HOLOMORPHIC FUNCTIONS AND REGULAR QUATERNIONIC
FUNCTIONS ON THE HYPERKÄHLER SPACE $\mathbb{H}$

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Let $\mathbb{H}$ be the space of quaternions, with its standard hypercomplex structure. Let $\mathcal{R}(\Omega)$ be the module of $\psi$-regular functions on $\Omega$. For every $p \in \mathbb{H}$, $p^2 = -1$, $\mathcal{R}(\Omega)$ contains the space of holomorphic functions w.r.t. the complex structure $J_p$ induced by $p$. We prove the existence, on any bounded domain $\Omega$, of $\psi$-regular functions that are not $J_p$-holomorphic for any $p$. Our starting point is a result of Chen and Li concerning maps between hyperkähler manifolds, where a similar result is obtained for a less restricted class of quaternionic maps. We give a criterion, based on the energy-minimizing property of holomorphic maps, that distinguishes $J_p$-holomorphic functions among $\psi$-regular functions.

1. Introduction

Let $\mathbb{H}$ be the space of quaternions, with its standard hypercomplex structure given by the complex structures $J_1, J_2$ on $T^\mathbb{H} \simeq \mathbb{H}$ defined by left multiplication by $i$ and $j$. Let $J^*_1, J^*_2$ be the dual structures on $T^*\mathbb{H}$.

We consider the module $\mathcal{R}(\Omega) = \{f = f_1 + f_2 j \mid \overline{\partial} f_1 = J^*_2(\partial \overline{f}_2) \text{ on } \Omega\}$ of left $\psi$-regular functions on $\Omega$. These functions are in a simple correspondence with Fueter left regular functions, since they can be obtained from them by means of a real coordinate reflection in $\mathbb{H}$. They have been studied by many authors (see for instance Sudbery\textsuperscript{7}, Shapiro and Vasilevski\textsuperscript{5} and Nono\textsuperscript{4}). The space $\mathcal{R}(\Omega)$ contains the identity mapping and any holomorphic mapping $(f_1, f_2)$ on $\Omega$ defines a $\psi$-regular function $f = f_1 + f_2 j$. This is no more true if we replace the class of $\psi$-regular functions with that of regular functions. The definition of $\psi$-regularity is also equivalent to that of $q$-holomorphicity given by Joyce\textsuperscript{2} in the setting of hypercomplex manifolds.

For every $p \in \mathbb{H}$, $p^2 = -1$, $\mathcal{R}(\Omega)$ contains the space $\text{Hol}_p(\Omega, \mathbb{H}) = \{f : \Omega \to \mathbb{H} \mid df + p J_p(df) = 0 \text{ on } \Omega\}$ of holomorphic functions w.r.t. the complex structure

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\[ J_p = p_1 J_1 + p_2 J_2 + p_3 J_3 \] on \( \Omega \) and to the structure induced on \( \mathbb{H} \) by left-multiplication by \( p \) (\( J_p \)-holomorphic functions on \( \Omega \)).

We show that on every domain \( \Omega \) there exist \( \psi \)-regular functions that are not \( J_p \)-holomorphic for any \( p \). A similar result was obtained by Chen and Li\(^1\) for the larger class of \( q \)-maps between hyperkähler manifolds.

This result is a consequence of a criterion (cf. Theorem 4.1) of \( J_p \)-holomorphicity, which is obtained using the energy-minimizing property of \( \psi \)-regular functions (cf. Proposition 4.1) and ideas of Lichnerowicz\(^3\) and Chen and Li\(^1\).

In Sec. 4.4 we give some other applications of the criterion. In particular, we show that if \( \Omega \) is connected, then the intersection \( \text{Hol}_p(\Omega, \mathbb{H}) \cap \text{Hol}_{p'}(\Omega, \mathbb{H}) \) \( (p \neq \pm p') \) contains only affine maps. This result is in accord with what was proved by Sommese\(^6\) about quaternionic maps (cf. Sec. 3.2 for definitions).

2. Fueter-regular and \( \psi \)-regular functions

2.1. Notations and definitions

We identify the space \( \mathbb{C}^2 \) with the set \( \mathbb{H} \) of quaternions by means of the mapping that associates the pair \((z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)\) with the quaternion \( q = z_1 + z_2j = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H} \). Let \( \Omega \) be a bounded domain in \( \mathbb{H} \simeq \mathbb{C}^2 \). A quaternionic function \( f = f_1 + f_2j \in C^1(\Omega) \) is \((left) regular\) on \( \Omega \) (in the sense of Fueter) if

\[ \mathcal{D}f = \frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1} + j \frac{\partial f}{\partial x_2} + k \frac{\partial f}{\partial x_3} = 0 \text{ on } \Omega \]

Given the “structural vector” \( \psi = (1, i, j, -k) \), \( f \) is called \((left) \psi\)-regular on \( \Omega \) if

\[ \mathcal{D}'f = \frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1} + j \frac{\partial f}{\partial x_2} - k \frac{\partial f}{\partial x_3} = 0 \text{ on } \Omega \]

We recall some properties of regular functions, for which we refer to the papers of Sudbery\(^7\), Shapiro and Vasilevski\(^5\) and Nōno\(^4\):

1. \( f \) is \( \psi \)-regular \( \Leftrightarrow \) \( \frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2} \), \( \frac{\partial f_1}{\partial \bar{z}_1} = -\frac{\partial f_2}{\partial \bar{z}_2} \)
2. Every holomorphic map \((f_1, f_2)\) on \( \Omega \) defines a \( \psi \)-regular function \( f = f_1 + f_2j \).
3. The complex components are both holomorphic or both non-holomorphic.
4. Every regular or \( \psi \)-regular function is harmonic.
5. If \( \Omega \) is pseudoconvex, every complex harmonic function is the complex component of a \( \psi \)-regular function on \( \Omega \).

\[ *\partial f_1 = -\frac{1}{2} \partial(\overline{f_2} d\bar{z}_1 \wedge dz_2) \]

6. The space \( \mathcal{R}(\Omega) \) of \( \psi \)-regular functions on \( \Omega \) is a right \( \mathbb{H} \)-module with integral representation formulas.
2.2. q-holomorphic functions

A definition equivalent to ψ-regularity has been given by Joyce in the setting of hypercomplex manifolds. Joyce introduced the module of \( q \)-holomorphic functions on a hypercomplex manifold. On this module he defined a (commutative) product.

A hypercomplex structure on the manifold \( \mathbb{H} \) is given by the complex structures \( J_1, J_2 \) on \( \mathbb{H} \simeq \mathbb{H} \) defined by left multiplication by \( i \) and \( j \). Let \( J_3^* \) be the dual structures on \( \mathbb{T} \mathbb{H} \). In complex coordinates

\[
\begin{align*}
J_1^* dz_1 &= idz_1, & J_1^* dz_2 &= idz_2 \\
J_2^* dz_1 &= -d\bar{z}_2, & J_2^* dz_2 &= d\bar{z}_1 \\
J_3^* dz_1 &= id\bar{z}_2, & J_3^* dz_2 &= -id\bar{z}_1
\end{align*}
\]

where we make the choice \( J_3^* = J_1^* J_2^* \Rightarrow J_3 = -J_1 J_2 \).

A function \( f \) is ψ-regular if and only if \( f \) is \( q \)-holomorphic, i.e.

\[
df + iJ_1^* (df) + jJ_2^* (df) + kJ_3^* (df) = 0.
\]

In complex components \( f = f_1 + f_2 j \), we can rewrite the equations of ψ-regularity as

\[
\bar{\partial} f_1 = J_2^* (\bar{\partial} f_2).
\]

3. Holomorphic maps

3.1. Holomorphic functions w.r.t. a complex structure \( J_p \)

Let \( J_p = p_1 J_1 + p_2 J_2 + p_3 J_3 \) be the complex structure on \( \mathbb{H} \) defined by a unit imaginary quaternion \( p = p_1 i + p_2 j + p_3 k \) in the sphere \( S^2 = \{ p \in \mathbb{H} \mid p^2 = -1 \} \). It is well-known that every complex structure compatible with the standard hyperkähler structure of \( \mathbb{H} \) is of this form. If \( f = f^0 + if^1 : \Omega \to \mathbb{C} \) is a \( J_p \)-holomorphic function, i.e. \( df^0 = J_p^* (df^1) \) or, equivalently, \( df + iJ_1^*(df) = 0 \), then \( f \) defines a ψ-regular function \( \tilde{f} = f^0 + pf^1 \) on \( \Omega \). We can identify \( \tilde{f} \) with a holomorphic function

\[
\tilde{f} : (\Omega, J_p) \to (\mathbb{C}_p, L_p)
\]

where \( \mathbb{C}_p = \langle 1, p \rangle \) is a copy of \( \mathbb{C} \) in \( \mathbb{H} \) and \( L_p \) is the complex structure defined on \( \mathbb{T} \mathbb{C}_p \simeq \mathbb{C}_p \) by left multiplication by \( p \).

More generally, we can consider the space of holomorphic maps from \( (\Omega, J_p) \) to \( (\mathbb{H}, L_p) \)

\[
Hol_p(\Omega, \mathbb{H}) = \{ f : \Omega \to \mathbb{H} \mid \bar{\partial}_p f = 0 \text{ on } \Omega \} = Ker \bar{\partial}_p
\]

(the \( J_p \)-holomorphic maps on \( \Omega \)) where \( \bar{\partial}_p \) is the Cauchy-Riemann operator w.r.t. the structure \( J_p \)

\[
\bar{\partial}_p = \frac{1}{2} (d + p J_p^* \circ d).
\]
For any positive orthonormal basis \( \{1, p, q, pq\} \) of \( \mathbb{H} \) \((p, q \in S^2)\), the equations of \( \psi \)-regularity can be rewritten in complex form as
\[
\overline{\partial}_p f_1 = J_p^* (\partial_p J_2)
\]
where \( f = (f^0 + pf^1) + (f^2 + pf^3)q = f_1 + f_2q \). Then every \( f \in Hol_p(\Omega, \mathbb{H}) \) is a \( \psi \)-regular function on \( \Omega \).

**Remark 3.1.**
1) The identity map is in \( Hol_i(\Omega, \mathbb{H}) \cap Hol_j(\Omega, \mathbb{H}) \) but not in \( Hol_k(\Omega, \mathbb{H}) \).
2) \( Hol_{-p}(\Omega, \mathbb{H}) = Hol_p(\Omega, \mathbb{H}) \)
3) If \( f \in Hol_p(\Omega, \mathbb{H}) \cap Hol_{p'}(\Omega, \mathbb{H}) \), with \( p \neq \pm p' \), then \( f \in Hol_{p''}(\Omega, \mathbb{H}) \) for every \( p'' = \frac{ap + bp'}{\|ap + bp'\|} \).
4) \( \psi \)-regularity distinguishes between holomorphic and anti-holomorphic maps: if \( f \) is an anti-holomorphic map from \((\Omega, J_p)\) to \((\mathbb{H}, L_p)\), then \( f \) can be \( \psi \)-regular or not. For example, \( f = \bar{z}_1 + \bar{z}_2j \in Hol_j(\Omega, \mathbb{H}) \cap Hol_k(\Omega, \mathbb{H}) \) is a \( \psi \)-regular function induced by the anti-holomorphic map \((\bar{z}_1, \bar{z}_2) : (\Omega, J_1) \rightarrow (\mathbb{H}, L_i)\) while \((\bar{z}_1, 0) : (\Omega, J_1) \rightarrow (\mathbb{H}, L_i)\) induces the function \( g = \bar{z}_1 \notin R(\Omega) \).

### 3.2. Quaternionic maps

A particular class of \( J_p \)-holomorphic maps is constituted by the quaternionic maps on the quaternionic manifold \( \Omega \). Sommese\(^6\) defined quaternionic maps between hypercomplex manifolds: a quaternionic map is a map
\[
f : (X, J_1, J_2) \rightarrow (Y, K_1, K_2)
\]
that is holomorphic from \((X, J_1)\) to \((Y, K_1)\) and from \((X, J_2)\) to \((Y, K_2)\).

In particular, a quaternionic map
\[
f : (\Omega, J_1, J_2) \rightarrow (\mathbb{H}, J_1, J_2)
\]
is an element of \( Hol_i(\Omega, \mathbb{H}) \cap Hol_j(\Omega, \mathbb{H}) \) and then a \( \psi \)-regular function on \( \Omega \). Sommese showed that quaternionic maps are affine. They appear for example as transition functions for 4-dimensional quaternionic manifolds.

### 4. Non-holomorphic \( \psi \)-regular maps

A natural question can now be raised: can \( \psi \)-regular maps always be made holomorphic by rotating the complex structure or do they constitute a new class of harmonic maps? In other words, does the space \( R(\Omega) \) contain the union \( \bigcup_{p \in S^2} Hol_p(\Omega, \mathbb{H}) \) properly?

Chen and Li\(^1\) posed and answered the analogous question for the larger class of \( q \)-maps between hyperkähler manifolds. In their definition, the complex structures of the source and target manifold can rotate independently. This implies that also anti-holomorphic maps are \( q \)-maps.
4.1. Energy and regularity

The energy (w.r.t. the euclidean metric $g$) of a map $f : \Omega \to \mathbb{C}^2 \simeq \mathbb{H}$, of class $C^1(\Omega)$, is the integral

$$E(f) = \frac{1}{2} \int_{\Omega} \|df\|^2 dV = \frac{1}{2} \int_{\Omega} (g, f^*g) dV = \frac{1}{2} \int_{\Omega} \text{tr}(JC(f)JC(f)^T) dV$$

where $JC(f)$ is the Jacobian matrix of $f$ with respect to the coordinates $\bar{z}_1, z_1, \bar{z}_2, z_2$.

Lichnerowicz proved that holomorphic maps between Kähler manifolds minimize the energy functional in their homotopy classes. Holomorphic maps $f$ smooth on $\partial \Omega$ minimize energy in the homotopy class constituted by maps $u$ with $u|_{\partial \Omega} = f|_{\partial \Omega}$ which are homotopic to $f$ relative to $\partial \Omega$.

From the theorem, functions $f \in Hol_p(\Omega, \mathbb{H})$ minimize the energy functional in their homotopy classes (relative to $\partial \Omega$). More generally:

**Proposition 4.1.** If $f$ is $\psi$-regular on $\Omega$, then it minimizes energy in its homotopy class (relative to $\partial \Omega$).

**Proof.** We repeat arguments of Lichnerowicz, Chen and Li. Let $i_1 = i, i_2 = j, i_3 = k$ and let

$$K(f) = \int_{\Omega} \sum_{\alpha=1}^{3} \langle J_{i_\alpha}, f^*L_{i_\alpha} \rangle dV; \quad I(f) = \frac{1}{2} \int_{\Omega} \|df + \sum_{\alpha=1}^{3} L_{i_\alpha} \circ df \circ J_{i_\alpha}\|^2 dV.$$  

Then $K(f)$ is a homotopy invariant of $f$ and $I(f) = 0$ if and only if $f \in R(\Omega)$. A computation similar to that made by Chen and Li gives

$$E(f) + K(f) = \frac{1}{4} I(f) \geq 0.$$  

From this the result follows immediately.

4.2. A criterion for holomorphicity

We now come to our main result. Let $f : \Omega \to \mathbb{H}$ be a function of class $C^1(\bar{\Omega})$.

**Theorem 4.1.** Let $A = (a_{\alpha\beta})$ be the $3 \times 3$ matrix with entries $a_{\alpha\beta} = -\int_{\Omega} \langle J_{i_\alpha}, f^*L_{i_\beta} \rangle dV$. Then

1. $f$ is $\psi$-regular if and only if $E(f) = \text{tr}A$.
2. If $f \in R(\Omega)$, then $A$ is real, symmetric and $\text{tr}A \geq \lambda_1 = \max\{\text{eigenvalues of A}\}$.

It follows that $\det(A - (\text{tr}A)I_3) \leq 0$.

3. If $f \in R(\Omega)$, then $f$ belongs to some space $\text{Hol}_p(\Omega, \mathbb{H})$ if and only if $E(f) = \text{tr}A = \lambda_1$ or, equivalently, $\det(A - (\text{tr}A)I_3) = 0$.

4. If $E(f) = \text{tr}A = \lambda_1$, $X_p = (p_1, p_2, p_3)$ is a unit eigenvector of $A$ relative to the largest eigenvalue $\lambda_1$ if and only if $f \in \text{Hol}_p(\Omega, \mathbb{H})$. 


4.3. The existence of non-holomorphic $\psi$-regular maps

The criterion can be applied to show that on every domain $\Omega$ in $\mathbb{H}$, there exist $\psi$-regular functions that are not holomorphic.

**Example 4.1.** Let $f = z_1 + z_2 + \bar{z}_1 + (z_1 + z_2 + \bar{z}_2)j$. Then $f$ is $\psi$-regular, but not holomorphic, since on the unit ball $B$ in $\mathbb{C}^2$, $f$ has energy $E(f) = 6$ and the matrix $A$ of the theorem is

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$ 

Therefore $E(f) = tr A > 2 = \lambda_1$.

In the preceding example, the Jacobian matrix of the function has even rank, a necessary condition for a holomorphic map. In the case when the rank is odd, the non-holomorphicity follows immediately. For example, $g = z_1 + \bar{z}_1 + \bar{z}_2 j$ is $\psi$-regular (on any $\Omega$) but not $J_p$-holomorphic, for any $p$, since $rk J_C(f)$ is odd.

**Example 4.2.** The linear, $\psi$-regular functions constitute a $\mathbb{H}$-module of dimension 3 over $\mathbb{H}$, generated e.g. by the set \{z_1 + z_2 j, z_2 + z_1 j, \bar{z}_1 + \bar{z}_2 j\}. An element

$$f = (z_1 + z_2)q_1 + (z_2 + z_1)q_2 + (\bar{z}_1 + \bar{z}_2)q_3$$

is holomorphic if and only if the coefficients $q_1 = a_1 + a_2 j$, $q_2 = b_1 + b_2 j$, $q_3 = c_1 + c_2 j$ satisfy the 6th-degree real homogeneous equation

$$\det(A - (tr A)I_3) = 0$$

obtained after integration on $B$. The explicit expression of this equation is given in the Appendix. So “almost all” (linear) $\psi$-regular functions are non-holomorphic.

**Example 4.3.** A positive example (with $p \neq i, j, k$). Let $h = \bar{z}_1 + (z_1 + \bar{z}_2)j$. On the unit ball $h$ has energy 3 and the matrix $A$ is

$$A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

then $E(h) = tr A$ is equal to the (simple) largest eigenvalue, with unit eigenvector $X = \frac{1}{\sqrt{2}}(1, 0, 2)$. It follows that $h$ is $J_p$-holomorphic with $p = \frac{1}{\sqrt{2}}(i + 2k)$, i.e. it satisfies the equation

$$df + \frac{1}{2}(i + 2k)(J_1^* + 2J_3^*)(df) = 0.$$
Example 4.4. We give a quadratic example. Let \( f = |z_1|^2 - |z_2|^2 + \bar{z}_1 \bar{z}_2 j \). \( f \) has energy 2 on \( B \) and the matrix \( A \) is

\[
A = \begin{bmatrix}
-2/3 & 0 & 0 \\
0 & 4/3 & 0 \\
0 & 0 & 4/3
\end{bmatrix}
\]

Then \( f \) is \( \psi \)-regular but not holomorphic w.r.t. any complex structure \( J_p \).

4.4. Other applications of the criterion

1) If \( f \in Hol_p(\Omega, \mathbb{H}) \cap Hol_{p'}(\Omega, \mathbb{H}) \) for two \( \mathbb{R} \)-independent \( p, p' \), then \( X_p, X_{p'} \) are independent eigenvectors relative to \( \lambda_1 \). Therefore the eigenvalues of the matrix \( A \) are \( \lambda_1 = \lambda_2 = -\lambda_3 \).

   If \( f \in Hol_p(\Omega, \mathbb{H}) \cap Hol_{p'}(\Omega, \mathbb{H}) \cap Hol_{p''}(\Omega, \mathbb{H}) \) for three \( \mathbb{R} \)-independent \( p, p', p'' \) then \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \Rightarrow A = 0 \) and therefore \( f \) has energy 0 and \( f \) is a (locally) constant map.

2) If \( \Omega \) is connected, then \( Hol_p(\Omega, \mathbb{H}) \cap Hol_{p'}(\Omega, \mathbb{H}) \) (\( p \neq \pm p' \)) contains only affine maps (cf. Sommese\(^6\)).

   We can assume \( p = i, p' = j \) since in view of property 3) of Remark 3.1 we can suppose \( p \) and \( p' \) orthogonal quaternions and then we can rotate the space of imaginary quaternions. Let \( f \in Hol_i(\Omega, \mathbb{H}) \cap Hol_j(\Omega, \mathbb{H}) \) and \( a = \left( \frac{\partial f_1}{\partial z_1}, \frac{\partial f_2}{\partial z_1} \right) \), \( b = \left( \frac{\partial f_2}{\partial z_2}, -\frac{\partial f_1}{\partial z_2} \right) \). Since \( f \in Hol_i(\Omega, \mathbb{H}) \), the matrix \( A \) is obtained after integration on \( \Omega \) of the matrix

\[
\begin{bmatrix}
|a|^2 + |b|^2 & 0 & 0 \\
0 & 2Re(a,b) & -2Im(a,b) \\
0 & -2Im(a,b) & -2Re(a,b)
\end{bmatrix}
\]

where \( \langle a, b \rangle \) denotes the standard hermitian product of \( \mathbb{C}^2 \).

Since \( f \in Hol_j(\Omega, \mathbb{H}) \), we have \( \int_{\Omega} Im(a,b)dV = 0 \) and \( \int_{\Omega} |a - b|^2 dV = 0 \). Therefore \( a = b \) on \( \Omega \). Then \( a \) is holomorphic and anti-holomorphic w.r.t. the standard structure \( J_1 \). This means that \( a \) is constant on \( \Omega \) and \( f \) is an affine map with linear part of the form

\[
(a_1 z_1 - \bar{a}_2 z_2) + (a_2 z_1 + \bar{a}_1 z_2)j
\]

i.e. the right multiplication of \( q = z_1 + z_2 j \) by the quaternion \( a_1 + a_2 j \).

3) We can give a classification of \( \psi \)-regular functions based on the dimension of the set of complex structures w.r.t. which the function is holomorphic. Let \( \Omega \) be connected. Given a function \( f \in R(\Omega) \), we set

\[ J(f) = \{ p \in S^2 \mid f \in Hol_p(\Omega, \mathbb{H}) \}. \]

The space \( R(\Omega) \) of \( \psi \)-regular functions is the disjoint union of subsets of functions of the following four types:
(i) $f$ is $J_p$-holomorphic for three $\mathbb{R}$-independent structures  
$\implies f$ is a constant and $\mathcal{J}(f) = S^2$.

(ii) $f$ is $J_p$-holomorphic for exactly two $\mathbb{R}$-independent structures  
$\implies f$ is a $\psi$-regular, invertible affine map and $\mathcal{J}(f)$ is an equator $S^1 \subset S^2$.

(iii) $f$ is $J_p$-holomorphic for exactly one structure $J_p$ (up to sign of $p$) $\implies \mathcal{J}(f)$ is a two-point set $S^0$.

(iv) $f$ is $\psi$-regular but not $J_p$-holomorphic w.r.t. any complex structure $\implies \mathcal{J}(f) = \emptyset$.

5. Sketch of proof of Theorem 4.1

If $f \in \mathcal{R}(\Omega)$, then $\mathcal{E}(f) = -\mathcal{K}(f) = trA$. Let

$$\mathcal{I}_p(f) = \frac{1}{2} \int_{\Omega} ||df + L_p \circ df \circ J_p||^2 dV.$$  

Then we obtain, as in Chen and Li

$$\mathcal{E}(f) + \int_{\Omega} (J_p, f^* L_p) dV = \frac{1}{4} \mathcal{I}_p(f).$$

If $X_p = (p_1, p_2, p_3)$, then

$$XAX^T = \sum_{a,\beta} p_a p_\beta a_{a\beta} = -\int_{\Omega} \left( \sum_{\alpha} p_\alpha J_\alpha, f^* \sum_{\beta} p_\beta L_{i_\beta} \right) dV$$

$$= -\int_{\Omega} (J_p, f^* L_p) dV = \mathcal{E}(f) - \frac{1}{4} \mathcal{I}_p(f).$$

Then $trA = \mathcal{E}(f) = XAX^T + \frac{1}{4} \mathcal{I}_p(f) \geq XAX^T$, with equality if and only if $\mathcal{I}_p(f) = 0$ i.e if and only if $f$ is a $J_p$-holomorphic map.

Let $M_\alpha$ ($\alpha = 1, 2, 3$) be the matrix associated to $J_\alpha^*$ w.r.t. the basis $\{dz_1, dz_2, dz_3, dz_4\}$. The entries of the matrix $A$ can be computed by the formula

$$a_{\alpha\beta} = -\int_{\Omega} (J_\alpha, f^* L_{i_\beta}) dV = \frac{1}{2} \int_{\Omega} tr(B_\alpha^T C_\beta) dV$$

where $B_\alpha = M_\alpha J_\mathcal{C}(f)^T$ for $\alpha = 1, 2$, $B_3 = -M_3 J_\mathcal{C}(f)^T$ for $\alpha = 3$ and $C_\beta = J_\mathcal{C}(f)^T M_\beta$ for $\beta = 1, 2, 3$.

A direct computation shows how from the particular form of the Jacobian matrix of a $\psi$-regular function it follows the symmetry property of $A$.

Appendix

We give the explicit expression of the $6^{th}$-degree real homogeneous equation satisfied by the complex coefficients of a linear $J_p$-holomorphic $\psi$-regular function.

$$\frac{1}{3!} \det(A - (trA)I_3) = a_{11}a_{22}a_{33} - a_{11}a_{22}b_1c_1 - a_{11}a_{22}c_1b_1 - a_{22}a_{33}b_1c_1$$

$$- a_{11}a_{33}b_1c_1 - a_{11}b_1c_1a_2 - a_{22}b_1c_1a_2 - a_{33}b_1c_1a_2 - a_{11}b_1c_2$$

$$- a_{22}b_1c_2 - a_{33}b_1c_2 - a_{11}b_2c_1 - a_{22}b_2c_1 - a_{33}b_2c_1 - a_{11}b_3c_1 - a_{22}b_3c_1$$

$$- a_{33}b_3c_1 - a_{11}b_1c_2 - a_{22}b_1c_2 - a_{33}b_1c_2 - a_{11}b_2c_2 - a_{22}b_2c_2 - a_{33}b_2c_2$$

$$- a_{11}b_1c_3 - a_{22}b_1c_3 - a_{33}b_1c_3 - a_{11}b_2c_3 - a_{22}b_2c_3 - a_{33}b_2c_3.$$
\[ a_2 x^2 \bar{a}_1 b_1 b_2 - a_1 c_1 c_2 \bar{a}_1 b_1 b_2 - a_2 c_1 c_2 \bar{a}_2 b_1 b_2 - a_1 c_2 \bar{a}_2 b_1 b_2 - a_2 c_2 \bar{a}_1 b_1 b_2 + a_1 a_2 b_1 b_2 c_1 - a_1 b_2^2 c_1 c_1 - a_1 a_2 b_2^2 c_1 + a_2^2 b_1 b_2 c_1 - 2a_1 b_1 c_1 a_1 b_1 c_1 - a_1 b_2 c_1 a_2 b_1 c_1 - a_1 b_1 c_2 a_2 b_1 c_1 - a_2 b_1 c_1 a_1 b_2 c_1 - 2a_1 b_2 c_1 a_1 b_2 c_1 - 2a_1 b_2 c_1 a_2 b_2 c_1 + a_2 b_1 c_2 a_2 b_2 c_1 + a_2 b_2 c_2 a_2 b_2 c_1 + c_1 a_1 a_2 b_1 b_2 c_1 + c_2 a_2 b_1 b_2 c_1 - a_1 b_1 a_2 b_2 c_1 - c_2 a_1 a_2 b_2 c_1 - a_1 b_1 \bar{a}_1 c_1^2 - a_1 b_2 \bar{a}_2 c_1^2 - b_1 a_1 a_2 b_2 c_1^2 + b_2 a_1 a_2 b_2 c_1^2 + a_1 a_2 b_2 c_1 c_2 - a_1 a_2^2 b_2 c_2 + a_1 b_1 b_2 c_2 c_2 - a_2 b_1 c_1 a_1 b_1 c_2 - a_2 b_1 c_1 a_2 b_1 c_2 - 2a_1 b_1 c_2 a_1 b_1 c_2 + a_2 b_1 a_2 b_1 c_2 - 2a_2 b_1 c_2 a_2 b_1 c_2 - a_1 b_2 c_2 a_2 b_1 c_2 - c_1 a_1 a_2 b_1 b_2 c_2 - c_2 a_1 a_2 b_1 b_2 c_2 - a_2 b_2 c_1 a_1 b_2 c_2 - a_2 b_1 c_2 a_1 b_2 c_2 - 2a_2 b_2 c_2 a_2 b_2 c_2 + c_1 a_1 a_2 b_1 b_2 c_2 + c_2 a_1 a_2 b_1 b_2 c_2 - a_2 b_2 \bar{a}_1 a_1 c_1 c_2 - a_1 b_2 \bar{a}_2 c_1 c_2 - a_1 b_1 a_2 b_1 c_1 c_2 - b_2 a_2 b_1 c_1 c_2 - b_1 a_1 a_2 b_2 c_1 c_2 - b_2 a_1 a_2 b_2 c_1 c_2 - a_2 b_1 a_1 c_2^2 - a_2 b_2 a_2 c_2^2 + b_1 a_1 a_2 b_1 c_2 + b_2 a_1 a_2 b_1 c_2 = 0 \]

References