ON REGULAR HARMONICS OF ONE QUATERNIONIC VARIABLE

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ABSTRACT. We prove some results about the Fueter-regular homogeneous polynomials, which appear as components in the power series of any quaternionic regular function. We obtain a differential condition that characterizes the homogeneous polynomials whose trace on the unit sphere extends as a regular polynomial. We apply this result to define an injective linear operator from the space of complex spherical harmonics to the module of regular homogeneous polynomials of a fixed degree k.

1. Introduction

Let $\mathbb{H}$ be the algebra of quaternions. Let B denote the unit ball in $\mathbb{C}^2 \simeq \mathbb{H}$ and $S = \partial B$ the group of unit quaternions. In §3.1 we obtain a differential condition that characterizes the homogeneous polynomials whose restriction to S coincides with the restriction of a regular polynomial. This result generalizes a similar characterization for holomorphic extensions of polynomials proved by Kytmanov (cf. [2] and [3]).

In §3.2 we show how to define an injective linear operator $R : \mathcal{H}_k(S) \to U_k^\psi$ from the space $\mathcal{H}_k(S)$ of complex-valued spherical harmonics of degree k to the $\mathbb{H}$-module U_k^ψ of ψ-regular homogeneous polynomials of the same degree (cf. §2.1 and §3.2 for precise definitions). As an application, we show how to construct bases of the module of regular homogeneous polynomials of a fixed degree starting from any choice of $\mathbb{C}$-bases of the spaces of complex harmonic homogeneous polynomials.

2. Notations and definitions

2.1. Let $\Omega = \{z \in \mathbb{C}^2 : \rho(z) < 0\}$ be a bounded domain in $\mathbb{C}^2$ with smooth boundary. Let ν denote the outer unit normal to $\partial\Omega$ and $\tau = i\nu$. For every $F \in C^1(\overline{\Omega})$, let $\overline{\partial}_n F = \frac{1}{2}\left(\frac{\partial F}{\partial \nu} + i\frac{\partial F}{\partial \tau}\right)$ be the normal component of $\overline{\partial} F$ (see for example Kytmanov [2]§§3.3 and 14.2). It can be expressed by means of the Hodge $*$-operator and the Lebesgue surface measure as $\overline{\partial}_n f d\sigma = *\overline{\partial} f_{|\partial\Omega}$. In a neighbourhood of $\partial\Omega$ we have the decomposition of $\overline{\partial} F$ in the tangential

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and the normal parts: $\overline{\partial} F = \overline{\partial}_b F + \overline{\partial}_n F \frac{\overline{\partial}\rho}{|\overline{\partial}\rho|}$. We denote by L the tangential Cauchy-Riemann operator $L = \frac{1}{|\partial\rho|}\left(\frac{\partial\rho}{\partial\bar z_2}\frac{\partial}{\partial\bar z_1} - \frac{\partial\rho}{\partial\bar z_1}\frac{\partial}{\partial\bar z_2}\right)$.

Let $\mathbb{H}$ be the algebra of quaternions $q = x_0 + ix_1 + jx_2 + kx_3$, where x_0, x_1, x_2, x_3 are real numbers and i, j, k denote the basic quaternions. We identify the space $\mathbb{C}^2$ with $\mathbb{H}$ by means of the mapping that associates the quaternion $q = z_1 + z_2 j$ with the element $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$. We refer to Sudbery [8] for the basic facts of quaternionic analysis. We will denote by $\mathcal{D}$ the left Cauchy-Riemann-Fueter operator

$$\mathcal{D} = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}.$$

A quaternionic C^1 function $f = f_1 + f_2 j$, is *(left-)regular* on a domain $\Omega \subseteq \mathbb{H}$ if $\mathcal{D}f = 0$ on Ω. We prefer to work with another class of regular functions, which is more explicitely connected with the hyperkähler structure of $\mathbb{H}$. It is defined by the Cauchy-Riemann-Fueter operator associated with the structural vector $\psi = \{1, i, j, -k\}$:

$$\mathcal{D}' = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3} = 2\left(\frac{\partial}{\partial \bar z_1} + j\frac{\partial}{\partial \bar z_2}\right).$$

A quaternionic C^1 function $f = f_1 + f_2 j$, is called *(left-)ψ-regular* on a domain Ω, if $\mathcal{D}'f = 0$ on Ω. This condition is equivalent to the following system of complex differential equations:

$$\frac{\partial f_1}{\partial \bar z_1} = \frac{\partial \overline{f_2}}{\partial z_2}, \qquad \frac{\partial f_1}{\partial \bar z_2} = -\frac{\partial \overline{f_2}}{\partial z_1}.$$

The identity mapping is ψ-regular, and any holomorphic mapping (f_1, f_2) on Ω defines a ψ-regular function $f = f_1 + f_2 j$. This is no more true if we replace ψ-regularity with regularity. Moreover, the complex components of a ψ-regular function are either both holomorphic or both non-holomorphic (cf. Vasilevski [9], Mitelman et al [4] and Perotti [5]). Let γ be the transformation of $\mathbb{C}^2$ defined by $\gamma(z_1, z_2) = (z_1, \bar z_2)$. Then a C^1 function f is regular on the domain Ω if, and only if, $f \circ \gamma$ is ψ-regular on $\gamma^{-1}(\Omega)$.

2.2. The two-dimensional Bochner-Martinelli form $U(\zeta, z)$ is the first complex component of the Cauchy-Fueter kernel $G'(p - q)$ associated with ψ-regular functions (cf. Fueter [1], Vasilevski [9], Mitelman et al [4]). Let $q = z_1 + z_2 j$, $p = \zeta_1 + \zeta_2 j$, $\sigma(q) = dx[0] - idx[1] + jdx[2] + kdx[3]$, where $dx[k]$ denotes the product of dx_0, dx_1, dx_2, dx_3 with dx_k deleted. Then $G'(p - q)\sigma(p) = U(\zeta, z) + \omega(\zeta, z)j$, where $\omega(\zeta, z)$ is the complex $(1, 2)$-form

$$\omega(\zeta, z) = -\frac{1}{4\pi^2}|\zeta - z|^{-4}((\bar\zeta_1 - \bar z_1)d\zeta_1 + (\bar\zeta_2 - \bar z_2)d\zeta_2) \wedge \overline{d\zeta}.$$

Here $\overline{d\zeta} = \overline{d\zeta_1} \wedge \overline{d\zeta_2}$ and we choose the orientation of $\mathbb{C}^2$ given by the volume form $\frac{1}{4} dz_1 \wedge dz_2 \wedge \overline{dz_1} \wedge \overline{dz_2}$. Given $g(\zeta, z) = \frac{1}{4\pi^2}|\zeta - z|^{-2}$, we can also write $U(\zeta, z) = -2 * \partial_\zeta g(\zeta, z)$ and $\omega(\zeta, z) = -\partial_\zeta (g(\zeta, z)\overline{d\zeta})$.

3. Regular polynomials

3.1. In this section we will obtain a differential condition that characterizes the homogeneous polynomials whose restrictions to the unit sphere extend regularly or ψ-regularly. We will use a computation made by Kytmanov in [3] (cf. also [2] Corollary 23.4), where the analogous result for holomorphic extensions is proved.

Let Ω be the unit ball B in $\mathbb{C}^2$, $S = \partial B$ the unit sphere. In this case the operators $\overline{\partial}_n$ and L have the following forms:

$$\overline{\partial}_n = \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_2}, \qquad L = z_2 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial \bar{z}_2}$$

and they preserve harmonicity. Let $\Delta = \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \frac{\partial^2}{\partial z_2 \partial \bar{z}_2}$ be the Laplacian in $\mathbb{C}^2$ and D_k the differential operator

$$D_k = \sum_{0 \le l \le k/2-1} \frac{(k-2l-1)!(2l-1)!!}{k!(l+1)!} 2^l \Delta^{l+1}.$$

Theorem 1. *Let $f = f_1 + f_2 j$ be a $\mathbb{H}$-valued, homogeneous polynomial of degree k. Then its restriction to S extends as a ψ-regular function into B if, and only if,*

$$(\overline{\partial}_n - D_k) f_1 + \overline{L(f_2)} = 0 \quad \text{on } S.$$

Proof. In the first part we can proceed as in [3]. The harmonic extension $\tilde{f}_1$ of $f_{1|S}$ into B is given by Gauss's formula: $\tilde{f}_1 = \sum_{s \ge 0} g_{k-2s}$, where g_{k-2s} is the homogeneous harmonic polynomial of degree $k-2s$ defined by

$$g_{k-2s} = \frac{k-2s+1}{s!(k-s+1)!} \sum_{j \ge 0} \frac{(-1)^j (k-j-2s)!}{j!} |z|^{2j} \Delta^{j+s} f_1. \qquad (*)$$

Then $\overline{\partial}_n \tilde{f}_1 = \overline{\partial}_n f_1 - D_k f_1$ on S (cf. [2] §23). Let $\tilde{f}_2$ be the harmonic extension of f_2 into B and $\tilde{f} = \tilde{f}_1 + \tilde{f}_2 j$. Then $(\overline{\partial}_n - D_k) f_1 + \overline{L(f_2)} = 0$ on S is equivalent to $\overline{\partial}_n \tilde{f}_1 + \overline{L(f_2)} = 0$ on S. We now show that this implies the ψ-regularity of $\tilde{f}$. Let F^+ and F^- be the ψ-regular functions defined respectively on B and on $\mathbb{C}^2 \setminus \overline{B}$ by the Cauchy-Fueter integral of $\tilde{f}$:

$$F^{\pm}(z) = \int_S U(\zeta, z) \tilde{f}(\zeta) + \int_S \omega(\zeta, z) j \tilde{f}(\zeta).$$

From the equalities $U(\zeta, z) = -2 {*}\partial_\zeta g(\zeta, z)$, $\omega(\zeta, z) = -\partial_\zeta (g(\zeta, z) d\bar{\zeta})$, we get that

$$F^-(z) = -2 \int_S (\tilde{f}_1(\zeta) + f_2(\zeta) j) {*}\partial_\zeta g(\zeta, z) - \int_S \partial_\zeta (g(\zeta, z) \overline{d\zeta}) (\overline{\tilde{f}_1} j - \overline{\tilde{f}_2})$$

for every $z \notin \overline{B}$. From the complex Green formula and Stokes' Theorem and from the equality $\overline{\partial} \tilde{f}_2 \wedge d\zeta_{|S} = 2 L(f_2) d\sigma$ on S, we get that the first complex component of $F^-(z)$ is

$$-2\int_S \tilde{f}_1 \partial_n g d\sigma + \int_S \overline{\tilde{f}_2} \partial_\zeta g \wedge \overline{d\zeta} = -2\int_S g\overline{\partial}_n \tilde{f}_1 d\sigma - \int_S g \partial_\zeta \overline{\tilde{f}_2} \wedge \overline{d\zeta}$$
$$= -2\int_S g(\overline{\partial}_n \tilde{f}_1 + \overline{L(f_2)}) d\sigma$$

and then it vanishes on $\mathbb{C}^2 \setminus \overline{B}$. Therefore, $F^- = F_2 j$, with F_2 a holomorphic function that can be holomorphically continued to the whole space. Let $\tilde{F}^- = \tilde{F}_2 j$ be such extension. Then $F = F^+ - \tilde{F}^-_{|B}$ is a ψ-regular function on B (indeed a polynomial of the same degree k), continuous on $\overline{B}$, such that $F_{|S} = f_{|S}$. The converse is immediate from the equations of ψ-regularity. □

Let N and T be the differential operators

$$N = \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + z_2 \frac{\partial}{\partial z_2}, \qquad T = \bar{z}_2 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial z_2}.$$

T is a tangential operator w.r.t. S, while N is non-tangential, such that $N(\rho) = |\overline{\partial}\rho|^2$, $\mathrm{Re}(N) = |\overline{\partial}\rho| \,\mathrm{Re}(\overline{\partial}_n)$, where $\rho = |z_1|^2 + |z_2|^2 - 1$. Let γ be the reflection introduced at the end of §2.1. The operator D_k is γ-invariant, i.e. $D_k(f \circ \gamma) = D_k(f) \circ \gamma$, since Δ is invariant. It follows a criterion for regularity of homogeneous polynomials.

Corollary 2. *Let $f = f_1 + f_2 j$ be a $\mathbb{H}$-valued, homogeneous polynomial of degree k. Then its restriction to S extends as a regular function into B if, and only if,*

$$(N - D_k) f_1 + \overline{T(f_2)} = 0 \quad \text{on } S.$$

Let $g = \sum_k g^k$ be the homogeneous decomposition of a polynomial g. After replacing $D_k g$ by $\sum_k D_k g^k$, we can extend the preceding results also to non-homogeneous polynomials.

3.2. Let $\mathcal{P}_k$ denote the space of homogeneous complex-valued polynomials of degree k on $\mathbb{C}^2$, and $\mathcal{H}_k$ the space of harmonic polynomials in $\mathcal{P}_k$. The space $\mathcal{H}_k$ is the sum of the pairwise $L^2(S)$-orthogonal spaces $\mathcal{H}_{p,q}$ $(p+q=k)$, whose elements are the harmonic homogeneous polynomials of degree p in z_1, z_2 and q in $\bar{z}_1, \bar{z}_2$ (cf. for example Rudin [7]§12.2). The spaces $\mathcal{H}_k$ and $\mathcal{H}_{p,q}$ can be identified with the spaces of the restrictions of their elements to S (*spherical harmonics*). These spaces will be denoted by $\mathcal{H}_k(S)$ and $\mathcal{H}_{p,q}(S)$ respectively.

Let U_k^ψ be the right $\mathbb{H}$-module of (left) ψ-regular homogeneous polynomials of degree k. The elements of the modules U_k^ψ can be identified with their restrictions to S, which we will call *regular harmonics*.

Theorem 3. *For every $f_1 \in \mathcal{P}_k$, there exists $f_2 \in \mathcal{P}_k$ such that the trace of $f = f_1 + f_2 j$ on S extends as a ψ-regular polynomial of degree at most k on $\mathbb{H}$. If $f_1 \in \mathcal{H}_k$, then $f_2 \in \mathcal{H}_k$ and $f = f_1 + f_2 j \in U_k^\psi$.*

Proof. We can suppose that f_1 has degree p in z and q in $\bar{z}$, $p+q=k$, and then extend by linearity. Let $\tilde{f_1} = \sum_{s\geq 0} g_{p-s,q-s}$ be the harmonic extension of f_1 into B, where $g_{p-s,q-s} \in \mathcal{H}_{p-s,q-s}$ is given by formula (*). Then $\overline{\partial}_n \overline{L(g_{p-s,q-s})} = (p-s+1)\overline{L(g_{p-s,q-s})}$. We set

$$\tilde{f_2} = \sum_{s\geq 0} \frac{1}{p-s+1} \overline{L(g_{p-s,q-s})} \in \bigoplus_{s\geq 0} \mathcal{H}_{k-2s}.$$

Then $\overline{\partial}_n \tilde{f_2} = \overline{L(f_1)}$ on S and we can conclude as in the proof of Theorem 1 that $\tilde{f} = \tilde{f_1} + \tilde{f_2} j$ is a ψ-regular polynomial of degree at most k. Now it suffices to define

$$f_2 = \sum_{s\geq 0} \frac{|z|^{2s}}{p-s+1} \overline{L(g_{p-s,q-s})} \in \mathcal{P}_k$$

to get a homogeneous polynomial $f = f_1 + f_2 j$, of degree k, that has the same restriction to S as $\tilde{f}$. If $f_1 \in \mathcal{H}_k$, then $\tilde{f_1} = f_1$, $\tilde{f_2} = f_2$ and therefore $f \in U_k^\psi$. □

Let $C : U_k^\psi \to \mathcal{H}_k(S)$ be the complex-linear operator that associates to $f = f_1 + f_2 j$ the restriction to S of its first complex component f_1. The function $\tilde{f}$ in the preceding proof gives a right inverse $R : \mathcal{H}_k(S) \to U_k^\psi$ of the operator C. The function $R(f_1)$ is uniquely determined by the orthogonality condition with respect to the functions holomorphic on a neighbourhood of $\overline{B}$:

$$\int_S (R(f_1) - f_1)\overline{h} d\sigma = 0 \quad \forall h \in \mathcal{O}(\overline{B}).$$

Corollary 4. *(i) The restriction operator C defined on U_k^ψ induces isomorphisms of real vector spaces*

$$\frac{U_k^\psi}{\mathcal{H}_{k,0} j} \simeq \mathcal{H}_k(S), \qquad \frac{U_k^\psi}{\mathcal{H}_{k,0} + \mathcal{H}_{k,0} j} \simeq \frac{\mathcal{H}_k(S)}{\mathcal{H}_{k,0}(S)}.$$

(ii) U_k^ψ has dimension $\frac{1}{2}(k+1)(k+2)$ over $\mathbb{H}$.

Proof. The first part follows from $\ker C = \{f = f_1 + f_2 j \in U_k^\psi \ : \ f_1 = 0 \text{ on } S\} = \mathcal{H}_{k,0} j$. Part (ii) can be obtained from any of the above isomorphisms, since $\mathcal{H}_{k,0}$ (as every space $\mathcal{H}_{p,q}, p+q=k$) and $\mathcal{H}_k(S)$ have real dimensions respectively $2(k+1)$ and $2(k+1)^2$. □

As an application of Corollary 2, we have another proof of the known result (cf. Sudbery [8] Theorem 7) that the right $\mathbb{H}$-module U_k of left-regular homogeneous polynomials of degree k has dimension $\frac{1}{2}(k+1)(k+2)$ over $\mathbb{H}$.

3.3. The operator $R : \mathcal{H}_k(S) = \bigoplus_{p+q=k} \mathcal{H}_{p,q}(S) \to U_k^\psi$ can also be used to obtain $\mathbb{H}$-bases for U_k^ψ starting from bases of the complex spaces $\mathcal{H}_{p,q}(S)$. On $\mathcal{H}_{p,q}(S)$, R acts in the following way:

$$R(h) = h + M(h)j, \quad \text{where } M(h) = \frac{1}{p+1}\overline{L(h)} \in \mathcal{H}_{q-1,p+1} \quad (h \in \mathcal{H}_{p,q})$$

Note that $M \equiv 0$ on $\mathcal{H}_{k,0}(S)$. If $q > 0$, $M^2 = -Id$ on $\mathcal{H}_{p,q}(S)$, since $qh = \overline{\partial}_n h = -\overline{L(M(h))}$ on S, and therefore

$$h = -\frac{1}{q}\overline{L(M(h))} = -\frac{1}{q(p+1)}\overline{L}L(h) = -M^2(h).$$

If $k = 2m+1$ is odd, then M is a complex conjugate isomorphism of $\mathcal{H}_{m,m+1}(S)$. Then M induces a quaternionic structure on this space, which has real dimension $4(m+1)$. We can find complex bases of $\mathcal{H}_{m,m+1}(S)$ of the form

$$\{h_1, M(h_1), \dots, h_{m+1}, M(h_{m+1})\}.$$

Theorem 5. *Let $\mathcal{B}_{p,q}$ denote a complex base of the space $\mathcal{H}_{p,q}(S)$ $(p+q = k)$. Then:*
(i) if $k = 2m$ is even, a basis of U_k^ψ over $\mathbb{H}$ is given by the set

$$\mathcal{B}_k = \{R(h) : h \in \mathcal{B}_{p,q}, p+q = k, 0 \le q \le p \le k\}.$$

(ii) if $k = 2m+1$ is odd, a basis of U_k^ψ over $\mathbb{H}$ is given by

$$\mathcal{B}_k = \{R(h) : h \in \mathcal{B}_{p,q}, p+q = k, 0 \le q < p \le k\} \cup \{R(h_1), \dots, R(h_{m+1})\},$$

where $h_1, \dots, h_{m+1}$ are chosen such that the set

$$\{h_1, M(h_1), \dots, h_{m+1}, M(h_{m+1})\}$$

forms a complex basis of $\mathcal{H}_{m,m+1}(S)$.

If the bases $\mathcal{B}_{p,q}$ are orthogonal in $L^2(S)$ and $h_1, \dots, h_{m+1} \in \mathcal{H}_{m,m+1}(S)$ are mutually orthogonal, then $\mathcal{B}_k$ is orthogonal, with norms

$$\|R(h)\|_{L^2(S,\mathbb{H})} = \left(\frac{p+q+1}{p+1}\right)^{1/2} \|h\|_{L^2(S)} \quad (h \in \mathcal{B}_{p,q})$$

w.r.t. the scalar product of $L^2(S, \mathbb{H})$.

Proof. From dimension count, it suffices to prove that the sets $\mathcal{B}_k$ are linearly independent. When $q \le p$, $q' \le p'$, $p+q = p'+q' = k$, the spaces $\mathcal{H}_{p,q}$ and $\mathcal{H}_{q'-1,p'+1}$ are distinct. Since $R(h) = h + M(h)j \in \mathcal{H}_{p,q} \oplus \mathcal{H}_{q-1,p+1}j$, this implies the independence over $\mathbb{H}$ of the images $\{R(h) : h \in \mathcal{B}_{p,q}\}$. It remains to consider the case when $k = 2m+1$ is odd. If $h \in \mathcal{H}_{m,m+1}(S)$, the complex components h and $M(h)$ of $R(h)$ belong to the same space. The independence of $\{R(h_1), \dots, R(h_{m+1})\}$ over $\mathbb{H}$ follows from the particular form of the complex basis chosen in $\mathcal{H}_{m,m+1}(S)$.

The scalar product of $L(h)$ and $L(h')$ in $\mathcal{H}_{p,q}(S)$ is

$$(L(h), L(h')) = (h, L^* L(h')) = -(h, \overline{L} L(h')) = q(p+1)(h, h'),$$

since the adjoint L^* is equal to $-\overline{L}$ (cf. [7]§18.2.2) and $\overline{L}L = q(p+1)M^2 = -q(p+1)Id$. Therefore, if h, h' are orthogonal, $M(h)$ and $M(h')$ are orthogonal in $\mathcal{H}_{q-1,p+1}$ and then also $R(h)$ and $R(h')$. Finally, the norm of $R(h)$, $h \in \mathcal{H}_{p,q}(S)$, is

$$\|R(h)\|^2 = \|h\|^2 + \|M(h)\|^2 = \|h\|^2 + \frac{1}{(p+1)^2}\|L(h)\|^2 = \frac{p+q+1}{p+1}\|h\|^2$$

and this concludes the proof. □

From Theorem 5 it is immediate to obtain also bases of the right $\mathbb{H}$-module U_k of left-regular homogeneous polynomials of degree k.

Examples. (i) The case $k = 2$. Starting from the orthogonal bases $\mathcal{B}_{2,0} = \{z_1^2, 2z_1z_2, z_2^2\}$ of $\mathcal{H}_{2,0}$ and $\mathcal{B}_{1,1} = \{z_1\bar{z}_2, |z_1|^2 - |z_2|^2, z_2\bar{z}_1\}$ of $\mathcal{H}_{1,1}$ we get the orthogonal basis of regular harmonics

$$\mathcal{B}_2 = \{z_1^2, 2z_1z_2, z_2^2, z_1\bar{z}_2 - \frac{1}{2}\bar{z}_1^2 j, |z_1|^2 - |z_2|^2 + \bar{z}_1\bar{z}_2 j, z_2\bar{z}_1 + \frac{1}{2}\bar{z}_2^2 j\}$$

of the six-dimensional right $\mathbb{H}$-module U_2^ψ.

(ii) The case $k = 3$. From the orthogonal bases

$\mathcal{B}_{3,0} = \{z_1^3, 3z_1^2z_2, 3z_1z_2^2, z_2^3\}$, $\quad \mathcal{B}_{2,1} = \{z_1^2\bar{z}_2, 2z_1|z_2|^2 - z_1|z_1|^2, 2z_2|z_1|^2 - z_2|z_2|^2, z_2^2\bar{z}_1\}$,

$\mathcal{B}_{1,2} = \{h_1 = z_1\bar{z}_2^2, M(h_1) = -z_2\bar{z}_1^2, h_2 = -2\bar{z}_2|z_1|^2 + \bar{z}_2|z_2|^2, M(h_2) = -2\bar{z}_1|z_2|^2 + \bar{z}_1|z_1|^2\}$,

we get the orthogonal basis of regular harmonics

$$\mathcal{B}_3 = \{z_1^3, 3z_1^2z_2, 3z_1z_2^2, z_2^3, z_1^2\bar{z}_2 - \frac{1}{3}\bar{z}_1^3 j, 2z_1|z_2|^2 - z_1|z_1|^2 - \bar{z}_1^2\bar{z}_2 j, 2z_2|z_1|^2 - z_2|z_2|^2 + \bar{z}_1\bar{z}_2^2 j,$$
$$z_2^2\bar{z}_1 + \frac{1}{3}\bar{z}_2^3 j, z_1\bar{z}_2^2 - z_2\bar{z}_1^2 j, -2\bar{z}_2|z_1|^2 + \bar{z}_2|z_2|^2 + (\bar{z}_1|z_1|^2 - 2\bar{z}_1|z_2|^2 +)j\}.$$

of the ten-dimensional right $\mathbb{H}$-module U_3^ψ.

In general, for any k, an orthogonal basis of $\mathcal{H}_{p,q}$ $(p + q = k)$ is given by the polynomials $\{P_{q,l}^k\}_{l=0,\dots,k}$ defined by formula (6.14) in Sudbery [8]. The basis of U_k obtained from these bases by means of Theorem 5 and applying the reflection γ is essentially the same given in Proposition 8 of Sudbery [8].

Another spanning set of the space $\mathcal{H}_{p,q}$ is given by the functions

$$g_\alpha^{p,q}(z_1, z_2) = (z_1 + \alpha z_2)^p(\bar{z}_2 - \alpha\bar{z}_1)^q \quad (\alpha \in \mathbb{C})$$

(cf. Rudin [7]§12.5.1). Since $M(g_\alpha^{p,q}) = \frac{(-1)^q q\bar{\alpha}^{p+q}}{p+1} g_{-1/\bar{\alpha}}^{q-1,p+1}$ for $\alpha \neq 0$ and $M(g_0^{p,q}) = -\frac{q}{p+1} z_2^{q-1}\bar{z}_1^{p+1}$, where we set $g_\alpha^{p,q} \equiv 0$ if $p < 0$, from Theorem 5 we get that U_k^ψ is spanned over $\mathbb{H}$ by the polynomials

$$R(g_\alpha^{p,q}) = \begin{cases} g_\alpha^{p,q} + \frac{(-1)^q q\bar{\alpha}^{p+q}}{p+1} g_{-1/\bar{\alpha}}^{q-1,p+1} j & \text{for } \alpha \neq 0 \\ z_1^p\bar{z}_2^q - \frac{q}{p+1} z_2^{q-1}\bar{z}_1^{p+1} j & \text{for } \alpha = 0 \end{cases} \qquad (\alpha \in \mathbb{C},\ p + q = k)$$

Any choice of $k+1$ distinct numbers $\alpha_0, \alpha_1, \ldots, \alpha_k$ gives rise to a basis of U_k^ψ.

The results obtained in this paper enabled the writing of a *Mathematica* package [6], named `RegularHarmonics`, which implements efficient computations with regular and ψ-regular functions and with harmonic and holomorphic functions of two complex variables.

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