha_{t_n}^{(\mathcal{F})}(a), b]| \leq 2||a - a_\epsilon|| \, ||b_\epsilon|| + 2||a - a_\epsilon|| \, ||b - b_\epsilon|| + 2||a|| \, ||b - b_\epsilon|| + \limsup_n |[\alpha_{t_n}^{(\mathcal{F})}(a_\epsilon), b_\epsilon]|$.
The last term on the right-hand side converges to 0 whereas the remaining terms are arbitrarily small. Therefore $|[\alpha_{t_n}^{(\mathcal{F})}(a), b]| \to 0$ for any sequence $t_n \to +\infty$, i.e. (22) is valid.
(b) For the sake of simplicity we indicate by Π and $\mathfrak{H}$ respectively the GNS representation Π_ω and GNS Hilbert space $\mathfrak{H}_\omega$. Since ω is pure, $\Pi(\mathcal{W}(\Im^+))$ is irreducible. As a consequence, its commutant $\Pi(\mathcal{W}(\Im^+))'$ contains only the elements cI with $c \in \mathbb{C}$. Thus the double commutant $(\Pi(\mathcal{W}(\Im^+))')'$ coincides with $\mathfrak{B}(\mathfrak{H})$. Finally applying double commutant von Neumann's theorem, for that $\overline{\Pi(\mathcal{W}(\Im^+))}^s = (\Pi(\mathcal{W}(\Im^+))')'$, we conclude that $\overline{\Pi(\mathcal{W}(\Im^+))}^s = \mathfrak{B}(\mathfrak{H})$, $\overline{X}^s$ denoting the closure in the strong topology on $\mathfrak{B}(\mathfrak{H})$ of any $X \subset \mathfrak{B}(\mathfrak{H})$. To go on, fix $a \in \mathcal{W}(\Im^+)$, $B \in \mathfrak{B}(\mathfrak{H})$ and take $\Psi_1, \Psi_2 \in \mathfrak{H}$. By $\overline{\Pi(\mathcal{W}(\Im^+))}^s = \mathfrak{B}(\mathfrak{H})$, for each $\epsilon > 0$ there is $b_\epsilon \in \mathcal{W}(\Im^+)$ with $||(B - \Pi(b_\epsilon))\Psi_1|| < \epsilon$ and $||(B - \Pi(b_\epsilon))\Psi_2|| < \epsilon$. With those choices, also using the fact that $||\Pi(\alpha_t^{(\mathcal{F})}(a))|| \leq ||\alpha_t^{(\mathcal{F})}(a)|| = ||a||$, one has

$$\left| \left\langle \Psi_1, \left[\Pi\left(\alpha_t^{(\mathcal{F})}(a)\right), B \right] \Psi_2 \right\rangle \right| \leq \left| \left\langle \Psi_1, \Pi\left(\left[\alpha_t^{(\mathcal{F})}(a), b_\epsilon \right] \right) \Psi_2 \right\rangle \right| + \epsilon ||a|| \left(||\Psi_1|| + ||\Psi_2|| \right)$$

Now, using asymptotic commutativity and continuity of Π, one concludes that the first term on the right-hand side vanishes as $t \to +\infty$. Since $\epsilon > 0$ is arbitrarily small, adapting the procedure, based on standard properties of $\limsup$ and $\liminf$, employed in the proof of 2.2, one obtains that the limit of the left-hand side of the inequality above vanishes as $t \to +\infty$. $\square$

B   Spacetime infinities

From [Wa84] we have the following definition (due to Ashtekar).

Definition B.1.   *A time-oriented four-dimensional smooth spacetime (M, g) satisfying vacuum Einstein equations is called* **vacuum spacetime asymptotically flat at null and spatial infinity**, *if there exists a spacetime $(\tilde{M}, \tilde{g})$ with $\tilde{g}$ smooth everywhere except possibly a point i^0 (called* **spatial infinity**) *where it is $C^{>0}$ (see p.227 of [Wa84]), a diffeomorphism $\psi : M \to \psi(M) \subset \tilde{M}$ and a map $\Omega : \psi(M) \to [0, +\infty)$ so that $\tilde{g} = \Omega^2 \psi^* g$ and the following facts hold. (We omit to write explicitly ψ and ψ^* in the following)*
(1) $\overline{J^+(i^0)} \cup \overline{J^-(i^0)} = \tilde{M} \backslash M$ the closure and causal sets being referred to $(\tilde{M}, \tilde{g})$. Thus i^0 is spacelike related with all the points of M and the boundary ∂M consists of the union of $\{i^0\}$, the **future null infinity** $\Im^+ = (\partial J^+(i^0)) \backslash \{i^0\}$ *and and the* **past null infinity** $\Im^- = (\partial J^-(i^0)) \backslash \{i^0\}$.

(2) *There is a open neighborhood V of ∂M such that $(V, \tilde{g})$ is strongly causal (see [Wa84]).*
(3) Ω *can be extended to a function on $\tilde{M}$ which C^2 at least at i^0 and smooth elsewhere.*
(4) *(a) $\Omega\!\upharpoonright_{\Im^+ \cup \Im^-} = 0$ but $d\Omega(x) \neq 0$ for $x \in \Im^+ \cup \Im^-$. (b) $\Omega(i^0) = 0$ and the limits toward i^0 of $d\Omega$ and $\hat{\nabla}_\mu \hat{\nabla}_\nu \Omega$ are respectively 0 and $2\tilde{g}_{\mu\nu}(i^0)$.*
(5) *(a) The map of null directions at i^0 into the space of integral curves of $n^\mu := \tilde{\nabla}^\mu \Omega$ on $\Im^+$ and $\Im^-$ is a diffeomorphism. (b) For a strictly positive smooth function ω defined in a neighborhood of $\Im^+ \cup \Im^-$ which satisfies $\tilde{\nabla}_\mu(\omega^4 n^\mu) = 0$ on $\Im^+ \cup \Im^-$ the integral curves of $\omega^{-1} n$ are complete on $\Im^+ \cup \Im^-$.*

C   Weyl algebras and Quasifree states.

C.1. *GNS theorem, Weyl algebras, regular states, field operators.*   Quantum theories can be formulated in a more advanced version within the algebraic approach where the fundamental object consist of a C^* algebra $\mathcal{A}$ whose Hermitian elements are interpreted as (bounded) observables characterizing the given physical system. States are introduced as follows. An *(algebraic) state* [KW91, Ha92] ω on a C^*-algebra $\mathcal{A}$ with unit $\mathbb{I}$ is a linear map $\omega : \mathcal{A} \to \mathbb{C}$ which is positive ($\omega(a^*a) \geq 0$ for $a \in \mathcal{A}$) and normalized ($\omega(\mathbb{I}) = 1$). It results that a state on a C^*-algebra is always continuous [BR021]. The celebrated *GNS* theorem [KW91, Ha92] proves that, for a pair $(\mathcal{A}, \omega)$ where, $\mathcal{A}$ is a C^*-algebra with unit, there is a triple $(\mathfrak{H}_\omega, \Pi_\omega, \Upsilon_\omega)$ such that: (i) $\mathfrak{H}_\omega$ is a non zero Hilbert space, (ii) Π_ω is a representation (a $*$ homomorphism) of $\mathcal{A}$ in terms of bounded operators on $\mathfrak{H}$, the involution $*$ corresponding to the Hermitean conjugation †, (ii) $\Upsilon_\omega \in \mathfrak{H}_\omega$ is cyclic (i.e. $\overline{\Pi_\omega(\mathcal{A})\Upsilon_\omega} = \mathfrak{H}_\omega$) and (iv) the expectation values $\omega(a)$, for every $a \in \mathcal{A}$, satisfy $\omega(a) = \langle \Upsilon_\omega, \Pi_\omega(a)\Upsilon_\omega \rangle$. Finally a GNS triple $(\mathfrak{H}_\omega, \Pi_\omega, \Upsilon_\omega)$ for an assigned pair $(\omega, \mathcal{A})$ is unique up to the unitary equivalences which preserve the corresponding cyclic vectors. As further useful results one also has $\Pi_\omega(\mathbb{I}) = I$ (the identity operator on $\mathfrak{H}_\omega$) and $||\Pi_\omega(a)|| \leq ||a||$ for $a \in \mathcal{A}$ [Ha92, BR021]. Therefore, GNS representations are always continuous but, generally speaking, a GNS representation of a C^* algebra is not isometric. It is possible to show that a GNS representation of a C^*-algebra is isometric if and only if it is faithful (i.e. injective). A **pure** (algebraic) state ω is defined as the extremal state in the convex set of the algebraic states. The state which are not pure are said to be **mixed**. It is possible to show [Ha92] that a state ω is extremal if and only if its GNS representation is irreducible.

In QFT (also in curved spacetime) for bosonic fields with linear field equations (see [KW91] for details), elementary observables are encoded by a certain C^* algebra called the *Weyl algebra* associated with the fields. This is built upon a real space of solutions of field equations equipped with a nondegenerate symplectic form dynamically invariant. From a general point of view, consider a generic real vector space S equipped with a nondegenerate symplectic form σ. A C^*-algebra $\mathcal{W}_{(\mathsf{S},\sigma)}$ is called **Weyl algebra** associated with (S, σ) if it contains a class of nonvanishing elements $W(\psi)$ for all $\psi \in \mathsf{S}$, called **Weyl generators**, satisfying **Weyl relations**[3]:

$$(W1) \qquad W(-\psi) = W(\psi)^*, \qquad\qquad (W2) \qquad W(\psi)W(\psi') = e^{i\sigma(\psi,\psi')/2}W(\psi + \psi');$$

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[3]Notice that in [KW91] a different convention for the sign of σ in (W2) is employed.

and $\mathcal{W}_{(S,\sigma)}$ coincides with the closure of the $*$-algebra (finitely) generated by Weyl generators. $\mathcal{W}_{(S,\sigma)}$ is *uniquely* determined by (S,σ) (theorem 5.2.8 in [BR022]): two different realizations admit a unique $*$ isomorphism which transform the former into the latter preserving Weyl generators and the norm on $\mathcal{W}_{(S,\sigma)}$ is unique since $*$ isomorphisms of C^* algebras are isometric. $\mathcal{W}_{(S,\sigma)}$ can always be realized in terms of bounded operators on $\ell^2(S)$, viewing S as a Abelian group and defining the generators as $(W(\psi)F)(\psi') := e^{-i\sigma(\psi,\psi')/2}F(\psi+\psi')$ for every $F \in \ell^2(S)$. In this realization (and thus in every realization) it turns out evident that generators $W(\psi)$ are *linearly independent*. As a consequence of (W1) and (W2), one gets: $W(0) = \mathbb{I}$ (the unit element), $W(\psi)^* = W(\psi)^{-1}$, $||W(\psi)|| = 1$ and, using non degenerateness of σ, $W(\psi) = W(\psi')$ iff $\psi = \psi'$.

Concerning *GNS* representations of a Weyl algebra $\mathcal{W}_{S,\sigma}$ we have that operators $\Pi_\omega(W(\psi))$ verify (W1) and (W2) and cannot vanish because $\Pi_\omega(W(\psi))\Pi_\omega(W(-\psi)) = I$ and so they generate an operatorial realization of $\mathcal{W}_{S,\sigma}$. Such a realization is made of *unitary* operators. By uniqueness of the norm of a Weyl algebra, $||\Pi_\omega(a)|| = ||a||$ for all $a \in \mathcal{W}_{(S,\sigma)}$. We conclude that every GNS representation of a Weyl algebra is always faithful and isometric. Strong continuity of the unitary group implementing a $*$-automorphism representation β of a topological group $G \ni g \mapsto \beta_g$ for a β-invariant state ω on a Weyl algebra $\mathcal{W}(S,\sigma)$, is equivalent to $\lim_{g\to\mathbb{I}} \omega(W(-\psi)\beta_g W(\psi)) = 1$ for all $\psi \in S$. The proof follows immediately from $||\Pi_\omega \left(\beta_{g'} W(\psi)\right) \Upsilon_\omega - \Pi_\omega \left(\beta_g W(\psi)\right) \Upsilon_\omega||^2 = 2 - \omega\left(W(-\psi)\beta_{g'^{-1}g}W(\psi)\right) - \omega\left(W(-\psi)\beta_{g^{-1}g'}W(\psi)\right)$ and $\overline{\Pi_\omega(\mathcal{W}(S,\sigma))\Upsilon_\omega} = \mathfrak{H}_\omega$.

A state ω on $\mathcal{W}_{(S,\sigma)}$, with GNS triple $(\mathfrak{H}_\omega, \Pi_\omega, \Upsilon_\omega)$, is called **regular** if the maps $\mathbb{R} \ni t \mapsto \Pi_\omega(W(t\psi))$ are strongly continuous. Then, in accordance with Stone theorem, one can write $\Pi_\omega(W(\psi)) = e^{i\sigma(\psi,\Psi_\omega)}$, $\sigma(\psi,\Psi_\omega)$ being the (self-adjoint) **field operator symplectically-smeared** with ψ. In this way field operators enters the theory in Weyl algebra scenario. Working formally, by Stone theorem (W2) implies $\mathbb{R}$-linearity and standard CCR:

$$(L) \quad \sigma(a\psi+b\psi',\Psi_\omega) = a\sigma(\psi,\Psi_\omega)+b\sigma(\psi',\Psi_\omega), \quad (CCR) \quad [\sigma(\psi,\Psi_\omega),\sigma(\psi',\Psi_\omega)] = -i\sigma(\psi,\psi')I,$$

for $a,b \in \mathbb{R}$ and $\psi,\psi' \in S$. Actually (L) and (CCR) hold rigorously in an invariant dense set of analytic vectors by Lemma 5.2.12 in [BR022] (it holds if ω is quasifree by proposition C.2).

A useful result on the assignment of states on $\mathcal{W}_{S,\sigma}$ is the following.

Proposition C.1. *Consider a Weyl algebra $\mathcal{W}_{S,\sigma}$ and let $\mathcal{W}_0$ be the finitely-generated linear span of Weyl generators $W(\psi)$, $\psi \in S$. The following holds.*
(a) *Two states ω,ω' on $\mathcal{W}_{S,\sigma}$ coincide if $\omega(W(\psi)) = \omega'(W(\psi))$ for all $\psi \in S$.*
(b) *The complex numbers $\omega(W(\psi)) \in \mathbb{C}$ with $\psi \in S$ determine a state on $\mathcal{W}_{S,\sigma}$ iff $\omega(W(0)) = 1$ and the linear functional on $\mathcal{W}_0$ uniquely associated to the values $\omega(W(\psi))$ is positive.*
(c) *A state ω on $\mathcal{W}_{S,\sigma}$ is regular iff $\mathbb{R} \ni t \mapsto \omega(W(t\psi))$ is continuous at $t = 0$ for all $\psi \in S$.*

Proof. (a) $\mathcal{W}_0$ is closed with respect to both the involution $*$ and the algebra product due to (W1) and (W2) and $\overline{\mathcal{W}_0} = \mathcal{W}_{S,\sigma}$. Since states on C^* algebras are continuous, states with the same values $\omega(W(\psi))$ coincide. (b) Assignation of values $\omega(W(\psi))$ determines a well-defined

linear functional $\omega_0 : \mathcal{W}_0 \to \mathbb{C}$ because generators $W(\psi)$ are linearly independent. Assume that ω_0 is normalized and positive: $\omega_0(a^*a) \geq 0$ for all $a \in \mathcal{W}_0$. If $\mathcal{I}_\omega$ is the *Gelfand (left $*$-) ideal* $\{a \in \mathcal{W}_{\mathsf{S},\sigma} \mid \omega(a^*a) = 0\}$, dealing with as in the proof of GNS theorem, the completion $\mathcal{H}$ of the quotient $\mathcal{H}_0 := \mathcal{W}_{\mathsf{S},\sigma}/\mathcal{I}_\omega$ acquires the structure of a Hilbert space with (strictly positive defined!) scalar product $\langle [a], [b] \rangle := \omega_0(a^*b)$. ($\mathcal{H} \neq \{0\}$ because $\mathbb{I} \notin \mathcal{N}$ since $1 = \omega(\mathbb{I}) = \omega(\mathbb{III}) = \omega(\mathbb{I}^*\mathbb{I})$.) The operators $\Pi(W(\psi)) : \mathcal{H}_0 \to \mathcal{H}_0$ with $\psi \in \mathsf{S}$ and $\Pi(W(\psi))[a] := [W(\psi)a]$ for every $a \in \mathcal{W}_0$ are bounded ($||\Pi(W(\psi))[a]||^2 = \omega_0(a^*W(\psi)^*W(\psi)a) = \omega_0(a^*a) = ||[a]||^2$) and thus uniquely extendible to bounded operators on $\mathcal{H}$ and satisfy Weyl relations (W1) and (W2) where $*$ is the Hermitean conjugation of bounded operators on $\mathcal{H}$. They cannot vanish because $\Pi(W(\psi))\Pi(W(-\psi))[a] = [W(\psi)W(-\psi)a] = I[a]$. Hence they generate an operator realization $\Pi(\mathcal{W}_{\mathsf{S},\sigma})$ of the Weyl algebra associated with (S,σ). Therefore there is a unique isometric $*$-algebra isomorphism $\chi : \Pi(\mathcal{W}_{\mathsf{S},\sigma}) \to \mathcal{W}_{\mathsf{S},\sigma}$ such that $\chi(\Pi(W(\psi))) = W(\psi)$ for every $\psi \in \mathsf{S}$ and thus $\Pi = \chi^{-1}$ is isometric. The identity $\omega_0(W(\psi)) = \omega(\mathbb{I}^*W(\psi)\mathbb{I}) = \langle [\mathbb{I}], \Pi(W(\psi))[\mathbb{I}] \rangle$ yields, by linearity, $\omega_0(a) = \langle [\mathbb{I}], \Pi(a)[\mathbb{I}] \rangle$ for every $a \in \mathcal{W}_0$, which in turn implies continuity of ω_0 it resulting to be a composition of continuous functions. Hence there is a unique, positive by continuity, linear bounded extension of ω_0, $\omega : \overline{\mathcal{W}_0} = \mathcal{W}_{\mathsf{S},\sigma} \to \mathbb{C}$. It defines a state on $\mathcal{W}_{\mathsf{S},\sigma}$ and coincides with the unique linear continuous extension of ω_0. (c) Let $\mathbb{R} \ni t \mapsto U_t$ be a one-parameter group of unitary operators on a Hilbert space H and $D \subset H$ a subset whose (finitely generated) span is dense in H. In these hypotheses U_t is strongly continuous if and only if $\langle \psi, U_t\psi \rangle \to \langle \psi, \psi \rangle$ as $t \to 0$ for every vector $\psi \in D$. In our case: $H = \mathfrak{H}_\omega$, $D := \{\Phi := \Pi_\omega(W(\phi))\Upsilon_\omega \mid \phi \in \mathsf{S}\}$ and $U_t := \Pi_\omega(W(t\psi))$. One has the identity relying on Weyl relations and GNS theorem: $\langle \Phi, \Pi_\omega(W(t\psi))\Phi \rangle = \langle \Upsilon_\omega, \Pi_\omega(W(-\phi))\Pi_\omega(W(t\psi))\Pi_\omega(W(\phi))\Upsilon_\omega \rangle = \omega_0(W(t\psi))e^{it\sigma(\psi,\phi)}$. It is now obvious that $t \mapsto \omega_0(W(t\psi))$ is continuous at $t = 0$ iff $\langle \Phi, U_t\Phi \rangle \to \langle \Phi, \Phi \rangle$ for $t \to 0$. $\square$

C.2. *Quasifree states.* In the standard approach of QFT, based on bosonic real scalar field operators Ψ a, *either vector or density matrix*, state is *quasifree* if the associated n-point functions (expectation values of a product of n fields) satisfy (i) $\langle \sigma(\psi, \Psi) \rangle = 0$ for all $\psi \in \mathsf{S}$ and (ii) the n-point functions $\langle \sigma(\psi_1, \Psi) \cdots \sigma(\psi_n, \Psi) \rangle$ are determined from the functions $\langle \sigma(\psi_i, \Psi)\sigma(\psi_j, \Psi) \rangle$, with $i, j = 1, 2, \cdots, n$, using standard Wick's expansion. A technically different but substantially equivalent definition, completely based on the Weyl algebra was presented in [KW91]. It relies on the following three observations. (a) Working formally with (i) and (ii), one finds that it holds $\langle e^{i\sigma(\psi,\Psi)} \rangle = e^{-\langle \sigma(\psi,\Psi)\sigma(\psi,\Psi) \rangle/2}$. In turn, at least formally, that identity determines the n-point functions (reproducing Wick's rule) by Stone theorem and (W2). (b) From (CCR) it holds $\langle \sigma(\psi, \Psi)\sigma(\psi', \Psi) \rangle = \mu(\psi, \psi') - (i/2)\sigma(\psi, \psi')$, where $\mu(\psi, \psi')$ is the symmetrized two-point function $(1/2)(\langle \sigma(\psi, \Psi)\sigma(\psi', \Psi) \rangle + \langle \sigma(\psi', \Psi)\sigma(\psi, \Psi) \rangle)$ which defines a symmetric positive-semidefined bilinear form on S. (c) $\langle A^\dagger A \rangle \geq 0$ for elements $A := [e^{i\sigma(\psi,\Psi)} - I] + i[e^{i\sigma(\psi,\Psi)} - I]$ entails:

$$|\sigma(\psi, \psi')|^2 \leq 4\,\mu(\psi, \psi)\mu(\psi', \psi')\,, \qquad \text{for every } \psi, \psi' \in \mathsf{S}\,, \tag{48}$$

which, in turn, implies that μ *is strictly positive defined* because σ is nondegenerated. Reversing the procedure, the general definition of quasifree states on Weyl algebras is the following.

Definition C.1. *Let $\mathcal{W}_{\mathsf{S},\sigma}$ be a Weyl algebra and μ a real scalar product on S satisfying (48). A state ω_μ on $\mathcal{W}_{\mathsf{S},\sigma}$ is called* **quasifree state** *associated with μ if*

$$\omega_\mu(W(\psi)) := e^{-\mu(\psi,\psi)/2}, \quad \text{for all } \psi \in \mathsf{S}. \tag{49}$$

Lemma C.1. *Let (S,σ) be a real symplectic space with σ nondegenerated and μ a real scalar product on S satisfying (48). There is a complex Hilbert space $\mathcal{H}_\mu$ and a map $K_\mu : \mathsf{S} \to \mathcal{H}_\mu$ with:*
  (i) K_μ is $\mathbb{R}$-linear with dense complexified range, i.e. $\overline{K_\mu(\mathsf{S}) + iK_\mu(\mathsf{S})} = \mathcal{H}_\mu$,
  (ii) for all $\psi, \psi' \in \mathsf{S}$, $\langle K_\mu\psi, K_\mu\psi' \rangle = \mu(\psi,\psi') - (i/2)\sigma(\psi,\psi')$.
Conversely, if the pair $(\mathcal{H}, K)$ satisfies (i) and $\sigma(\psi,\psi') = -2Im\langle K\psi, K\psi'\rangle_{\mathcal{H}}$, with $\psi, \psi' \in \mathsf{S}$, the unique real scalar product μ on S satisfying (ii) verifies (48).

The last statement arises by Cauchy-Schwarz inequality, the remaining part being in Proposition 3.1 in [KW91]. Notice that K_μ is always injective due to (ii) and non degenerateness of σ. Now existence of quasifree states can be proved using the lemma above with the following proposition. Therein, uniqueness and regularity of the state follows from proposition C.1 in this paper, the remaining part is contained in Lemma A.2 and Proposition 3.1 in [KW91].

Proposition C.2. *For every μ as in definition C.1 the following hold.*
(a) *there is a unique quasifree state ω_μ associated with μ and it is regular.*
(b) *The GNS triple $(\mathfrak{H}_{\omega_\mu}, \Pi_{\omega_\mu}, \Upsilon_{\omega_\mu})$ is determined as follows with respect to $(\mathcal{H}_\mu, K_\mu)$ in lemma (C.1). (i) $\mathfrak{H}_{\omega_\mu}$ is the symmetric Fock space with one-particle space $\mathcal{H}_\mu$. (ii) The cyclic vector Υ_{ω_μ} is the vacuum vector of $\mathfrak{H}_\omega$. (iii) Π_{ω_μ} is determined by $\Pi_{\omega_\mu}(W(\psi)) = e^{\overline{i\sigma(\psi,\Psi)}}$, the bar denoting the closure, where[4]*

$$\sigma(\psi, \Psi) := ia(K_\mu\psi) - ia^\dagger(K_\mu\psi), \quad \text{for all } \psi \in \mathsf{S} \tag{50}$$

$a(\phi)$ and $a^\dagger(\phi)$, $\phi \in \mathcal{H}_\mu$, being the usual annihilation (antilinear in ϕ) and creation operators defined in the dense linear manifold spanned by the states with finite number of particles.
(c) *A pair $(\mathcal{H}, K) \neq (\mathcal{H}_\mu, K_\mu)$ satisfies (i) and (ii) in lemma C.1 for μ, determining the same quasifree state ω_μ, if and only if there is a unitary operator $U : \mathcal{H}_\mu \to \mathcal{H}$ such that $UK_\mu = K$.*
(d) *ω_μ is pure (i.e. its GNS representation is irreducible) if and only if $\overline{K_\mu(\mathsf{S})} = \mathcal{H}_\mu$[5].*

D   Fourier-Plancherel transform on $\mathbb{R} \times \mathbb{S}^2$.

Define $\mathcal{S}(\mathfrak{I}^+; \mathbb{C}) := \mathcal{S}(\mathfrak{I}^+) + i\mathcal{S}(\mathfrak{I}^+)$ (i.e. the complex linear space of the complex-valued smooth functions $\psi : \mathfrak{I}^+ \to \mathbb{R}$ such that, in that Bondi frame, ψ with all derivatives vanish as $|u| \to +\infty$,

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[4]The field operator $\Phi(f)$, with f in the complex Hilbert space $\mathfrak{h}$, used in [BR022] in propositions 5.2.3 and 5.2.4 is related to $\sigma(\psi,\Psi)$ by means of $\sigma(\psi,\Psi) = \sqrt{2}\Phi(iK_\mu\psi)$ assuming $\mathcal{H} := \mathfrak{h}$.

[5]In turn this is equivalent (see p.77 in [KW91]) to $4\mu(\psi',\psi') = \sup_{\psi \in \mathsf{S}\setminus\{0\}} |\sigma(\psi,\psi')|/\mu(\psi,\psi)$ for every $\psi' \in \mathsf{S}$.

uniformly in $\zeta, \overline{\zeta}$, faster than $|u|^{-k}$, $\forall k \in \mathbb{N}$). The space $\mathcal{S}(\mathfrak{I}^+; \mathbb{C})$ generalizes straightforwardly Schwartz' space on $\mathbb{R}^n$. It can be equipped with the Hausdorff topology induced from the countable class seminorms – *they depend on the Bondi frame but the topology does not* – $p, q, m, n \in \mathbb{N}$,

$$||\psi||_{p,q,m,n} := \sup_{(u,\zeta,\overline{\zeta}) \in \mathfrak{I}^+} \left| |u|^p \partial_u^q \partial_\zeta^m \partial_{\overline{\zeta}}^n \psi(u, \zeta, \overline{\zeta}) \right| .$$

$\mathcal{S}(\mathfrak{I}^+; \mathbb{C})$ is dense in both $L^1(\mathbb{R} \times \mathbb{S}^2, du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}))$ and $L^2(\mathbb{R} \times \mathbb{S}^2, du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}))$ (with the topology of these spaces which are weaker than that of $\mathcal{S}(\mathfrak{I}^+; \mathbb{C})$), because it includes the dense space $C_c^\infty(\mathbb{R} \times \mathbb{S}^2; \mathbb{C})$ of smooth compactly supported complex-valued functions. We also define the space of *distributions* $\mathcal{S}'(\mathfrak{I}^+; \mathbb{C})$ containing all the linear functionals from $\mathbb{R} \times \mathbb{S}^2$ to $\mathbb{C}$ which are weakly continuous with respect to the topology of $\mathcal{S}(\mathfrak{I}^+; \mathbb{C})$. Obviously $\mathcal{S}(\mathfrak{I}^+; \mathbb{C}) \subset \mathcal{S}'(\mathfrak{I}^+; \mathbb{C})$ and $L^p(\mathbb{R} \times \mathbb{S}^2, du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta})) \subset \mathcal{S}'(\mathfrak{I}^+; \mathbb{C})$ for $p = 1, 2$. We introduce the Fourier transforms $\mathcal{F}_\pm(\psi)$ of $f \in \mathcal{S}(\mathfrak{I}^+; \mathbb{C})$

$$\mathcal{F}_\pm(\psi)(k, \zeta, \overline{\zeta}) := \int_{\mathbb{R} \times \mathbb{S}^2} \frac{e^{\pm iku}}{\sqrt{2\pi}} f(u, \zeta, \overline{\zeta}) du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}) , \quad (k, \zeta, \overline{\zeta}) \in \mathbb{R} \times \mathbb{S}^2 .$$

$\mathcal{F}_\pm$ the properties listed in the theorem below whose proof is a straightforward extensions of the analog for standard Fourier transform in $\mathbb{R}^n$ (theorems IX.1, IX.2, IX.6, IX.7 in [1]). In (4) $C_\infty(\mathfrak{I}^+)$ denotes the Banch space, with respect to the supremum norm $||\cdot||_\infty$, of the continuous complex valued functions on $\mathbb{R} \times \mathbb{S}^2$ *vanishing at infinity*, i.e. $f \in C_\infty(\mathbb{R} \times \mathbb{S}^2)$ iff f is continuous and, for every $\epsilon > 0$ there is a compact set $K_\epsilon \subset \mathbb{R} \times \mathbb{S}^2$ with $|f(x)| < \epsilon$ for $x \notin K_\epsilon$.

Theorem D.1. *The maps $\mathcal{F}_\pm$ satisfy the following properties.*
(1) *for all $p, m, n \in \mathbb{N}$ and every $\psi \in \mathcal{S}(\mathfrak{I}^+; \mathbb{C})$ it holds*

$$\mathcal{F}_\pm \left(\partial_u^p \partial_\zeta^m \partial_{\overline{\zeta}}^n \psi \right)(k, \zeta, \overline{\zeta}) = (\pm i)^p k^p \partial_\zeta^m \partial_{\overline{\zeta}}^n \psi \mathcal{F}_\pm(\psi)(k, \zeta, \overline{\zeta}) .$$

(2) $\mathcal{F}_\pm$ *are continuous bijections onto $\mathcal{S}(\mathfrak{I}^+; \mathbb{C})$ and $\mathcal{F}_- = (\mathcal{F}_+)^{-1}$.*
(3) *If $\psi, \phi \in \mathcal{S}(\mathfrak{I}^+; \mathbb{C})$ one has*

$$\int_\mathbb{R} \overline{\mathcal{F}_\pm(\psi)}(k, \zeta, \overline{\zeta}) \mathcal{F}_\pm(\phi)(k, \zeta, \overline{\zeta}) dk = \int_\mathbb{R} \overline{\psi}(u, \zeta, \overline{\zeta}) \phi(u, \zeta, \overline{\zeta}) du , \textit{ for all } (\zeta, \overline{\zeta}) \in \mathbb{S}^2 ,$$

$$\int_{\mathbb{R} \times \mathbb{S}^2} \overline{\mathcal{F}_\pm(\psi)(k, \zeta, \overline{\zeta})} \mathcal{F}_\pm(\phi)(k, \zeta, \overline{\zeta}) dk \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}) = \int_{\mathbb{R} \times \mathbb{S}^2} \overline{\psi(u, \zeta, \overline{\zeta})} \phi(u, \zeta, \overline{\zeta}) du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}) .$$

(4) *If $T \in \mathcal{S}'(\mathfrak{I}^+; \mathbb{C})$ the definition $\mathcal{F}_\pm T(f) := T(\mathcal{F}_\pm(f))$, for all $f \in \mathcal{S}'(\mathfrak{I}^+; \mathbb{C})$ is well-posed, gives rise to the unique weakly continuous linear extension of $\mathcal{F}_\pm$ to $\mathcal{S}'(\mathfrak{I}^+; \mathbb{C})$ and one has, with the usual definition of derivative of a distribution,*

$$\mathcal{F}_\pm \left(\partial_u^p \partial_\zeta^m \partial_{\overline{\zeta}}^n T \right) = (\pm i)^p k^p \partial_\zeta^m \partial_{\overline{\zeta}}^n \mathcal{F}_\pm(T) , \quad \textit{for all } p, m, n \in \mathbb{N} .$$

(5) Plancherel theorem. *From (3) and reminding that $\mathcal{S}(\mathfrak{I}^+; \mathbb{C})$ is dense in the Hilbert space $L^2(\mathbb{R} \times \mathbb{S}^2, du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}))$, $\mathcal{F}_\pm$ extend uniquely to unitary transformations from the Hilbert space*

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$L^2(\mathbb{R} \times \mathbb{S}^2, du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}))$ to $L^2(\mathbb{R} \times \mathbb{S}^2, du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}))$ and the extension of $\mathcal{F}_-$ is the inverse of that of $\mathcal{F}_+$.

These extensions coincide respectively with the restrictions to $L^2(\mathbb{R} \times \mathbb{S}^2, du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}))$ of the action of $\mathcal{F}_\pm$ on distributions as in (4) when reminding that $L^2(\mathbb{R} \times \mathbb{S}^2, du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta})) \subset \mathcal{S}'(\Im^+; \mathbb{C})$.
(6) Riemann-Lebesgue lemma. *Reminding that* $\mathcal{S}(\Im^+; \mathbb{C})$ *is dense in* $L^1(\mathbb{R} \times \mathbb{S}^2, du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}))$, $\mathcal{F}_\pm$ *uniquely extend to a bounded operator from* $L^1(\mathbb{R} \times \mathbb{S}^2, du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}))$ *to* $C_\infty(\mathbb{R} \times \mathbb{S}^2)$. *In particular one has, for* $f \in L^1(\mathbb{R} \times \mathbb{S}^2, du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}))$

$$||\mathcal{F}_\pm(f)||_\infty \leq (2\pi)^{-1/2} ||f||_1$$

These extensions coincide respectively with the restrictions to $L^1(\mathbb{R} \times \mathbb{S}^2, du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}))$ *of the action of* $\mathcal{F}_\pm$ *on distributions as in (4) when reminding that* $L^1(\mathbb{R} \times \mathbb{S}^2, du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta})) \subset \mathcal{S}'(\Im^+; \mathbb{C})$.

From now on $\mathcal{F} : \mathcal{S}'(\Im^+; \mathbb{C}) \to \mathcal{S}'(\Im^+; \mathbb{C})$ denotes the extension to distributions of $\mathcal{F}_+$ as stated in (4) in theorem D.1 whose inverse, $\mathcal{F}^{-1}$, is the analogous extension of $\mathcal{F}_-$. We call $\mathcal{F}$ **Fourier-Plancherel transformation**, also if, properly speaking this name should be reserved to its restriction to $L^2(\mathbb{R} \times \mathbb{S}^2, du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}))$ defined in (5) in theorem D.1. We also use the formal distributional notation for $\mathcal{F}$ (and the analog for $\mathcal{F}^{-1}$)

$$\mathcal{F}(\psi)(k, \zeta, \overline{\zeta}) := \int_{\mathbb{R} \times \mathbb{S}^2} \frac{e^{iku}}{\sqrt{2\pi}} f(u, \zeta, \overline{\zeta}) du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}) \,,$$

regardless if f is a function or a distribution. We have the following final proposition whose proof is immediate from (4) and (5) in theorem D.1.

Proposition D.1. *Let* $m \in \mathbb{N}$. *The Fourier-Plancherel transform* $\mathcal{F}(T)$ *of a distribution* $T \in \mathcal{S}'(\Im^+; \mathbb{C})$ *is a measurable function satisfying*

$$\int_{\mathbb{R} \times \mathbb{S}^2} (1 + |k|^2)^m |\mathcal{F}(T)|^2 dk \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}) < +\infty$$

if and only if the u-*derivatives of* T *in the sense of distributions, are measurable functions and satisfy*

$$\partial_u^n T \in L^2(\mathbb{R} \times \mathbb{S}^2, du \wedge \epsilon_{\mathbb{S}^2}) \,, \quad \text{for } \mathbb{N} \ni n \leq m.$$

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