HOLOMORPHIC FUNCTIONS AND REGULAR QUATERNIONIC FUNCTIONS ON THE HYPERKÄHLER SPACE $\mathbb{H}$

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Let $\mathbb{H}$ be the space of quaternions, with its standard hypercomplex structure. Let $\mathcal{R}(\Omega)$ be the module of *ψ-regular* functions on Ω. For every $p \in \mathbb{H}$, $p^2 = -1$, $\mathcal{R}(\Omega)$ contains the space of holomorphic functions w.r.t. the complex structure J_p induced by p. We prove the existence, on any bounded domain Ω, of ψ-regular functions that are not J_p-holomorphic for any p. Our starting point is a result of Chen and Li concerning maps between hyperkähler manifolds, where a similar result is obtained for a less restricted class of quaternionic maps. We give a criterion, based on the energy-minimizing property of holomorphic maps, that distinguishes J_p-holomorphic functions among ψ-regular functions.

1. Introduction

Let $\mathbb{H}$ be the space of quaternions, with its standard hypercomplex structure given by the complex structures J_1, J_2 on $T\mathbb{H} \simeq \mathbb{H}$ defined by left multiplication by i and j. Let J_1^*, J_2^* be the dual structures on $T^*\mathbb{H}$.

We consider the module $\mathcal{R}(\Omega) = \{f = f_1 + f_2 j \mid \overline{\partial} f_1 = J_2^*(\partial \overline{f_2}) \text{ on } \Omega\}$ of left *ψ-regular* functions on Ω. These functions are in a simple correpondence with Fueter left regular functions, since they can be obtained from them by means of a real coordinate reflection in $\mathbb{H}$. They have been studied by many authors (see for instance Sudbery[7], Shapiro and Vasilevski[5] and Nōno[4]). The space $\mathcal{R}(\Omega)$ contains the identity mapping and any holomorphic mapping (f_1, f_2) on Ω defines a ψ-regular function $f = f_1 + f_2 j$. This is no more true if we replace the class of ψ-regular functions with that of regular functions. The definition of ψ-regularity is also equivalent to that of *q-holomorphicity* given by Joyce[2] in the setting of hypercomplex manifolds.

For every $p \in \mathbb{H}$, $p^2 = -1$, $\mathcal{R}(\Omega)$ contains the space $Hol_p(\Omega, \mathbb{H}) = \{f : \Omega \to \mathbb{H} \mid df + pJ_p(df) = 0 \text{ on } \Omega\}$ of holomorphic functions w.r.t. the complex structure

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$J_p = p_1 J_1 + p_2 J_2 + p_3 J_3$ on Ω and to the structure induced on $\mathbb{H}$ by left-multiplication by p (*J_p-holomorphic* functions on Ω).

We show that on every domain Ω there exist ψ-regular functions that are not J_p-holomorphic for any p. A similar result was obtained by Chen and Li[1] for the larger class of *q-maps* between hyperkähler manifolds.

This result is a consequence of a criterion (cf. Theorem 4.1) of J_p-holomorphicity, which is obtained using the energy-minimizing property of ψ-regular functions (cf. Proposition 4.1) and ideas of Lichnerowicz[3] and Chen and Li[1].

In Sec. 4.4 we give some other applications of the criterion. In particular, we show that if Ω is connected, then the intersection $Hol_p(\Omega, \mathbb{H}) \cap Hol_{p'}(\Omega, \mathbb{H})$ ($p \neq \pm p'$) contains only affine maps. This result is in accord with what was proved by Sommese[6] about *quaternionic maps* (cf. Sec. 3.2 for definitions).

2. Fueter-regular and ψ-regular functions

2.1. *Notations and definitions*

We identify the space $\mathbb{C}^2$ with the set $\mathbb{H}$ of quaternions by means of the mapping that associates the pair $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$ with the quaternion $q = z_1 + z_2 j = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}$. Let Ω be a bounded domain in $\mathbb{H} \simeq \mathbb{C}^2$. A quaternionic function $f = f_1 + f_2 j \in C^1(\Omega)$ is *(left) regular* on Ω (in the sense of Fueter) if

$$\mathcal{D}f = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} + k\frac{\partial f}{\partial x_3} = 0 \quad \text{on } \Omega$$

Given the "structural vector" $\psi = (1, i, j, -k)$, f is called *(left) ψ-regular* on Ω if

$$\mathcal{D}'f = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} - k\frac{\partial f}{\partial x_3} = 0 \quad \text{on } \Omega.$$

We recall some properties of regular functions, for which we refer to the papers of Sudbery[7], Shapiro and Vasilevski[5] and Nōno[4]:

(1) f is ψ-regular $\Leftrightarrow \quad \dfrac{\partial f_1}{\partial \bar{z}_1} = \dfrac{\partial \overline{f_2}}{\partial z_2}, \quad \dfrac{\partial f_1}{\partial \bar{z}_2} = -\dfrac{\partial \overline{f_2}}{\partial z_1}$
(2) Every holomorphic map (f_1, f_2) on Ω defines a ψ-regular function $f = f_1 + f_2 j$.
(3) The complex components are both holomorphic or both non-holomorphic.
(4) Every regular or ψ-regular function is harmonic.
(5) If Ω is pseudoconvex, every complex harmonic function is the complex component of a ψ-regular function on Ω.

$$*\overline{\partial} f_1 = -\tfrac{1}{2}\partial(\overline{f_2} d\bar{z}_1 \wedge d\bar{z}_2)$$

(6) The space $\mathcal{R}(\Omega)$ of ψ-regular functions on Ω is a *right* $\mathbb{H}$-module with integral representation formulas.

2.2. *q-holomorphic functions*

A definition equivalent to ψ-regularity has been given by Joyce[2] in the setting of hypercomplex manifolds. Joyce introduced the module of *q-holomorphic* functions on a hypercomplex manifold. On this module he defined a (commutative) product. A hypercomplex structure on the manifold $\mathbb{H}$ is given by the complex structures J_1, J_2 on $T\mathbb{H} \simeq \mathbb{H}$ defined by left multiplication by i and j. Let J_1^*, J_2^* be the dual structures on $T^*\mathbb{H}$. In complex coordinates

$$\begin{cases} J_1^* dz_1 = i\, dz_1, & J_1^* dz_2 = i\, dz_2 \\ J_2^* dz_1 = -d\bar{z}_2, & J_2^* dz_2 = d\bar{z}_1 \\ J_3^* dz_1 = i\, d\bar{z}_2, & J_3^* dz_2 = -i\, d\bar{z}_1 \end{cases}$$

where we make the choice $J_3^* = J_1^* J_2^* \Rightarrow J_3 = -J_1 J_2$.

A function f is ψ-regular if and only if f is *q-holomorphic*, i.e.

$$df + iJ_1^*(df) + jJ_2^*(df) + kJ_3^*(df) = 0.$$

In complex components $f = f_1 + f_2 j$, we can rewrite the equations of ψ-regularity as

$$\overline{\partial} f_1 = J_2^*(\partial \overline{f}_2).$$

3. Holomorphic maps

3.1. *Holomorphic functions w.r.t. a complex structure J_p*

Let $J_p = p_1 J_1 + p_2 J_2 + p_3 J_3$ be the complex structure on $\mathbb{H}$ defined by a unit imaginary quaternion $p = p_1 i + p_2 j + p_3 k$ in the sphere $S^2 = \{p \in \mathbb{H} \mid p^2 = -1\}$. It is well-known that every complex structure compatible with the standard hyperkähler structure of $\mathbb{H}$ is of this form. If $f = f^0 + i f^1 : \Omega \to \mathbb{C}$ is a J_p*-holomorphic* function, i.e. $df^0 = J_p^*(df^1)$ or, equivalently, $df + iJ_p^*(df) = 0$, then f defines a ψ-regular function $\tilde{f} = f^0 + p f^1$ on Ω. We can identify $\tilde{f}$ with a holomorphic function

$$\tilde{f} : (\Omega, J_p) \to (\mathbb{C}_p, L_p)$$

where $\mathbb{C}_p = \langle 1, p \rangle$ is a copy of $\mathbb{C}$ in $\mathbb{H}$ and L_p is the complex structure defined on $T^*\mathbb{C}_p \simeq \mathbb{C}_p$ by left multiplication by p.

More generally, we can consider the space of holomorphic maps from (Ω, J_p) to $(\mathbb{H}, L_p)$

$$Hol_p(\Omega, \mathbb{H}) = \{f : \Omega \to \mathbb{H} \mid \overline{\partial}_p f = 0 \text{ on } \Omega\} = Ker \overline{\partial}_p$$

(the J_p*-holomorphic maps* on Ω) where $\overline{\partial}_p$ is the Cauchy-Riemann operator w.r.t. the structure J_p

$$\overline{\partial}_p = \frac{1}{2}\left(d + pJ_p^* \circ d\right).$$

For any positive orthonormal basis $\{1, p, q, pq\}$ of $\mathbb{H}$ ($p, q \in S^2$), the equations of ψ-regularity can be rewritten in complex form as

$$\overline{\partial}_p f_1 = J_q^*(\partial_p \overline{f}_2)$$

where $f = (f^0 + pf^1) + (f^2 + pf^3)q = f_1 + f_2 q$. Then every $f \in Hol_p(\Omega, \mathbb{H})$ is a ψ-regular function on Ω.

Remark 3.1. 1) The *identity* map is in $Hol_i(\Omega, \mathbb{H}) \cap Hol_j(\Omega, \mathbb{H})$ but not in $Hol_k(\Omega, \mathbb{H})$.

2) $Hol_{-p}(\Omega, \mathbb{H}) = Hol_p(\Omega, \mathbb{H})$

3) If $f \in Hol_p(\Omega, \mathbb{H}) \cap Hol_{p'}(\Omega, \mathbb{H})$, with $p \neq \pm p'$, then $f \in Hol_{p''}(\Omega, \mathbb{H})$ for every $p'' = \frac{\alpha p + \beta p'}{\|\alpha p + \beta p'\|}$.

4) ψ-regularity distinguishes between holomorphic and anti-holomorphic maps: if f is an *anti-holomorphic* map from (Ω, J_p) to $(\mathbb{H}, L_p)$, then f can be ψ-regular or not. For example, $f = \bar{z}_1 + \bar{z}_2 j \in Hol_j(\Omega, \mathbb{H}) \cap Hol_k(\Omega, \mathbb{H})$ is a ψ-regular function induced by the anti-holomorphic map

$$(\bar{z}_1, \bar{z}_2) : (\Omega, J_1) \to (\mathbb{H}, L_i)$$

while $(\bar{z}_1, 0) : (\Omega, J_1) \to (\mathbb{H}, L_i)$ induces the function $g = \bar{z}_1 \notin \mathcal{R}(\Omega)$.

3.2. *Quaternionic maps*

A particular class of J_p-holomorphic maps is constituted by the *quaternionic maps* on the quaternionic manifold Ω. Sommese[6] defined quaternionic maps between hypercomplex manifolds: a quaternionic map is a map

$$f : (X, J_1, J_2) \to (Y, K_1, K_2)$$

that is holomorphic from (X, J_1) to (Y, K_1) *and* from (X, J_2) to (Y, K_2).

In particular, a quaternionic map

$$f : (\Omega, J_1, J_2) \to (\mathbb{H}, J_1, J_2)$$

is an element of $Hol_i(\Omega, \mathbb{H}) \cap Hol_j(\Omega, \mathbb{H})$ and then a ψ-regular function on Ω. Sommese showed that quaternionic maps are affine. They appear for example as transition functions for 4-dimensional *quaternionic manifolds*.

4. Non-holomorphic ψ-regular maps

A natural question can now be raised: can ψ-regular maps always be made holomorphic by rotating the complex structure or do they constitute a new class of harmonic maps? In other words, does the space $\mathcal{R}(\Omega)$ contain the union $\bigcup_{p \in S^2} Hol_p(\Omega, \mathbb{H})$ properly?

Chen and Li[1] posed and answered the analogous question for the larger class of *q-maps* between hyperkähler manifolds. In their definition, the complex structures of the source and target manifold can rotate *independently*. This implies that also anti-holomorphic maps are q-maps.

4.1. *Energy and regularity*

The *energy* (w.r.t. the euclidean metric g) of a map $f : \Omega \to \mathbb{C}^2 \simeq \mathbb{H}$, of class $C^1(\overline{\Omega})$, is the integral

$$\mathcal{E}(f) = \frac{1}{2}\int_\Omega \|df\|^2 dV = \frac{1}{2}\int_\Omega \langle g, f^*g\rangle dV = \frac{1}{2}\int_\Omega tr(J_\mathbb{C}(f)\overline{J_\mathbb{C}(f)}^T)dV$$

where $J_\mathbb{C}(f)$ is the Jacobian matrix of f with respect to the coordinates $\bar{z}_1, z_1, \bar{z}_2, z_2$.

Lichnerowicz[3] proved that holomorphic maps between Kähler manifolds minimize the energy functional in their homotopy classes. Holomorphic maps f smooth on $\overline{\Omega}$ minimize energy in the homotopy class constituted by maps u with $u_{|\partial\Omega} = f_{|\partial\Omega}$ which are homotopic to f relative to $\partial\Omega$.

From the theorem, functions $f \in Hol_p(\Omega, \mathbb{H})$ minimize the energy functional in their homotopy classes (relative to $\partial\Omega$). More generally:

Proposition 4.1. *If f is ψ-regular on Ω, then it minimizes energy in its homotopy class (relative to $\partial\Omega$).*

Proof. We repeat arguments of Lichnerowicz, Chen and Li. Let $i_1 = i, i_2 = j, i_3 = k$ and let

$$\mathcal{K}(f) = \int_\Omega \sum_{\alpha=1}^{3} \langle J_\alpha, f^* L_{i_\alpha}\rangle dV, \quad \mathcal{I}(f) = \frac{1}{2}\int_\Omega \|df + \sum_{\alpha=1}^{3} L_{i_\alpha} \circ df \circ J_\alpha\|^2 dV.$$

Then $\mathcal{K}(f)$ is a homotopy invariant of f and $\mathcal{I}(f) = 0$ if and only if $f \in \mathcal{R}(\Omega)$. A computation similar to that made by Chen and Li[1] gives

$$\mathcal{E}(f) + \mathcal{K}(f) = \frac{1}{4}\mathcal{I}(f) \geq 0.$$

From this the result follows immediately. □

4.2. *A criterion for holomorphicity*

We now come to our main result. Let $f : \Omega \to \mathbb{H}$ be a function of class $C^1(\overline{\Omega})$.

Theorem 4.1. *Let $A = (a_{\alpha\beta})$ be the 3×3 matrix with entries $a_{\alpha\beta} = -\int_\Omega \langle J_\alpha, f^* L_{i_\beta}\rangle dV$. Then*

(1) f is ψ-regular if and only if $\mathcal{E}(f) = trA$.

(2) If $f \in \mathcal{R}(\Omega)$, then A is real, symmetric and

$$trA \geq \lambda_1 = \max\{\textit{eigenvalues of } A\}.$$

It follows that $\det(A - (trA)I_3) \leq 0$.

(3) If $f \in \mathcal{R}(\Omega)$, then f belongs to some space $Hol_p(\Omega, \mathbb{H})$ if and only if $\mathcal{E}(f) = trA = \lambda_1$ or, equivalently, $\det(A - (trA)I_3) = 0$.

(4) If $\mathcal{E}(f) = trA = \lambda_1$, $X_p = (p_1, p_2, p_3)$ is a unit eigenvector of A relative to the largest eigenvalue λ_1 if and only if $f \in Hol_p(\Omega, \mathbb{H})$.

4.3. *The existence of non-holomorphic ψ-regular maps*

The criterion can be applied to show that on every domain Ω in $\mathbb{H}$, there exist ψ-regular functions that are not holomorphic.

Example 4.1. Let $f = z_1 + z_2 + \bar{z}_1 + (z_1 + z_2 + \bar{z}_2)j$. Then f is ψ-regular, but not holomorphic, since on the unit ball B in $\mathbb{C}^2$, f has energy $\mathcal{E}(f) = 6$ and the matrix A of the theorem is

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Therefore $\mathcal{E}(f) = trA > 2 = \lambda_1$.

In the preceding example, the Jacobian matrix of the function has even rank, a necessary condition for a holomorphic map. In the case when the rank is odd, the non-holomorphicity follows immediately. For example, $g = z_1 + \bar{z}_1 + \bar{z}_2 j$ is ψ-regular (on any Ω) but not J_p-holomorphic, for any p, since $rkJ_{\mathbb{C}}(f)$ is odd.

Example 4.2. The linear, ψ-regular functions constitute a $\mathbb{H}$-module of dimension 3 over $\mathbb{H}$, generated e.g. by the set $\{z_1 + z_2 j, z_2 + z_1 j, \bar{z}_1 + \bar{z}_2 j\}$. An element

$$f = (z_1 + z_2 j)q_1 + (z_2 + z_1 j)q_2 + (\bar{z}_1 + \bar{z}_2 j)q_3$$

is holomorphic if and only if the coefficients $q_1 = a_1 + a_2 j$, $q_2 = b_1 + b_2 j$, $q_3 = c_1 + c_2 j$ satisfy the 6^{th}-degree real homogeneous equation

$$\det(A - (trA)I_3) = 0$$

obtained after integration on B. The explicit expression of this equation is given in the Appendix. So "almost all" (linear) ψ-regular functions are non-holomorphic.

Example 4.3. A positive example (with $p \neq i, j, k$). Let $h = \bar{z}_1 + (z_1 + \bar{z}_2)j$. On the unit ball h has energy 3 and the matrix A is

$$A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

then $\mathcal{E}(h) = trA$ is equal to the (simple) largest eigenvalue, with unit eigenvector $X = \frac{1}{\sqrt{5}}(1, 0, 2)$. It follows that h is J_p-holomorphic with $p = \frac{1}{\sqrt{5}}(i + 2k)$, i.e. it satisfies the equation

$$df + \tfrac{1}{5}(i + 2k)(J_1^* + 2J_3^*)(df) = 0.$$

Example 4.4. We give a quadratic example. Let $f = |z_1|^2 - |z_2|^2 + \bar{z}_1\bar{z}_2 j$. f has energy 2 on B and the matrix A is

$$A = \begin{bmatrix} -2/3 & 0 & 0 \\ 0 & 4/3 & 0 \\ 0 & 0 & 4/3 \end{bmatrix}$$

Then f is ψ-regular but not holomorphic w.r.t. any complex structure J_p.

4.4. *Other applications of the criterion*

1) If $f \in Hol_p(\Omega, \mathbb{H}) \cap Hol_{p'}(\Omega, \mathbb{H})$ for two $\mathbb{R}$-independent p, p', then $X_p, X_{p'}$ are independent eigenvectors relative to λ_1. Therefore the eigenvalues of the matrix A are $\lambda_1 = \lambda_2 = -\lambda_3$.

If $f \in Hol_p(\Omega, \mathbb{H}) \cap Hol_{p'}(\Omega, \mathbb{H}) \cap Hol_{p''}(\Omega, \mathbb{H})$ for three $\mathbb{R}$-independent p, p', p'' then $\lambda_1 = \lambda_2 = \lambda_3 = 0 \Rightarrow A = 0$ and therefore f has energy 0 and f is a (locally) *constant* map.

2) If Ω is connected, then $Hol_p(\Omega, \mathbb{H}) \cap Hol_{p'}(\Omega, \mathbb{H})$ $(p \neq \pm p')$ contains only *affine* maps (cf. Sommese[6]).

We can assume $p = i$, $p' = j$ since in view of property 3) of Remark 3.1 we can suppose p and p' orthogonal quaternions and then we can rotate the space of imaginary quaternions. Let $f \in Hol_i(\Omega, \mathbb{H}) \cap Hol_j(\Omega, \mathbb{H})$ and $a = \left(\frac{\partial f_1}{\partial z_1}, \frac{\partial f_2}{\partial z_1}\right)$, $b = \left(\overline{\frac{\partial f_2}{\partial z_2}}, -\overline{\frac{\partial f_1}{\partial z_2}}\right)$. Since $f \in Hol_i(\Omega, \mathbb{H})$, the matrix A is obtained after integration on Ω of the matrix

$$\begin{bmatrix} |a|^2 + |b|^2 & 0 & 0 \\ 0 & 2Re\langle a, b\rangle & -2Im\langle a, b\rangle \\ 0 & -2Im\langle a, b\rangle & -2Re\langle a, b\rangle \end{bmatrix}$$

where $\langle a, b\rangle$ denotes the standard hermitian product of $\mathbb{C}^2$.

Since $f \in Hol_j(\Omega, \mathbb{H})$, we have $\int_\Omega Im\langle a, b\rangle dV = 0$ and $\int_\Omega |a - b|^2 dV = 0$. Therefore $a = b$ on Ω. Then a is holomorphic and anti-holomorphic w.r.t. the standard structure J_1. This means that a is constant on Ω and f is an affine map with linear part of the form

$$(a_1 z_1 - \bar{a}_2 z_2) + (a_2 z_1 + \bar{a}_1 z_2) j$$

i.e. the right multiplication of $q = z_1 + z_2 j$ by the quaternion $a_1 + a_2 j$.

3) We can give a classification of ψ-regular functions based on the dimension of the set of complex structures w.r.t. which the function is holomorphic. Let Ω be connected. Given a function $f \in \mathcal{R}(\Omega)$, we set

$$\mathcal{J}(f) = \{p \in S^2 \mid f \in Hol_p(\Omega, \mathbb{H})\}.$$

The space $\mathcal{R}(\Omega)$ of ψ-regular functions is the disjoint union of subsets of functions of the following four types:

(i) f is J_p-holomorphic for three $\mathbb{R}$-independent structures
$\Longrightarrow f$ is a constant and $\mathcal{J}(f) = S^2$.
(ii) f is J_p-holomorphic for exactly two $\mathbb{R}$-independent structures
$\Longrightarrow f$ is a ψ-regular, invertible affine map and $\mathcal{J}(f)$ is an equator $S^1 \subset S^2$.
(iii) f is J_p-holomorphic for exactly one structure J_p (up to sign of p) $\Longrightarrow \mathcal{J}(f)$ is a two-point set S^0.
(iv) f is ψ-regular but not J_p-holomorphic w.r.t. any complex structure $\Longrightarrow \mathcal{J}(f) = \emptyset$.

5. Sketch of proof of Theorem 4.1

If $f \in \mathcal{R}(\Omega)$, then $\mathcal{E}(f) = -\mathcal{K}(f) = trA$. Let

$$\mathcal{I}_p(f) = \frac{1}{2}\int_\Omega \|df + L_p \circ df \circ J_p\|^2 dV.$$

Then we obtain, as in Chen and Li[1]

$$\mathcal{E}(f) + \int_\Omega \langle J_p, f^* L_p\rangle dV = \frac{1}{4}\mathcal{I}_p(f).$$

If $X_p = (p_1, p_2, p_3)$, then

$$XAX^T = \sum_{\alpha,\beta} p_\alpha p_\beta a_{\alpha\beta} = -\int_\Omega \langle \sum_\alpha p_\alpha J_\alpha, f^* \sum_\beta p_\beta L_{i_\beta}\rangle dV$$

$$= -\int_\Omega \langle J_p, f^* L_p\rangle dV = \mathcal{E}(f) - \frac{1}{4}\mathcal{I}_p(f).$$

Then $trA = \mathcal{E}(f) = XAX^T + \frac{1}{4}\mathcal{I}_p(f) \geq XAX^T$, with equality if and only if $\mathcal{I}_p(f) = 0$ i.e if and only if f is a J_p-holomorphic map.

Let M_α ($\alpha = 1,2,3$) be the matrix associated to J^*_α w.r.t. the basis $\{d\bar{z}_1, dz_1, d\bar{z}_2, dz_2\}$. The entries of the matrix A can be computed by the formula

$$a_{\alpha\beta} = -\int_\Omega \langle J_\alpha, f^* L_{i_\beta}\rangle dV = \frac{1}{2}\int_\Omega tr(\overline{B_\alpha}^T C_\beta) dV$$

where $B_\alpha = M_\alpha J_{\mathbb{C}}(f)^T$ for $\alpha = 1,2$, $B_\alpha = -M_\alpha J_{\mathbb{C}}(f)^T$ for $\alpha = 3$ and $C_\beta = J_{\mathbb{C}}(f)^T M_\beta$ for $\beta = 1,2,3$.

A direct computation shows how from the particular form of the Jacobian matrix of a ψ-regular function it follows the symmetry property of A.

Appendix

We give the explicit expression of the 6^{th}-degree real homogeneous equation satisfied by the complex coefficients of a linear J_p-holomorphic ψ-regular function.

$\frac{1}{16}det(A - (trA)I_3) = a_1a_2b_2c_1^2\bar{b}_1 - a_1a_2b_1c_1c_2\bar{b}_1 - a_1^2b_2c_1c_2\bar{b}_1 + a_1^2b_1c_2^2\bar{b}_1 - a_1c_1^2\bar{a}_1\bar{b}_1^2 - a_1c_1c_2\bar{a}_2\bar{b}_1^2 + a_2^2b_2c_1^2\bar{b}_2 - a_2^2b_1c_1c_2\bar{b}_2 - a_1a_2b_2c_1c_2\bar{b}_2 + a_1a_2b_1c_2^2\bar{b}_2 -$

$$
\begin{aligned}
&a_2c_1^2\bar{a}_1\bar{b}_1\bar{b}_2 - a_1c_1c_2\bar{a}_1\bar{b}_1\bar{b}_2 - a_2c_1c_2\bar{a}_2\bar{b}_1\bar{b}_2 - a_1c_2^2\bar{a}_2\bar{b}_1\bar{b}_2 - a_2c_1c_2\bar{a}_1\bar{b}_2^2 - a_2c_2^2\bar{a}_2\bar{b}_2^2 + \\
&a_1a_2b_1b_2c_1\bar{c}_1 - a_1^2b_2^2c_1\bar{c}_1 - a_1a_2b_1^2c_2\bar{c}_1 + a_1^2b_1b_2c_2\bar{c}_1 - 2a_1b_1c_1\bar{a}_1\bar{b}_1\bar{c}_1 - a_1b_2c_1\bar{a}_2\bar{b}_1\bar{c}_1 - \\
&a_1b_1c_2\bar{a}_2\bar{b}_1\bar{c}_1 - a_2b_1c_1\bar{a}_1\bar{b}_2\bar{c}_1 - 2a_1b_2c_1\bar{a}_1\bar{b}_2\bar{c}_1 + a_1b_1c_2\bar{a}_1\bar{b}_2\bar{c}_1 - 2a_2b_2c_1\bar{a}_2\bar{b}_2\bar{c}_1 + \\
&a_2b_1c_2\bar{a}_2\bar{b}_2\bar{c}_1 - a_1b_2c_2\bar{a}_2\bar{b}_2\bar{c}_1 + c_1\bar{a}_1\bar{a}_2\bar{b}_1\bar{b}_2\bar{c}_1 + c_2\bar{a}_2^2\bar{b}_1\bar{b}_2\bar{c}_1 - c_1\bar{a}_1^2\bar{b}_2^2\bar{c}_1 - c_2\bar{a}_1\bar{a}_2\bar{b}_2^2\bar{c}_1 - \\
&a_1b_1^2\bar{a}_1\bar{c}_1^2 - a_1b_1b_2\bar{a}_2\bar{c}_1^2 + b_1\bar{a}_1\bar{a}_2\bar{b}_2\bar{c}_1^2 + b_2\bar{a}_2^2\bar{b}_2\bar{c}_1^2 + a_2^2b_1b_2c_1\bar{c}_2 - a_1a_2b_2^2c_1\bar{c}_2 - a_2^2b_1^2c_2\bar{c}_2 + \\
&a_1a_2b_1b_2c_2\bar{c}_2 - a_2b_1c_1\bar{a}_1\bar{b}_1\bar{c}_2 + a_1b_2c_1\bar{a}_1\bar{b}_1\bar{c}_2 - 2a_1b_1c_2\bar{a}_1\bar{b}_1\bar{c}_2 + a_2b_2c_1\bar{a}_2\bar{b}_1\bar{c}_2 - \\
&2a_2b_1c_2\bar{a}_2\bar{b}_1\bar{c}_2 - a_1b_2c_2\bar{a}_2\bar{b}_1\bar{c}_2 - c_1\bar{a}_1\bar{a}_2\bar{b}_1^2\bar{c}_2 - c_2\bar{a}_2^2\bar{b}_1^2\bar{c}_2 - a_2b_2c_1\bar{a}_1\bar{b}_2\bar{c}_2 - a_2b_1c_2\bar{a}_1\bar{b}_2\bar{c}_2 - \\
&2a_2b_2c_2\bar{a}_2\bar{b}_2\bar{c}_2 + c_1\bar{a}_1^2\bar{b}_1\bar{b}_2\bar{c}_2 + c_2\bar{a}_1\bar{a}_2\bar{b}_1\bar{b}_2\bar{c}_2 - a_2b_1^2\bar{a}_1\bar{c}_1\bar{c}_2 - a_1b_1b_2\bar{a}_1\bar{c}_1\bar{c}_2 - \\
&a_2b_1b_2\bar{a}_2\bar{c}_1\bar{c}_2 - a_1b_2^2\bar{a}_2\bar{c}_1\bar{c}_2 - b_1\bar{a}_1\bar{a}_2\bar{b}_1\bar{c}_1\bar{c}_2 - b_2\bar{a}_2^2\bar{b}_1\bar{c}_1\bar{c}_2 - b_1\bar{a}_1^2\bar{b}_2\bar{c}_1\bar{c}_2 - b_2\bar{a}_1\bar{a}_2\bar{b}_2\bar{c}_1\bar{c}_2 - \\
&a_2b_1b_2\bar{a}_1\bar{c}_2^2 - a_2b_2^2\bar{a}_2\bar{c}_2^2 + b_1\bar{a}_1^2\bar{b}_1\bar{c}_2^2 + b_2\bar{a}_1\bar{a}_2\bar{b}_1\bar{c}_2^2 = 0
\end{aligned}
$$

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