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Introduction

One of the most challenging problems in physics is the connection between the macroscopic and the microscopic world, that is between classical and quantum mechanics. In principle a macroscopic system should be described as a collection of microscopic ones, so that classical mechanics should be derived from quantum theory by means of suitable approximations. At a first glance the solution of the problem is not straightforward: indeed there are deep differences between the classical and the quantum description of the physical world.
In classical mechanics the state of an elementary physical system, for instance a point particle, is given by specifying its position q (a point in its configuration space) and its velocity $\dot{q}$. The time evolution in the time interval $[t_0, t]$ is given by a a path $q(s)_{s\in[t_0,t]}$ in the configuration space, which is determined by the Hamilton's least action principle:

$$\delta S_t(q) = 0, \quad S_t(q) = \int_{t_0}^{t} \mathcal{L}(q(s), \dot{q}(s))ds.$$

$S(q)$ denotes the action functional, $\mathcal{L}$ is the Lagrangian of the system.
In quantum mechanics the state of a d-dimensional particle is represented by a unitary vector ψ in the complex separable Hilbert space $L^2(\mathbb{R}^d)$, the so-called "wave function", while is time evolution is described by the Sc-hrödinger equation:

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi + V\psi \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{1}$$

where $\hbar$ is the reduced Planck constant, $m > 0$ is the mass of the particle and $F = -\nabla V$ is an external force. It is important to recall that in quantum mechanics, because of Heisenberg's uncertainty principle, there are observables which are "incompatible": the measurement of one destroys the information about the measurement of the other. Position and velocity are the typical example of a couple of incompatible observables, as a consequence the concept of trajectory makes no sense in quantum theory.

In 1942 R.P. Feynman [59], following a suggestion by Dirac [54], proposed an heuristic but very suggestive representation for the solution of the Schrödinger equation. Feynman's original aim was to give a Lagrangian formulation of quantum mechanics and to introduce in it the concept of trajectory.

According to Feynman the wave function of the system at time t evaluated at the point $x \in \mathbb{R}^d$ is given as an "integral over histories", or as an integral over all possible paths γ in the configuration space of the system with finite energy passing in the point x at time t:

$$\psi(t,x) = \quad \text{“} \quad const \int_{\{\gamma|\gamma(t)=x\}} e^{\frac{i}{\hbar}S_t(\gamma)}\psi_0(\gamma(0))D\gamma \quad \text{”} \tag{2}$$

$S_t(\gamma)$ is the classical action of the system evaluated along the path γ

$$S_t(\gamma) \equiv S_t^\circ(\gamma) - \int_0^t V(\gamma(s))ds, \tag{3}$$

$$S_t^\circ(\gamma) \equiv \frac{m}{2}\int_0^t |\dot{\gamma}(s)|^2 ds, \tag{4}$$

and $D\gamma$ is an heuristic Lebesgue "flat" measure on the space of paths. Formula (2) lacks of rigor: indeed neither the "infinite dimensional Lebesgue measure", nor the normalization constant in front of the integral are well defined. Nevertheless even if more than 50 years have passed since Feynman's original proposal, formula (2) is still fascinating. First of all it creates a connection between the classical description of the physical world and the quantum one. Indeed it allows, at least heuristically, to associate a quantum evolution to each classical Lagrangian. Moreover an heuristic application of the stationary phase method for oscillatory integrals allows the study of the behavior of the solution of the Schrödinger equation taking into account that $\hbar$ is small. Indeed the integrand is strongly oscillating and the main contributions to the integral should come from those paths γ that make stationary the phase function $S(\gamma)$. These, by Hamilton's least action principle, are exactly the classical orbits of the system.
Inspired by Feynman's work some time later Kac [70, 71] noted that that by considering the heat equation

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta u(t,x) - Vu(t,x) \\ u(0,x) = u_0(x) \end{cases} \tag{5}$$

instead of equation (1), it is possible to replace the heuristic expression (2) with a well defined integral on the space of continuous paths with respect to the Wiener measure W:

$$u(t,x) = \int_{\omega(0)=0} u_0(\omega(t)+x)e^{-\int_0^t V(\omega(s)+x)ds}dW(\omega) \tag{6}$$

Such an interpretation is not possible for the heuristic "Feynman measure" $e^{\frac{i}{\hbar}S_t(\gamma)}D\gamma$. Indeed Cameron [40] proved that the latter cannot be realized as

a complex $\sigma-$ additive measure, even on very nice subsets.
As a consequence mathematicians tried to realize it as a linear continuous functional on a sufficiently rich Banach algebra of functions. In order to mirror the features of the heuristic Feynman measure, such a functional should have some properties:

1. It should behave in a simple way under "translations and rotations in path space", reflecting the fact that $D\gamma$ is a "flat" measure.

2. It should satisfy a Fubini type theorem, concerning iterated integrations in path space.

3. It should be approximable by finite dimensional oscillatory integrals, allowing a sequential approach in the spirit of Feynman's original work.

4. It should be related to probabilistic integrals with respect to the Wiener measure, allowing an "analytic continuation approach to Feynman path integrals from Wiener type integrals".

5. It should be sufficiently flexible to allow a rigorous mathematical implementation of an infinite dimensional version of the stationary phase method and the corresponding study of the semiclassical limit of the quantum mechanics.

Nowadays several implementation of this program can be found in the physical and in the mathematical literature, for instance by means of analytic continuation of Wiener integrals [40, 83, 69, 97, 72, 55, 77, 82, 100, 45, 95], or as an infinite dimensional distribution in the framework of Hida calculus [63, 52], via "complex Poisson measures" [78, 1], or via non standard analysis [7] or as a infinite dimensional oscillatory integral. The latter method is particularly interesting as it is the only one by which a development of an infinite dimensional stationary phase method has been performed. Such an approach has its roots in a work by Ito [67, 68] and was developed by S. Albeverio and R. Høegh-Krohn [12, 13], D.Elworthy and A.Truman [57], S. Albeverio and Z. Brzeźniak [4, 5]. Indeed, when the potential V is of the following form

$$V(x) = \frac{1}{2}x \cdot \Omega^2 x + V_1(x), \tag{7}$$

where Ω^2 is a positive definite symmetric $d \times d$ matrix and V_1 is the Fourier transform of a complex bounded variation measure on $\mathbb{R}^d$, Albeverio and Høegh-Krohn define in [12] the Feynman integral as a functional on a suitable Hilbert space of paths by means of a Parseval type formula (previous work in this direction is due to K. Ito). In [57] Elworthy and Truman define the Feynman functional by means of a sequential approach. The "infinite

dimensional oscillatory integral" they propose is defined as the limit of a sequence of finite dimensional oscillatory integrals. They can also prove that, for the class of function considered in [12], the infinite dimensional oscillatory integral can be explicitly computed by means of the Parseval type formula proposed by Albeverio and Høegh-Krohn. Such an approach allows a rigorous implementation of an infinite dimensional version of the stationary phase method and was further developed in [13, 5] and [2, 3] in connection with the study of the asymptotic behavior of the integral in the limit $\hbar \downarrow 0$.
In this thesis we show some new developments of the infinite dimensional oscillatory integrals-Feynman path integrals theory.

- In the first chapter we recall the definitions of finite and infinite dimensional oscillatory integrals and their application to the rigorous mathematical realization of the Feynman functional and the representation of the solution of the Schrödinger equation.

- In the second chapter we show that the infinite dimensional oscillatory integrals are a flexible tool and can be used to give a rigorous mathematical realization of an Hamiltonian version of the Feynman heuristic formula, a "phase space Feynman path integral". We prove that under suitable assumptions it represents the solution of a Schrödinger equation in which the classical potential V depends both on position and on momentum.

- In the third chapter we show that it is possible to generalize the definition of infinite dimensional oscillatory integrals in order to deal with complex-valued phase functions. We apply such a functional to the solution of a stochastic Schrödinger equation appearing in the theory of continuous quantum measurement: the Schrödinger-Belavkin equation.

- In the fourth chapter we focus on the finite dimensional case and prove that it is possible to enlarge the class of phase functions for which the corresponding finite dimensional oscillatory integral can be explicitly computed in terms of an absolutely convergent integral. In the particular case where the phase function is an homogeneous even polynomial, we give the detailed asymptotic expansion of the oscillatory integrals in fractional powers of the small parameter $\hbar$ and give conditions for either the convergence or the Borel summability of the expansion.

- In the fifth chapter we generalize some results of the fourth chapter to the infinite dimensional case. We show that when the phase function is the sum of a quadratic term plus a quartic perturbation, the corresponding infinite dimensional oscillatory integral can still be defined and computed in terms of an absolutely convergent integral with respect to a "true measure" on the space of paths. Such abstract result is then applied to the representation of the solution of the Schrödinger equation with a classical potential V of the type "harmonic oscillator plus quartic perturbation", that is $V(x) = \frac{\Omega^2}{2}x^2 + \lambda x^4$, $\lambda \geq 0$. Moreover under suitable assumptions, we prove the Borel summability of the asymptotic expansion of the solution in a power series of the coupling constant λ.

Chapter 1

Oscillatory integrals and the Schrödinger equation

In this chapter we recall some known results, that is the definitions of finite- and infinite dimensional oscillatory integrals and the main theorems about them, for more details we refer to [12, 57, 4]. In the following we will denote by $\mathcal{H}$ a (finite or infinite dimensional) real separable Hilbert space, whose elements are denoted by $x, y \in \mathcal{H}$ and the scalar product with $\langle x, y\rangle$. We are going to define the oscillatory integral on the Hilbert space $\mathcal{H}$

$$\int_{\mathcal{H}} e^{\frac{i}{\hbar}\Phi(x)} f(x)dx, \qquad (1.1)$$

where $\hbar$ is a non vanishing real parameter, Φ and f are respectively suitable real-valued and complex-valued smooth functions. We remark that even in finite dimensions ($\dim(\mathcal{H}) = n$) the integral (4.1) in general is not well defined in Lebesgue sense, unless $\int_{\mathbb{R}^N} |f(x)|dx < +\infty$. The study of finite dimensional oscillatory integrals of the above form is already a classical topic, largely developed in connections with various problems in mathematics and physics. Well known examples of simple integrals of the above form are the Fresnel integrals of the theory of wave diffraction and Airy's integrals of the theory of rainbow, see e.g. [34]. The theory of Fourier integral operators [64, 65, 78] also grew out of the investigation of oscillatory integrals. It allows the study of existence and regularity of a large class of elliptic and pseudoelliptic operators and provides constructive tools for the solutions of the corresponding equations. In particular one is interested in discussing the asymptotic behavior of the above integrals when the parameter $\hbar$ is sent to 0, in a mathematical idealization. The method of stationary phase provides a tool for such investigations and has many applications, such as the study of the classical limit of quantum mechanics (see [58, 13, 4]). In the general case of degenerate critical points of the phase function Φ, the theory of unfoldings of singularities is applied, see

[24, 56].
The extensions of the definition of oscillatory integrals on an infinite dimensional Hilbert space $\mathcal{H}$ and the implementation of a corresponding infinite-dimensional version of the stationary phase method has a particular interest in connection with the rigorous mathematical definition of the "Feynman path integrals".
The definition of such integrals is divided into two main steps: first of all a finite dimensional oscillatory integral is defined as the limit of a sequence of absolutely convergent integrals. In the second step the infinite dimensional oscillatory integral is defined as the limit of a sequence of finite dimensional oscillatory integrals.
In the first and in the second sections, we shall recall the main result on the finite and respectively infinite dimensional oscillatory integrals. In the third section we shall show how the latter can be applied to give a rigorous mathematical meaning to Feynman's heuristic formula (2) and to represent the solution of the Schrödinger equation (1).

1.1 Finite dimensional oscillatory integrals

Let us assume that $\mathcal{H} = \mathbb{R}^n$ and define the oscillatory integral [64, 65]

$$\int_{\mathbb{R}^N} e^{\frac{i}{\hbar}\Phi(x)} f(x) dx.$$

In the whole chapter $\hbar > 0$ is a fixed parameter (we call it $\hbar$ because of its interpretation in the context of applications to quantum mechanics). The following definition is taken from [57] and is a modification of one given in [65].

Definition 1. *The oscillatory integral of function $f : \mathbb{R}^n \to \mathbb{C}$ with respect to a phase function Φ is well defined if and only if for each test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\phi(0) = 1$ the limit*

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^N} e^{\frac{i}{\hbar}\Phi(x)} f(x) \phi(\epsilon x) dx \tag{1.2}$$

exists and is independent of ϕ. In this case the limit is called the oscillatory of f with respect to Φ and denoted by $\int_{\mathbb{R}^N} e^{\frac{i}{\hbar}\Phi(x)} f(x) dx$

The particular case in which the phase function Φ is a quadratic form is well studied. This particular type of oscillatory integrals are called "Fresnel integrals". In this case it is convenient to include into the definition of the oscillatory integral the "multiplication factor" $(2\pi i\hbar)^{-dim(\mathcal{H})/2}$, which will be useful in the extension of such a definition to the infinite dimensional case. Let us denote by Q an invertible symmetric operator from $\mathbb{R}^N$ to $\mathbb{R}^N$ and define the "Fresnel integral"

$$\int e^{\frac{i}{2\hbar}\langle x,Qx\rangle} f(x)dx$$

Definition 2. *A function $f:\mathbb{R}^n \to \mathbb{C}$ is called Fresnel integrable with respect to Q if and only if for each $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\phi(0)=1$ the limit*

$$\lim_{\epsilon\to 0}(2\pi i\hbar)^{-n/2}\int e^{\frac{i}{2\hbar}\langle x,Qx\rangle} f(x)\phi(\epsilon x)dx \tag{1.3}$$

exists and is independent of ϕ. In this case the limit is called the Fresnel integral of f with respect to Q and denoted by

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,Qx\rangle} f(x)dx \tag{1.4}$$

The description of the full class of Fresnel integrable functions is not easy, but one can find some interesting subsets of it.
Let us consider the space $\mathcal{M}(\mathbb{R}^N)$ of complex bounded variation measures on $\mathbb{R}^N$ endowed with the total variation norm. $\mathcal{M}(\mathbb{R}^N)$ is a Banach algebra, where the product of two measures $\mu * \nu$ is by definition their convolution:

$$\mu * \nu(E) = \int_{\mathbb{R}^N} \mu(E-x)\nu(dx), \qquad \mu,\nu \in \mathcal{M}(\mathbb{R}^N)$$

and the unit element is the Dirac measure δ_0.
Let $\mathcal{F}(\mathbb{R}^N)$ be the space of functions $f:\mathbb{R}^N \to \mathbb{C}$ which are the Fourier transforms of complex bounded variation measures $\mu_f \in \mathcal{M}(\mathbb{R}^N)$:

$$f(x) = \int_{\mathbb{R}^N} e^{ik\cdot x}\mu_f(dk), \qquad \mu_f \in \mathcal{M}(\mathbb{R}^N).$$

One can prove that when $\Phi(x) = \frac{1}{2}\langle x,Qx\rangle$ and $f \in \mathcal{F}(\mathbb{R}^n)$, then the Fresnel integral of f with respect to Q is well defined and can be computed by means of a well defined integral, which is convergent in Lebesgue's sense.

Theorem 1. *Let $f \in \mathcal{F}(\mathbb{R}^n)$, then f is Fresnel integrable and its Fresnel integral with respect to Q is given by:*

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,Qx\rangle} f(x)dx = (\det Q)^{-1/2}\int e^{\frac{-i\hbar}{2}\langle \alpha,Q^{-1}\alpha\rangle}\mu_f(d\alpha). \tag{1.5}$$

where $(\det Q)^{1/2} = (|\det Q|)^{1/2}e^{i\pi Ind(Q)/2}$, $Ind(Q)$ being the number of negative eigenvalues of the operator Q, counted with their multiplicity.

For the proof see [57], see also [12, 4].
In an analogous way one can define the "normalized Fresnel integral" by introducing a normalization factor:

Definition 3. *A function $f : \mathbb{R}^n \to \mathbb{C}$ is called Fresnel integrable with respect to Q if and only if for each $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\phi(0) = 1$ the limit*

$$\lim_{\epsilon\to 0}(2\pi i\hbar)^{-n/2}(\det Q)^{\frac{1}{2}} \int e^{\frac{i}{2\hbar}\langle x,Qx\rangle} f(x)\phi(\epsilon x)dx \tag{1.6}$$

exists and is independent of ϕ. In this case the limit is called the normalized Fresnel integral of f with respect to Q and denoted by

$$\widetilde{\int}^{Q} e^{\frac{i}{2\hbar}\langle x,Qx\rangle} f(x)dx \tag{1.7}$$

One can easily see that f is Fresnel integrable with respect to Q in the sense of definition 2 if and only if f is Fresnel integrable with respect to Q in the sense of definition 3 and the two Fresnel integrals are related by a multiplication factor:

$$\widetilde{\int}^{Q} e^{\frac{i}{2\hbar}\langle x,Qx\rangle} f(x)dx = (\det Q)^{\frac{1}{2}} \widetilde{\int} e^{\frac{i}{2\hbar}\langle x,Qx\rangle} f(x)dx \tag{1.8}$$

Theorem 1 in this case assumes the following form:

Theorem 2. *Let $f \in \mathcal{F}(\mathbb{R}^n)$, then f is Fresnel integrable and its normalized Fresnel integral with respect to Q is given by:*

$$\widetilde{\int}^{Q} e^{\frac{i}{2\hbar}\langle x,Qx\rangle} f(x)dx = \int e^{\frac{-i\hbar}{2}\langle \alpha,Q^{-1}\alpha\rangle}\mu_f(d\alpha). \tag{1.9}$$

Note that if we substitute into the latter the function $f = 1$, we have $\widetilde{\int}^{Q} e^{\frac{i}{2\hbar}\langle x,Qx\rangle} f(x)dx = 1$. For this reason the integral of definition 3 is called "normalized".
The choice of a suitable normalization factor, that is the choice between definition 2 and definition 3, will be important in the extension of the theory to the infinite dimensional case.

1.2 Infinite dimensional oscillatory integrals

Let us consider an infinite dimensional real separable Hilbert space $\mathcal{H}$ and an invertible, densely defined and self-adjoint operator Q on $\mathcal{H}$. The infinite dimensional oscillatory integral on $\mathcal{H}$ with quadratic phase function $\frac{1}{2}\langle x, Qx\rangle$ is defined as the limit of a sequence of finite dimensional oscillatory integrals (defined in the previous section) [57, 4].

Definition 4. *A function $f : \mathcal{H} \to \mathbb{C}$ is called Fresnel integrable with respect to Q if and only for each sequence $\{P_n\}_{n\in\mathbb{N}}$ of projectors onto n-dimensional subspaces of $\mathcal{H}$, such that $P_n \leq P_{n+1}$ and $P_n \to I$ strongly as $n \to \infty$, (I being the identity operator in $\mathcal{H}$), the finite dimensional approximations of the Fresnel integral of f with respect to Q*

$$\widetilde{\int_{P_n\mathcal{H}}} e^{\frac{i}{2\hbar}\langle P_n x, QP_n x\rangle} f(P_n x) d(P_n x),$$

are well defined (in the sense of definition 2) and the limit

$$\lim_{n\to\infty} \widetilde{\int_{P_n\mathcal{H}}} e^{\frac{i}{2\hbar}\langle P_n x, QP_n x\rangle} f(P_n x) d(P_n x) \tag{1.10}$$

exists and is independent on the sequence $\{P_n\}$.
In this case the limit is called the Fresnel integral of f with respect to Q and is denoted by

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x, Qx\rangle} f(x) dx$$

It is not easy to characterize the full class of integrable functions, but under suitable assumptions on the operator Q it is possible to generalize theorem 1 to the infinite dimensional case.
Let us denote by $\mathcal{M}(\mathcal{H})$ the Banach space of the complex bounded variation measures on $\mathcal{H}$, endowed with the total variation norm, that is:

$$\mu \in \mathcal{M}(\mathcal{H}), \qquad \|\mu\| = \sup \sum_i |\mu(E_i)|,$$

where the supremum is taken over all sequences $\{E_i\}$ of pairwise disjoint Borel subsets of $\mathcal{H}$, such that $\cup_i E_i = \mathcal{H}$. $\mathcal{M}(\mathcal{H})$ is a Banach algebra, where the product of two measures $\mu * \nu$ is by definition their convolution:

$$\mu * \nu(E) = \int_{\mathcal{H}} \mu(E - x)\nu(dx), \qquad \mu, \nu \in \mathcal{M}(\mathcal{H})$$

and the unit element is the vector δ_0.
Let $\mathcal{F}(\mathcal{H})$ be the space of complex functions on H which are Fourier transforms of measures belonging to $\mathcal{M}(\mathcal{H})$, that is:

$$f : \mathcal{H} \to \mathbb{C} \qquad f(x) = \int_H e^{i\langle x,\beta\rangle} \mu_f(d\beta) \equiv \hat{\mu}_f(x).$$

$\mathcal{F}(\mathcal{H})$ is a Banach algebra of functions, where the product is the pointwise one, the unit element is the function 1, i.e. $1(x) = 1\ \forall x \in \mathcal{H}$, and the norm is given by $\|f\| = \|\mu_f\|$.

The following result holds:

Theorem 3. *Let us assume that $f \in \mathcal{F}(\mathcal{H})$ and $(Q - I)$ is a trace class operator (I being the identity operator). Then f is Fresnel integrable with respect to Q and the corresponding Fresnel integral is given by the following Cameron Martin-Parseval type formula:*

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,Qx\rangle} f(x)dx = (\det Q)^{-1/2} \int_{\mathcal{H}} e^{-\frac{i\hbar}{2}\langle \alpha,Q^{-1}\alpha\rangle} \mu_f(d\alpha) \tag{1.11}$$

where $\det Q = |\det Q| e^{-\pi i \,\mathrm{Ind}\, Q}$ *is the Fredholm determinant of the operator Q,* $|\det Q|$ *its absolute value and Ind(Q) is the number of negative eigenvalues of the operator Q, counted with their multiplicity.*

For the proof see [4, 57].
In an analogous way it is possible to define the normalized infinite dimensional oscillatory integral as the limit of a sequence of finite dimensional oscillatory integrals (in the sense of definition 3)

Definition 5. *A function $f : \mathcal{H} \to \mathbb{C}$ is called Fresnel integrable with respect to Q if and only for each sequence $\{P_n\}_{n\in\mathbb{N}}$ of projectors onto n-dimensional subspaces of $\mathcal{H}$, such that $P_n \leq P_{n+1}$ and $P_n \to I$ strongly as $n \to \infty$, (I being the identity operator in $\mathcal{H}$), the finite dimensional approximations of the Fresnel integral of f with respect to Q*

$$\widetilde{\int_{P_n\mathcal{H}}^{P_nQP_n}} e^{\frac{i}{2\hbar}\langle P_nx,QP_nx\rangle} f(P_nx)d(P_nx),$$

are well defined (in the sense of definition 3) and the limit

$$\lim_{n\to\infty} \widetilde{\int_{P_n\mathcal{H}}^{P_nQP_n}} e^{\frac{i}{2\hbar}\langle P_nx,QP_nx\rangle} f(P_nx)d(P_nx) \tag{1.12}$$

exists and is independent on the sequence $\{P_n\}$.
In this case the limit is called the normalized Fresnel integral of f with respect to Q and is denoted by

$$\widetilde{\int_{\mathcal{H}}^{Q}} e^{\frac{i}{2\hbar}\langle x,Qx\rangle} f(x)dx$$

In this case, if $f \in \mathcal{F}(\mathcal{H})$, then it is possible to prove a formula similar to (1.11) even if $Q - I$ is not trace class:

Theorem 4. *Let us assume that $f \in \mathcal{F}(\mathcal{H})$. Then f is Fresnel integrable with respect to Q (in the sense of definition 5 and the corresponding normalized Fresnel integral is given by the following Cameron-Martin-Parseval type formula:*

$$\widetilde{\int_{\mathcal{H}}^{Q}} e^{\frac{i}{2\hbar}\langle x,Qx\rangle} f(x)dx = \int_{\mathcal{H}} e^{-\frac{i\hbar}{2}\langle \alpha,Q^{-1}\alpha\rangle} \mu_f(d\alpha). \tag{1.13}$$

Such a result shows that in the infinite dimensional case the normalization constant in the finite dimensional approximations plays a crucial role and definitions 4 and 5 are not equivalent. Indeed definition 5 and theorem 4 make sence even if the operator $Q-I$ is not trace class. In fact it is possible to introduce different normalization constants in the finite dimensional approximations and the properties of corresponding infinite dimensional oscillatory integrals are related to the trace properties of $Q-I$ and its powers [5]. More precisely, for any $p \in \mathbb{N}$, let us consider the Schatten class $\mathcal{T}_p(\mathcal{H})$ of bounded linear operators L in $\mathcal{H}$ such that

$$\|L\|_p = (\mathrm{Tr}(L^*L)^{p/2})^{1/p}$$

is finite. $(\mathcal{T}_p(\mathcal{H}), \|\cdot\|_p)$ is a Banach space. For any $p \in \mathbb{N}$ and $L \in \mathcal{T}_p(\mathcal{H})$ one defines the regularized Fredholm determinant $\det_{(p)} : I + \mathcal{T}_p(\mathcal{H}) \to \mathbb{R}$:

$$\det_{(p)}(Q) = \det\Big(Q \exp \sum_{j=0}^{p-1} \frac{(I-Q)^j}{j}\Big), \qquad Q = I - L,\ L \in \mathcal{T}_p(\mathcal{H}),$$

where det denotes the usual Fredholm determinant. $\det_{(2)}$ is called Carleman determinant.

For $p \in \mathbb{N}$, $(Q-I) \in \mathcal{T}_1(\mathcal{H})$ let us define the normalized quadratic form on $\mathcal{H}$:

$$N_p(Q)(x) = \langle x, Qx\rangle + i\hbar \mathrm{Tr} \sum_{j=0}^{p-1} \frac{(Q-I)^j}{j}, \qquad x \in \mathcal{H} \tag{1.14}$$

For generic $p \in \mathbb{N}$ let us define the *class p normalized oscillatory integral*:

Definition 6. *Let $p \in \mathbb{N}$, Q a bounded linear operator in $\mathcal{H}$, $f : \mathcal{H} \to \mathbb{C}$. The class p normalized oscillatory integral of the function f with respect to the operator Q is well defined if for each sequence $\{P_n\}_{n\in\mathbb{N}}$ of projectors onto n-dimensional subspaces of $\mathcal{H}$, such that $P_n \leq P_{n+1}$ and $P_n \to I$ strongly as $n \to \infty$, (I being the identity operator in $\mathcal{H}$), the normalized finite dimensional approximations*

$$\widetilde{\int_{P_n\mathcal{H}}} e^{\frac{i}{2\hbar} N_p(P_nQP_n)(P_nx)} f(P_nx)d(P_nx), \tag{1.15}$$

are well defined (in the sense of definition 2 and the limit

$$\lim_{n\to\infty} \widetilde{\int_{P_n\mathcal{H}}} e^{\frac{i}{2\hbar} N_p(P_nQP_n)(P_nx)} f(P_nx)d(P_nx) \tag{1.16}$$

exists and is independent on the sequence $\{P_n\}$.
In this case the limit is denoted by

$${}^p\widetilde{\int_{\mathcal{H}}^{Q}} e^{\frac{i}{2\hbar}\langle x, Qx\rangle} f(x)dx$$

If $Q-I$ is not a trace class operator, then the quadratic form (1.14) is not well defined and the right hand side of (1.15) makes sense thanks to the fact that all the function are restricted on finite dimensional subspaces. Nevertheless the limit (1.16) can make sense, as the following result shows.

Theorem 5. *Let us assume that $f \in \mathcal{F}(\mathcal{H})$, $(Q-I) \in \mathcal{T}_p(\mathcal{H})$ and $\det_{(p)}(Q) \neq 0$. Then the class-p normalized oscillatory integral of the function f with respect to the operator Q exists and is given by the following Cameron-Martin-Parseval type formula:*

$$ {}^{p}\widetilde{\int_{\mathcal{H}}^{Q}} e^{\frac{i}{2\hbar}\langle x, Qx\rangle} f(x)dx = [\det_{(p)}(Q)]^{-1/2} \int_{\mathcal{H}} e^{-\frac{i\hbar}{2}\langle \alpha, Q^{-1}\alpha\rangle} \mu_f(d\alpha). \tag{1.17} $$

1.3 Application to the Schrödinger equation

In the setting explained in section 1.2 one can give a rigorous mathematical interpretation of formula (2) in terms of an infinite dimensional oscillatory integral on a suitable Hilbert space of paths.

Let us consider the so-called Cameron-Martin space H_t, that is the space of absolutely continuous functions $\gamma : [0,t] \to \mathbb{R}^d$, $\gamma(t) = 0$, such that $\int_0^t |\dot\gamma(s)|^2 ds < \infty$, endowed with the following scalar product

$$ \langle \gamma_1, \gamma_2 \rangle = \int_0^t \dot\gamma_1(s) \cdot \dot\gamma_2(s) ds, $$

Let us consider the Schrödinger equation in $L^2(\mathbb{R}^d)$

$$ i\hbar \frac{\partial}{\partial t}\psi = H\psi \tag{1.18} $$

with initial datum $\psi_{|t=0} = \psi_0$ and quantum mechanical Hamiltonian $H = -\frac{\hbar^2}{2}\Delta + \frac{1}{2}x\Omega^2 x + V_1(x)$, where $x \in \mathbb{R}^d$, $\Omega^2 \geq 0$ is a positive $d \times d$ matrix, $V_1 \in \mathcal{F}(\mathbb{R}^d)$ and $\psi_0 \in \mathcal{F}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

By considering the operator $Q = I - L$ on H_t given by

$$ \langle \gamma, L\gamma \rangle \equiv \int_0^t \gamma(s)\Omega^2\gamma(s) ds, $$

and the function $v : H_t \to \mathbb{C}$

$$ v(\gamma) \equiv \int_0^t V_1(\gamma(s)+x)ds + 2x\Omega^2 \int_0^t \gamma(s)ds, \qquad \gamma \in H_t, $$

formula (2)

$$ \text{``} \quad const \int_{\{\gamma | \gamma(t)=x\}} e^{\frac{i}{\hbar}\int_0^t (\frac{1}{2}\dot\gamma(s)^2 - \frac{1}{2}\gamma(s)\Omega^2\gamma(s) - V_1(\gamma(s)))ds} \psi_0(\gamma(0)) D\gamma \quad \text{''} $$

can be interpreted as the infinite dimensional oscillatory integral on H_t (in the sense of definition 4)

$$\widetilde{\int_{H_t}} e^{\frac{i}{2\hbar}\langle\gamma,(I-L)\gamma\rangle} e^{-\frac{i}{\hbar}v(\gamma)}\psi_0(\gamma(0)+x)d\gamma. \tag{1.19}$$

By analyzing the spectrum of the operator L (see [57] for more details) one can easily verify that L is trace class and $I-L$ is invertible. The following holds:

Theorem 6. *Let $\psi_0 \in \mathcal{F}(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ and let $V_1 \in \mathcal{F}(\mathbb{R}^d)$. Then the function $f : H_t \to \mathbb{C}$ given by*

$$f(\gamma) = e^{-\frac{i}{\hbar}v(\gamma)}\psi_0(\gamma(0)+x)$$

is the Fourier transform of a complex bounded variation measure μ_f on H_t and the infinite dimensional oscillatory integral of f with respect to $Q \equiv I-L$

$$\widetilde{\int_{H_t}} e^{\frac{i}{2\hbar}\langle\gamma,(I-L)\gamma\rangle} e^{-\frac{i}{\hbar}v(\gamma)}\psi_0(\gamma(0)+x)d\gamma.$$

is well defined (in the sense of definition 4) and it is equal to

$$\det(I-L)^{-1/2}\int_{H_t} e^{\frac{-i\hbar}{2}\langle\gamma,(I-L)^{-1}\gamma\rangle}\mu_f(d\gamma).$$

Moreover it is a representation of the solution of equation (1.18) *evaluated at $x \in \mathbb{R}^d$ at time t.*

For a proof see [57].

Remark 1. *With the same tecnique it is possible to deal with potentials of the type "harmonic oscillator plus linear perturbation".*

Remark 2. *It is important to note that if $V_1 \in \mathcal{F}(\mathbb{R}^d)$, then V_1 is bounded. As a consequence the only unbounded potentials for which the Feynman functional of [12, 57, 4] can be rigorously defined are those of harmonic oscillator type. The extension to unbounded potentials which are Laplace transforms of bounded measures [6, 74] also does not cover the case of potentials which are polynomials of degree larger than 2.*

Remark 3. *The case of time-dependent potentials has been handled by means of analytic continuation in mass (see [69] Ch 14-18 for more details).*

Chapter 2

Phase Space Feynman Path integrals

Let us recall that Feynman's original aim was to give a Lagrangian formulation of quantum mechanics. On the other hand an Hamiltonian formulation could be preferable from many points of view. For instance the discussion of the approach from quantum mechanics to classical mechanics, i.e the study of the behavior of physical quantities taking into account that $\hbar$ is small, is more natural in an Hamiltonian setting (see, e.g. , [4, 78] for a discussion of this behavior). In other words the "phase space" rather then the "configuration space" is the natural framework of classical mechanics.
As a consequence one is tempted to propose a "phase space Feynman path integral" representation for the solution of the Schrödinger equation (1), that is the heuristic formula:

$$\text{``}\psi(t,x) = \text{const} \int_{q(t)=x} e^{\frac{i}{\hbar}S(q,p)}\psi_0(q(0))dqdp\text{''}. \tag{2.1}$$

Here the integral is meant on the space of paths $q(s), p(s),\ s \in [0,t]$ in the phase space of the system ($(q(s)_{s\in[0,t]}$ is the path in configuration space and $p(s)_{s\in[0,t]}$ is the path in momentum space) and S is the action functional in the Hamiltonian formulation:

$$S(q,p) = \int_0^t (\dot{q}(s)p(s) - H(q(s),p(s)))ds,$$

(H being the classical Hamiltonian of the system). The aim of this chapter is to give a rigorous mathematical realization of the heuristic formula (2.1) in terms of a well defined infinite dimensional oscillatory integral and to prove that, under suitable assumptions on the initial datum ψ_0 and on the classical potential V, it gives a representation of the solution of the Schrödinger equation (1). In particular we show that by means of this functional the case in

which the potential V depends explicitly both on position and on momentum can be handled.
We note that an approach of phase space Feynman path integrals via analytic continuation of "phase space Wiener integrals" has been presented by I. Daubechies and J. Klauder [49]. Analytic continuation was also used in other "path space" approaches, see [83, 69, 41] and references therein. Our approach is more direct in the spirit of [12].

2.1 Lie-Trotter product formula

We first recall an abstract version of the Lie-Trotter product formula.

Lemma 1. *Let A and B be self-adjoint operators in a Hilbert space $\mathcal{H}$ and let $A+B$ be essentially self-adjoint on $D(A)\cap D(B)$. Then*

$$s-\lim_{n\to\infty}(e^{itA/n}e^{itB/n})^n = e^{i(A+B)t}, \qquad t\in\mathbb{R} \tag{2.2}$$

Here $s-\lim$ is the strong operator limit[1]. For a proof and a discussion of this lemma see e.g. [43, 90].

Let $\mathcal{H}=L^2(\mathbb{R}^d)$ and let us consider a potential V depending both on the position and on the momentum in the following way: $V=V_1(x)+V_2(p)$. V_1 is defined as a self-adjoint operator is $\mathcal{H}$, with its natural domain as a multiplication operator. V_2 is the operator in $\mathcal{H}$ with domain

$$D(V_2(p))=\{\psi\in\mathcal{H}\mid \alpha\to V_2(\alpha)\hat{\psi}(\alpha)\in\mathcal{H}\}$$

where $\hat{\psi}$ is the Fourier transform of ψ. It coincides with the operator defined by functional calculus as $V_2(p)$, with p the self-adjoint operator $-i\hbar\nabla$ in $\mathcal{H}$. V is then the sum, as a self-adjoint operator in $\mathcal{H}$, of the self-adjoint operators V_1 and V_2. We assume that the functions V_1 and V_2 are such that the corresponding operators have a common dense domain of essentially self-adjointness D. This is the case, e.g., when $V_1\in L^2(\mathbb{R}^d)+L^\infty(\mathbb{R}^d)$, V_2 is bounded measurable, and $D=C_0^\infty(\mathbb{R}^d)$ or $D=\mathcal{S}(\mathbb{R}^d)$. We assume, in order to apply lemma 1, that V_1, V_2 are such that $-\frac{\hbar^2}{2m}\Delta+V_2$ and $-\frac{\hbar^2}{2m}\Delta+V_1+V_2$ are essentially self-adjoint on D. We denote by H the closure of the latter operator. H (which we also write simply as $-\frac{\hbar^2}{2m}\Delta+V_1+V_2$), is then the quantum Hamiltonian.
As a self-adjoint operator on $\mathcal{H}$, $H/\hbar$ is the generator of an one-parameter group $U(t)_{t\in\mathbb{R}}$ of unitary operators, denoted by

$$U(t)=e^{-\frac{it}{\hbar}H}=e^{-\frac{it(p^2/2m+V)}{\hbar}}.$$

[1] A sequence $(A_n)_{n\in\mathbb{N}}$ of linear operators $A_n: D\subseteq\mathcal{H}\to\mathcal{H}$ with a common domain D in a Hilbert space $(\mathcal{H},\|\cdot\|)$ converges strongly to an operator A is for each $\psi\in D$, one has $\lim_{n\to\infty}\|A_n\psi-A\psi\|=0$.

Given an initial vector $\psi_0 \in \mathcal{H}$, the solution of the Cauchy problem

$$\begin{cases} \dot{\psi} = -\frac{i}{\hbar} H\psi \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{2.3}$$

is given by $\psi(t) = e^{-\frac{it(p^2/2m+V)}{\hbar}}\psi_0$.
By lemma 1 we have

$$e^{-\frac{it(p^2/2m+V)}{\hbar}} = s - \lim_{n\to\infty} \left(e^{-\frac{i\epsilon(p^2/2m+V_2)}{\hbar}} e^{-\frac{i\epsilon(V_1)}{\hbar}} \right)^n, \qquad \epsilon \equiv \frac{t}{n}$$

$$\psi(t) = e^{-\frac{it(p^2/2m+V)}{\hbar}}\psi_0 = \lim_{n\to\infty} \left(e^{-\frac{i\epsilon(p^2/2m+V_2)}{\hbar}} e^{-\frac{i\epsilon V_1}{\hbar}} \right)^n \psi_0, \qquad \psi_0 \in C_0^\infty(\mathbb{R}^d),$$

(see e.g. [43, 91] for related uses of the Lie-Trotter formula).
By shifting from the position representation to the momentum representation and vice versa and assuming that V_1 and V_2 are continuous, we can write in the strong $L^2(\mathbb{R}^d)$-sense, for all $t > 0$:

$$\psi(t,x) = \lim_{n\to\infty} \int_{\mathbb{R}^d} e^{-\frac{i\epsilon(p_{n-1}^2/2m+V_2(p_{n-1}))}{\hbar}} \cdot$$

$$\left(e^{-\frac{i\epsilon V_1}{\hbar}} \widetilde{\left(e^{-\frac{i\epsilon(p^2/2m+V_2)}{\hbar}} e^{-\frac{i\epsilon(V_1)}{\hbar}} \right)^{n-1} \psi_0} \right)(p_1) \frac{e^{i\frac{xp_{n-1}}{\hbar}}}{(2\pi\hbar)^{d/2}} dp_{n-1}$$

$$= \lim_{n\to\infty} \int_{\mathbb{R}^{2d}} e^{-\frac{i\epsilon(p_{n-1}^2/2m+V_2(p_{n-1}))}{\hbar}} e^{-\frac{i\epsilon V_1(x_{n-1})}{\hbar}} \cdot$$

$$\left(\left(e^{-\frac{i\epsilon(p^2/2m+V_2)}{\hbar}} e^{-\frac{i\epsilon(V_1)}{\hbar}} \right)^{n-1} \psi_0 \right)(x_{n-1}) \frac{e^{i\frac{xp_{n-1}}{\hbar}}}{(2\pi\hbar)^{d/2}} \frac{e^{-i\frac{x_{n-1}p_{n-1}}{\hbar}}}{(2\pi\hbar)^{d/2}} dp_{n-1}dx_{n-1}$$

$$= \lim_{n\to\infty} \left(\frac{1}{\sqrt{2\pi\hbar}} \right)^{2nd} \cdot \tag{2.4}$$

$$\int_{\mathbb{R}^{2nd}} e^{-\frac{i\epsilon}{\hbar}\sum_{j=0}^{n-1}\left(\frac{p_j^2}{2m} + V_1(x_j) + V_2(p_j) - p_j\frac{(x_{j+1}-x_j)}{\epsilon} \right)} \psi_0(x_0) \prod_{j=0}^{n-1} dp_j dx_j,$$

where $x_n \equiv x$.

Remark 4. *The integrals above are to be understood as limits as $\Lambda \uparrow \mathbb{R}^d$, $n \to \infty$ in the $L^2(\mathbb{R}^{2nd})$ sense of the corresponding integrals over Λ^{2nd}, with Λ bounded (see [83]). Formula (2.4) holds first as a strong L^2-limit, but then(possibly by subsequences) also for Lebesgue a.e.x. in $\mathbb{R}^d$. It also follows from this that (2.4) gives the solution to the Cauchy problem (2.3).*

The latter expression suggests the following formula for the limit:

$$\psi(t,x) = \text{const} \int_{q(t)=x} e^{\frac{i}{\hbar}S(q,p)}\psi_0(q(0))dqdp \tag{2.5}$$

$$S(q,p) = \int_0^t p(s)\dot{q}(s) - H(q(s),p(s))ds$$

which does not yet have a mathematical meaning. It will be rigorously defined in the following sections.

2.2 Phase space Feynman functional

Let us consider again the expression (2.5) in the particular case of the free particle, namely when the Hamiltonian is just the kinetic energy: $H = p^2/2m$. In this case we have heuristically

$$\psi(t,x) = \text{const} \int_{q(t)=x} e^{\frac{i}{\hbar}\int_0^t (p(s)\dot{q}(s) - p(s)^2/2m)ds}\psi_0(q(0))dqdp \tag{2.6}$$

We can give to this expression a precise meaning: under suitable hypothesis on the initial wave function ψ_0, it is an infinite dimensional oscillatory integral. From now on we will assume for notational simplicity that $m = 1$, but but the whole discussion can be generalized to arbitrary m.
Following [98, 98], let us introduce the Hilbert space $\mathcal{H}_t \times \mathcal{L}_t$, namely the space of paths in the $d-$dimensional phase space $(q(s),p(s))_{s\in[0,t]}$, such that the path $(q(s))_{s\in[0,t]}$ belongs to the Cameron Martin space $\mathcal{H}_t$, namely to the space of the absolutely continuous functions q from $[0,t]$ to $\mathbb{R}^d$ such that $q(t) = 0$ and $\dot{q} \in \mathcal{L}_2([0,t],\mathbb{R}^d)$, with inner product $\langle q_1, q_2\rangle = \int_0^t \dot{q}_1(s)\dot{q}_2(s)ds$, while the path in the momentum space $(p(s))_{s\in[0,t]}$ belongs to $\mathcal{L}_t = \mathcal{L}_2([0,t],\mathbb{R}^d)$. $\mathcal{H}_t \times \mathcal{L}_t$ is an Hilbert space with the natural inner product

$$\langle q,p;Q,P\rangle = \int_0^t \dot{q}(s)\dot{Q}(s)ds + \int_0^t p(s)P(s)ds.$$

Let us introduce the following bilinear form:

$$[q,p;Q,P] =$$

$$\int_0^t \dot{q}(s)P(s)ds + \int_0^t p(s)\dot{Q}(s)ds - \int_0^t p(s)P(s)ds = \langle q,p;A(Q,P)\rangle,$$

where A is the following operator in $\mathcal{H}_t \times \mathcal{L}_t$,:

$$A(Q,P)(s) = (\int_t^s P(u)du, \dot{Q}(s) - P(s)). \tag{2.7}$$

$A(Q,P)$ is densely defined, e.g. on $C^1([0,t];\mathbb{R}^d)\times C^1([0,t];\mathbb{R}^d)$. Moreover $A(Q,P)$ is invertible with inverse given by

$$A^{-1}(Q,P)(s)=(\int_t^s P(u)du+Q(s),\dot{Q}(s)) \tag{2.8}$$

(on the range of A).
Now expression (2.5) can be realized rigorously as the normalized Fresnel integral (5):

$$\widetilde{\int_{\mathcal{H}_t\times\mathcal{L}_t}^{A}} e^{\frac{i}{2\hbar}\langle q,p;A(q,p)\rangle}\psi_0(q(0)+x)dqdp$$

where $q+x$ denotes the translated path $q(s)\to q(s)+x$.
In this case the heuristic expression (2.5) is well defined through Lie-Trotter product formula, namely as the limit of a sequence of finite dimensional integrals, as we saw in the previous section. We are going now to show that it is also the limit of a sequence of finite dimensional oscillatory integrals in the sense of definition 5.
Let us consider a sequence of partitions π_n of the interval $[0,t]$ into n subintervals of amplitude $\epsilon\equiv t/n$:

$$t_0=0,t_1=\epsilon,\dots,t_i=i\epsilon,\dots,t_n=n\epsilon=t.$$

To each π_n we associate a projector $P_n:\mathcal{H}_t\times L_t\to:\mathcal{H}_t\times L_t$ onto a finite dimensional subspace of $\mathcal{H}_t\times L_t$, namely the subspace of polygonal paths. In other words each projector P_n acts on a phase space path $(q,p)\in\mathcal{H}_t\times\mathcal{L}_t$ in the following way:

$$P_n(q,p)(s)=$$

$$\Big(\sum_{i=1}^n\chi_{[t_{i-1},t_i]}(s)\Big(q(t_{i-1})+\tfrac{(q(t_i)-q(t_{i-1})}{t_i-t_{i-1}}(s-t_{i-1})\Big),\sum_{i=1}^n\chi_{[t_{i-1},t_i]}(s)p_i\Big),$$

where $p_i=\frac{\int_{t_{i-1}}^{t_i}p(s)ds}{t_i-t_{i-1}}=\frac{1}{\epsilon}\int_{t_{i-1}}^{t_i}p(s)ds$.

Theorem 7. *For each $n\in\mathbb{N}$, P_n is a projector in $\mathcal{H}_t\times\mathcal{L}_t$. Moreover for $n\to\infty$ $P_n\to I$ as a bounded operator.*

Proof. • P_n is symmetric, indeed for all $(Q,P)\in\mathcal{H}_t\times\mathcal{L}_t$ and all $(q,p)\in\mathcal{H}_t\times\mathcal{L}_t$

$$\langle Q,P;P_n(q,p)\rangle=\int_0^t\dot{Q}(s)\sum_{i=1}^n\chi_{[t_{i-1},t_i]}(s)\frac{(q(t_i)-q(t_{i-1})}{t_i-t_{i-1}}ds+$$

$$+\int_0^t P(s)\sum_{i=1}^n \chi_{[t_{i-1},t_i]}(s)p_i ds = \sum_{i=1}^n \frac{(q(t_i)-q(t_{i-1})(Q(t_i)-Q(t_{i-1})}{t_i-t_{i-1}}+$$

$$+\sum_{i=1}^n \frac{\int_{t_{i-1}}^{t_i} p(s)ds \int_{t_{i-1}}^{t_i} P(s)ds}{t_i-t_{i-1}} = \langle P_n(Q,P);q,p\rangle$$

- $P_n^2 = P_n$, indeed

  $P_n^2(q,p)(s) =$

  $\Big(\sum_{i=1}^n \chi_{[t_{i-1},t_i]}(s)\Big(q(t_{i-1})+\frac{(q(t_i)-q(t_{i-1})}{t_i-t_{i-1}}(s-t_{i-1})\Big), \sum_{i=1}^n \chi_{[t_{i-1},t_i]}(s)p_i\Big)$

  $= P_n(q,p)(s)$

- $\forall (q,p)\in \mathcal{H}_t\times\mathcal{L}_t$, $\|P_n(q,p)-(q,p)\|\to 0$ as $n\to\infty$:
  Let us consider the subset $\mathcal{K}\subseteq \mathcal{H}_t\times\mathcal{L}_t$, $\mathcal{K}=\{(q,p)\in\mathcal{H}_t\times\mathcal{L}_t : \|P_n(q,p)-(q,p)\|\to 0\,,\, n\to\infty\}$. It is enough to prove that the closure of $\mathcal{K}$ is $\mathcal{H}_t\times\mathcal{L}_t$. To prove this it is sufficient to show that $\mathcal{K}$ is a closed subspace of $\mathcal{H}_t\times\mathcal{L}_t$ and contains a dense subset of $\mathcal{H}_t\times\mathcal{L}_t$. This follows from the density of the piecewise linear paths in $\mathcal{H}_t$ and the density of the piecewise constant paths in $\mathcal{L}_t$ (see e.g. [96]).

□

Theorem 8. *Let the function $(q,p)\to\psi_0(x+q(0))$, $\psi_0\in\mathcal{S}(\mathbb{R}^d)$, be Fresnel integrable with respect to A (with A defined by (2.7)). Then the phase space Feynman path integral, namely the limit*

$$\lim_{n\to\infty}(2\pi i\hbar)^{-nd}(\det(P_nAP_n))^{1/2}\int_{P_n(\mathcal{H}_t\times\mathcal{L}_t)} e^{\frac{i}{2\hbar}\langle P_n(q,p),AP_n(q,p)\rangle}\psi_0(x+q(0))dP_n(q,p) \tag{2.9}$$

coincides with the limit (2.4), namely with the solution of the Schrödinger equation with a free Hamiltonian.

Proof. The result follows by direct computation, indeed:

$$\widetilde{\int}_{P_n(\mathcal{H}_t\times\mathcal{L}_t)} e^{\frac{i}{2\hbar}\langle P_n(q,p),AP_n(q,p)\rangle}\psi_0(x+q(0))dP_n(q,p)$$

$$=\Big(\frac{1}{\sqrt{2\pi\hbar}}\Big)^{2nd}\int_{\mathbb{R}^{2nd}} e^{-\frac{i\epsilon}{\hbar}\sum_{j=0}^{n-1}\Big(\frac{p_j^2}{2}-p_j\frac{(x_{j+1}-x_j)}{\epsilon}\Big)}\psi_0(x_0)\prod_{j=0}^{n-1}dp_jdx_j,$$

and the two limits (2.4) and (2.9) coincide. Indeed (2.9) is a pointwise limit by hypothesis. On the other hand (2.4) is a limit in the L_2 sense, hence, passing if necessary to a subsequence, it is also a pointwise limit. □

Remark 5. *The latter result is equivalent to the "traditional" formulation of the Feynman path integral in the configuration space. Indeed it can be obtained by means of Fubini theorem [12] and an integration with respect to the momentum variables:*

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{2\pi\hbar}}\right)^{2nd}\int_{\mathbb{R}^{2nd}} e^{-\frac{i\epsilon}{\hbar}\sum_{j=0}^{n-1}\left(\frac{p_j^2}{2m}-p_j\frac{(x_{j+1}-x_j)}{\epsilon}\right)}\psi_0(x_0)\prod_{j=0}^{n-1}dp_j dx_j$$

$$=\lim_{n\to\infty}\left(\frac{1}{\sqrt{2\pi i\hbar}}\right)^{nd}\int_{\mathbb{R}^{nd}} e^{-\frac{i\epsilon}{\hbar}\sum_{j=0}^{n-1}m\frac{(x_{j+1}-x_j)^2}{2\epsilon^2}}\psi_0(x_0)\prod_{j=0}^{n-1}dx_j$$

The latter expression yields the Feynman functional on the configuration space, i.e. heuristically const $\int e^{\int_0^t\mathcal{L}(q(s),\dot{q}(s))ds}dq$, *($\mathcal{L}$ being the classical Lagrangian density).*

Remark 6. *The integration with respect to the momentum variables might seem to be superfluous, but it is very useful when we introduce a potential depending on the momentum.*

Theorem 9. *Let us consider a semibounded potential V depending explicitly on the momentum: $V = V(p)$ and the corresponding quantum mechanical Hamiltonian $H = -\frac{\hbar^2}{2}\Delta + V(p)$. Let us suppose H is an essentially self-adjoint operator on $\mathcal{L}_2(\mathbb{R}^d)$. Let the function $(q,p)\to e^{-\frac{i}{\hbar}\int_0^t V(P_n(p(s)))ds}\psi_0(x+q(0))$ be Fresnel integrable with respect to the operator A, with A defined by (2.7). Then the solution to the Schrödinger equation*

$$\begin{cases}\dot{\psi} = -\frac{i}{\hbar}H\psi \\ \psi(0,x)=\psi_0(x), \qquad \psi_0\in\mathcal{S}(\mathbb{R}^d)\end{cases} \tag{2.10}$$

is given by the phase space path integral

$$\lim_{n\to\infty}(2\pi i\hbar)^{-nd}(\det(P_nAP_n))^{1/2}\int_{P_n(\mathcal{H}_t\times\mathcal{L}_t)} e^{\frac{i}{2\hbar}\langle P_n(q,p),AP_n(q,p)\rangle} e^{-\frac{i}{\hbar}\int_0^t V(P_n(p(s)))ds}\psi_0(x+q(0))dP_n(q,p)$$

Proof. We can proceed in a completely analogous way as in the proof of theorem 8, therefore we shall omit the details. □

2.3 The phase space Feynman-Kac formula

Let us consider a classical potential V depending both on the position $Q\in\mathbb{R}^d$ and on the momentum $P\in\mathbb{R}^d$, but of the special form: $V = V(Q,P) =$

$V_1(Q) + V_2(P)$ (The general case presents problems due to the non commutativity of the quantized expression of Q and P), for a different approach with more general Hamiltonians see [93]. Moreover let us suppose the function $f : \mathcal{H}_t \times \mathcal{L}_t \to \mathbb{C}$

$$f(q,p) = \psi_0(x + q(0)) e^{-\frac{i}{\hbar}\int_0^t V(q(s)+x,p(s))ds}, \qquad \psi_0 \in \mathcal{S}(\mathbb{R}^d)$$

is the Fourier transform of a complex bounded variation measure μ_f on $\mathcal{H}_t \times \mathcal{L}_t$:

$$f(q,p) = \int_{\mathcal{H}_t \times \mathcal{L}_t} e^{i\langle q,p;Q,P\rangle} d\mu_f(Q,P).$$

Under additional assumptions on V_1 and V_2 we shall see that the phase space Feynman path integral of the function f can be computed and is given by

$$\widetilde{\int}_{\mathcal{H}_t \times \mathcal{L}_t} e^{\frac{i}{2\hbar}\langle q,p;A(q,p)\rangle} e^{-\frac{i}{\hbar}\int_0^t V(q(s)+x,p(s))ds} \psi(0, q(0) + x) dq dp =$$

$$= \int_{\mathcal{H}_t \times \mathcal{L}_t} e^{\frac{-i\hbar}{2}\langle q,p;A^{-1}(q,p)\rangle} d\mu_f(q,p). \tag{2.11}$$

This follows from the previous section together with the following

Lemma 2. *Let us consider a potential* $V(Q,P) = V_1(Q) + V_2(P)$ *and an initial wave function* ψ_0 *such that* $V_1, \psi_0 \in \mathcal{F}(\mathbb{R}^d)$ *and the function* $p(s)_{s\in[0,t]} \to \int_0^t V_2(p(s))ds \in \mathcal{F}(\mathcal{L}_t)$*. Then the functional*

$$f(q,p) = \psi_0(x + q(0)) e^{-\frac{i}{\hbar}\int_0^t V(q(s)+x,p(s))ds}$$

belongs to $\mathcal{F}(\mathcal{H}_t \times \mathcal{L}_t)$

Proof. $f(q,p)$ is the product of two functions: the first, say f_1, depends only on the first variable q, while the second f_2 depends only on the variable p, more precisely

$$f_1(q) = \psi_0(x + q(0)) e^{-\frac{i}{\hbar}\int_0^t V_1(q(s)+x)ds}, \qquad f_2(p) = e^{-\frac{i}{\hbar}\int_0^t V_2(p(s))ds}.$$

Under the given hypothesis on V_1 and ψ_0, f_1 belongs to $\mathcal{F}(\mathcal{H}_t)$. The proof is given for instance in [12]. For f_2 one must pay more attention: indeed the same proof given for f_1 does not work, as f_2 is defined on a different Hilbert space and we have to require explicitly that $\int_0^t V_2(p(s))ds \in \mathcal{F}(\mathcal{L}_t)$. Under this hypothesis one can easily prove that (see again [12]) $f_2 \in \mathcal{F}(\mathcal{L}_t)$.
Now if $f_1 = \hat{\mu}_{f_1} \in \mathcal{F}(\mathcal{H}_t)$, f_1 can be extended to a function, denoted again by f_1, in $\mathcal{F}(\mathcal{H}_t \times \mathcal{L}_t)$: it is the Fourier transform of the product measure on $\mathcal{H}_t \times \mathcal{L}_t$ of $\mu_{f_1}(dq)$ and $\delta_0(dp)$. The same holds for $f_2 = \hat{\mu}_{f_2}$: $f_2 = (\widehat{\delta_0(dq)\mu_{f_2}}(dp))$.

Finally, as $\mathcal{F}(\mathcal{H}_t \times \mathcal{L}_t)$ is a Banach algebra, the product of two elements $f_1 f_2$ is again an element of $\mathcal{F}(\mathcal{H}_t \times \mathcal{L}_t)$: more precisely it is the Fourier transform of the convolution of the two measures in $\mathcal{M}(\mathcal{H}_t \times \mathcal{L}_t)$ corresponding to f_1 and f_2 respectively, and the conclusion follows. □

The next theorem shows that the above oscillatory integral (2.11) gives the solution to the Schrödinger equation (2.12).

Theorem 10. *Let us consider the following Hamiltonian*

$$H(Q;P) = \frac{P^2}{2} + V_1(Q) + V_2(P)$$

in $L^2(\mathbb{R}^d)$ and the corresponding Schrödinger equation

$$\begin{cases} \dot{\psi} = -\frac{i}{\hbar} H\psi \\ \psi(0,x) = \psi_0(x), \quad x \in \mathbb{R}^d \end{cases} \tag{2.12}$$

Let us suppose that $V_1, \psi_0 \in \mathcal{F}(\mathbb{R}^d)$ and $\int_0^t V_2(p(s))ds \in \mathcal{F}(\mathcal{L}_t)$. Then the solution to the Cauchy problem (2.12) is given by the phase space Feynman path integral:

$$\widetilde{\int}_{\mathcal{H}_t \times \mathcal{L}_t} e^{\frac{i}{2\hbar}\langle q,p;A(q,p)\rangle} e^{-\frac{i}{\hbar}\int_0^t (V_1(q(s)+x)+V_2(p(s)))ds} \psi_0(q(0)+x) dq dp$$

Proof. We follow the proof given by Elworthy and Truman in [57].
For $0 \le u \le t$ let $\mu_u(V_1,x) \equiv \mu_u$, $\nu_u^t(V_1,x) \equiv \nu_u^t$, $\eta_u^t(V_2) \equiv \eta_u^t$ and $\mu_0(\psi)$ be the measures on $\mathcal{H}_t \times \mathcal{L}_t$, whose Fourier transforms when evaluated at $(q,p) \in \mathcal{H}_t \times \mathcal{L}_t$ are $V_1(x+q(u))$, $\exp\Big(-i\int_u^t V_1(x+q(s))ds\Big)$, $\exp\Big(-i\int_u^t V_2(p(s))ds\Big)$ and $\psi_0(q(0)+x)$.
We set

$$U(t)\psi_0(x) = \widetilde{\int}_{\mathcal{H}_t \times \mathcal{L}_t} e^{\frac{i}{2\hbar}\langle q,p;A(q,p)\rangle} e^{-\frac{i}{\hbar}\int_0^t (V_1(q(s)+x)+V_2(p(s)))ds} \psi_0(q(0)+x) dq dp$$

and

$$U_0(t)\psi_0(x) = \widetilde{\int}_{\mathcal{H}_t \times \mathcal{L}_t} e^{\frac{i}{2\hbar}\langle q,p;A(q,p)\rangle} e^{-\frac{i}{\hbar}\int_0^t V_2(p(s)))ds} \psi_0(q(0)+x) dq dp$$

By section 3 we have:

$$U(t)\psi_0(x) = \int_{\mathcal{H}_t \times \mathcal{L}_t} e^{\frac{-i\hbar}{2}\langle q,p;A^{-1}(q,p)\rangle} (\eta_0^t * \nu_0^t * \mu_0(\psi))(dq dp). \tag{2.13}$$

Now, if $\{\mu_u : a \leq u \leq t\}$ is a family in $\mathcal{M}(\mathcal{H}_t \times \mathcal{L}_t)$, we shall let $\int_a^b \mu_u du$ denote the measure on $\mathcal{H}_t \times \mathcal{L}_t$ given by :

$$f \to \int_a^b \int_{\mathcal{H}_t \times \mathcal{L}_t} f(q,p) d\mu_u(q,p) du$$

whenever it exists.
Since for any continuous path q we have

$$\exp\Big(-i \int_0^t V_1(q(s))ds \Big) = 1 - i \int_0^t V_1(q(u)) \exp\Big(-i \int_u^t V_1(q(s))ds \Big) du$$

the following relation holds

$$\nu_0^t = \delta_0 - i \int_0^t (\mu_u * \nu_u^t) du \tag{2.14}$$

where δ_0 is the Dirac measure at $0 \in \mathcal{H}_t$.
Applying this relation to (2.13) we obtain:

$$U(t)\psi_0(x) = \int_{\mathcal{H}_t \times \mathcal{L}_t} e^{\frac{-i\hbar}{2}\langle q,p;A^{-1}(q,p)\rangle} (\eta_0^t * \mu_0(\psi))(dqdp)$$

$$-i \int_0^t \int_{\mathcal{H}_t \times \mathcal{L}_t} e^{\frac{-i\hbar}{2}\langle q,p;A^{-1}(q,p)\rangle} (\eta_0^t * \mu_u(V_1,x) * \nu_u^t * \mu_0(\psi))(dqdp)du$$

$$= U_0(t)\psi_0(x) - i \int_0^t \widetilde{\int}_{\mathcal{H}_t \times \mathcal{L}_t} e^{\frac{i}{2\hbar}\langle q,p;A(q,p)\rangle} e^{-\frac{i}{\hbar}\int_u^t V_1(q(s)+x)ds}$$

$$e^{-\frac{i}{\hbar}\int_0^t V_2(p(s)))ds} V_1(q(u)+x)\psi_0(q(0)+x) dqdpdu$$

Now we have, by Fubini theorem for Fresnel integrals[12]

$$\widetilde{\int}_{\mathcal{H}_t \times \mathcal{L}_t} e^{\frac{i}{2\hbar}\langle q,p;A(q,p)\rangle} e^{-\frac{i}{\hbar}\int_u^t V_1(q(s)+x)ds} e^{-\frac{i}{\hbar}\int_0^t V_2(p(s)))ds}.$$

$$V_1(q(u)+x)\psi_0(q(0)+x)dqdp$$

$$= \widetilde{\int}_{\mathcal{H}_{t-u} \times \mathcal{L}_{t-u}} e^{\frac{i}{2\hbar}\langle q,p;A(q,p)\rangle_{\mathcal{H}_{t-u}\times\mathcal{L}_{t-u}}} e^{-\frac{i}{\hbar}\int_0^{t-u} V_1(q(s)+x)ds} e^{-\frac{i}{\hbar}\int_0^{t-u} V_2(p(s)))ds} V_1(q(0)+x)$$

$$\widetilde{\int}_{\mathcal{H}_u \times \mathcal{L}_u} e^{\frac{i}{2\hbar}\langle q_1,p_1;A(q_1,p_1)\rangle_{\mathcal{H}_u\times\mathcal{L}_u}} e^{-\frac{i}{\hbar}\int_0^u V_2(p_1(s)))ds} \psi_0(q_1(0)) dq_1 dp_1 dqdp$$

Here $q \in \mathcal{H}_{t-u}$ and $q_1 \in \mathcal{H}_u$ are the integration variables, and $\mathcal{H}_s$ denotes the Cameron-Martin space of paths $\gamma : [0,s] \to \mathbb{R}^d$.
We have:

$$U(t)\psi_0(x) = U_0(t)\psi_0(x) - i \int_0^t U(t-u)(V_1 U_0(u)\psi_0)(x) du$$

$$= U_0(t)\psi_0(x) - i\int_0^t U(u)(V_1 U_0(-u)U_0(t)\psi_0)(x)du$$

The iterative solution of the latter integral equation is the convergent Dyson perturbation series for $U(t)$ with respect to $U_0(t)$, which proves the theorem. $\square$

Chapter 3

Application to a stochastic Schrödinger equation

In this chapter we show that it is possible to generalize the definition of infinite dimensional oscillatory integrals in order to deal with complex-valued phase functions. We prove a Cameron-Martin-Parseval type formula which is the generalization of theorem 3 to the complex case. We apply these results to the representation of the solution of a particular type of stochastic Schrödinger equation, of some importance in the quantum theory of continuous measurements: the Schrödinger-Belavkin equation.

3.1 Oscillatory integrals with complex-valued phase function

Let $\mathcal{H}$ be a real separable Hilbert space. We shall denote by $\mathcal{H}^{\mathbb{C}}$ its complexification. An element $x \in \mathcal{H}^{\mathbb{C}}$ is a couple of vectors $x = (x_1, x_2)$, with $x_1, x_2 \in \mathcal{H}$, or with a different notation $x = x_1 + ix_2$. The multiplication of the vector $x \in \mathcal{H}^{\mathbb{C}}$ for the pure imaginary scalar $i = \sqrt{-1}$ is given by $ix = (-x_2, x_1)$.
A linear operator $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$ can be extended to a linear operator denoted again by A on $\mathcal{H}^{\mathbb{C}}$:

$$A : D(A) \subseteq H^{\mathbb{C}} \to H^{\mathbb{C}}, \qquad D(A) = D(A) + iD(A),$$
$$Ax = A(x_1, x_2) = (Ax_1, Ax_2).$$

In an analogous way a vector $y \in \mathcal{H}$ can be seen as the element $(y, 0) \in \mathcal{H}^{\mathbb{C}}$.
Let $dim(\mathcal{H}) = 1$, i.e. $\mathcal{H} = \mathbb{R}$, $\mathcal{H}^{\mathbb{C}} = \mathbb{C}$. Then, for any $f \in \mathcal{F}(\mathbb{R})$, $f = \hat{\mu}_f$, and any complex constant $\alpha \in \mathbb{C}$, $\alpha \neq 0$, $Im(\alpha) \geq 0$, one can easily prove the following equality

$$\widetilde{\int_{\mathbb{R}}} e^{\frac{i\alpha}{2\hbar}x^2} f(x)dx = \alpha^{-1/2} \int_{\mathbb{R}} e^{\frac{-i\hbar}{2\alpha}x^2} \mu_f(dx) \tag{3.1}$$

The proof is completely similar to the proof of theorem 1.
More generally, given $\alpha \in \mathbb{C}$, $\alpha \neq 0, Im(\alpha) > 0$ and $\beta \in \mathbb{R}$

$$\widetilde{\int_{\mathbb{R}}} e^{\frac{i\alpha}{2\hbar}x^2} e^{\beta x} f(x)dx = \alpha^{-1/2} \int_{\mathbb{R}} e^{\frac{-i\hbar}{2\alpha}(x-i\beta)^2} \mu_f(dx) \tag{3.2}$$

Such a result can be generalized to the infinite dimensional case [9]:

Theorem 11. *Let $\mathcal{H}$ be a real separable Hilbert space, let $y \in \mathcal{H}$ be a vector in $\mathcal{H}$ and let L_1 and L_2 be two self-adjoint, trace class commuting operators on $\mathcal{H}$ such that $I + L_1$ is invertible and L_2 is non negative. Let moreover $f : \mathcal{H} \to \mathbb{C}$ be the Fourier transform of a complex bounded variation measure μ_f on $\mathcal{H}$:*

$$f(x) = \hat{\mu}_f(x), \qquad f(x) = \int_{\mathcal{H}} e^{i\langle x,k\rangle} \mu_f(dk).$$

Then the infinite dimensional oscillatory integral (with complex phase)

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I+L)x\rangle} e^{\langle y,x\rangle} f(x)dx$$

is well defined and it is given by

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I+L)x\rangle} e^{\langle y,x\rangle} f(x)dx = \det(I+L)^{-1/2} \int_{\mathcal{H}} e^{\frac{-i\hbar}{2}\langle k-iy,(I+L)^{-1}(k-iy)\rangle} \mu_f(dk) \tag{3.3}$$

(L being the operator on the complexification $\mathcal{H}^{\mathbb{C}}$ of the real Hilbert space $\mathcal{H}$ given by $L = L_1 + iL_2$).

Proof. First of all one can notice that both sides of equation (3.3) are well defined. Indeed one can easily prove that $(I + L) : H_{\mathbb{C}} \to H_{\mathbb{C}}$ is invertible, if $(I + L_1)$ is invertible and that $\det(I + L)$ exists as L is trace class.
On the other hand the function $f : \mathcal{H} \to \mathbb{C}$

$$f(x) = e^{-\frac{1}{2\hbar}\langle x,L_2 x\rangle} e^{\langle y,x\rangle} g(x)$$

where $y \in \mathcal{H}$ and $g \in \mathcal{F}(\mathcal{H})$, $g(x) = \hat{\mu}_g(x)$, $\mu_g \in \mathcal{M}(\mathcal{H})$ is the Fourier transform of a complex bounded variation measure μ_f on $\mathcal{H}$, $f = \hat{\mu}_f$. In fact μ_f is the convolution of μ_g and the measure ν, with $\nu(dx) = e^{\frac{\hbar}{2}\langle y,L_2^{-1}y\rangle - i\hbar\langle y,L_2^{-1}x\rangle} \mu_{L_2}(dx)$, where μ_{L_2} is the Gaussian measure on $\mathcal{H}$ with covariance operator $L_2/\hbar$. By theorem 3 the Fresnel (or Feynman path) integral of f with respect to the operator $Q = (I + L_1)/\hbar$ is well defined and it is given by:

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I+L_1)x\rangle} f(x)dx = \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I+L_1)x\rangle} e^{-\frac{1}{2\hbar}\langle x,L_2 x\rangle} e^{\langle y,x\rangle} g(x)dx$$

$$= \det(I+L_1)^{-1/2}\int_H e^{\frac{-i\hbar}{2}\langle x,(I+L_1)^{-1}x\rangle}\mu_g * \nu(dx)$$

$$= \det(I+L_1)^{-1/2}\int_{\mathcal{H}}\int_{\mathcal{H}} e^{\frac{-i\hbar}{2}\langle x+z,(I+L_1)^{-1}(x+z)\rangle}\mu_g(dz)\nu(dx)$$

$$= \det(I+L_1)^{-1/2}\int_{\mathcal{H}}\int_{\mathcal{H}} e^{\frac{-i\hbar}{2}\langle x+z,(I+L_1)^{-1}(x+z)\rangle} e^{\frac{\hbar}{2}\langle y,L_2^{-1}y\rangle - i\hbar\langle y,L_2^{-1}x\rangle}\mu_{L_2}(dx)\mu_g(dz) \tag{3.4}$$

Equation (3.3) can be proved by taking the finite dimensional approximation of the last line of equation (3.4) and of the r.h.s. of (3.3) and showing they coincide. As L_1 and L_2 are two commuting symmetric trace class operators on $\mathcal{H}$, they have a common spectral decomposition. Thus there exists a complete orthonormal system $\{e_n\} \subset \mathcal{H}$ such that

$$L_1(x) = \sum_n a_n \langle e_n, x\rangle e_n, \qquad L_2(x) = \sum_n b_n \langle e_n, x\rangle e_n, \qquad \gamma \in \mathcal{H},$$

with $a_n, b_n \in \mathbb{R}$.

Let $\{P_m\}$ be the family of projectors onto the span of the first m eigenvectors $e_1, ..., e_m$, namely:

$$P_m(x) = \sum_{n=1}^{m} \langle e_n, x\rangle e_n$$

One can easily see that $P_m \to I$ as $m \to \infty$ and $L_1 P_m(\mathcal{H}) \subseteq P_m(\mathcal{H})$, $L_2 P_m(\mathcal{H}) \subseteq P_m(\mathcal{H})$. Moreover the infinite dimensional oscillatory integral

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I+L_1)x\rangle} e^{-\frac{1}{2\hbar}\langle x,L_2 x\rangle} e^{\langle y,x\rangle} g(x)dx$$

can be computed as

$$\lim_{m\to\infty}(2\pi i\hbar)^{-m/2}\int_{P_m\mathcal{H}} e^{\frac{i}{2\hbar}\langle P_m x,(I+L_1)P_m x\rangle} e^{-\frac{1}{2\hbar}\langle P_m x,L_2 P_m x\rangle} e^{\langle y,P_m x\rangle} g(P_m x)d(P_m x)$$

which, from the Cameron-Martin formula can be seen to be equal to

$$(\prod_{n=1}^{m} a_n b_n)^{-1/2}(2\pi\hbar)^{-m/2}\int_{P_m\mathcal{H}}\int_{P_m\mathcal{H}} e^{-i\hbar/2\sum_{n=1}^{m} a_n^{-1}(x_n+z_n)^2 - \hbar/2\sum_{n=1}^{m} b_n^{-1}x_n^2}$$

$$\cdot e^{-\hbar/2\sum_{n=1}^{m} b_n^{-1}y_n^2 - i\hbar\sum_{n=1}^{m} b_n^{-1}y_n x_n} d(P_m x)(\mu_g \circ P_m)(dz) \tag{3.5}$$

where $x_n = \langle x, e_n\rangle$, $z_n = \langle z, e_n\rangle$, $y_n = \langle y, e_n\rangle$, $d(P_m x)$ being the $m-$dimensional Lebesgue measure on $P_m\mathcal{H}$.

The finite dimensional approximation of the right hand side of equation (3.3) assumes the following form:

$$\Big(\prod_{n=1}^{m}(a_n+ib_n)\Big)^{-1/2}\int_{P_m\mathcal{H}} e^{-i\hbar/2\sum_{n=1}^{m}(a_n+ib_n)^{-1}(x_n-iy_n)^2}(\mu_g\circ P_m)(dx) \tag{3.6}$$

By a direct computation one can verify that expressions (3.6) and (3.5) coincide. Now we can pass to the limit and from Lebesgue's dominated convergence theorem we have

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I+L_1)x\rangle}e^{-\frac{1}{2\hbar}\langle x,L_2x\rangle}e^{\langle y,x\rangle}g(x)dx$$
$$=\det(I+L)^{-1/2}\int_{\mathcal{H}} e^{\frac{-i\hbar}{2}\langle k-iy,(I+L)^{-1}(k-iy)\rangle}\mu_g(dk) \tag{3.7}$$

□

An analogous result holds also for the normalized Fresnel integral with complex phase (in the sense of definition 5 in the first chapter).

Theorem 12. *Let $\mathcal{H}$ be a real separable Hilbert space, let $y\in\mathcal{H}$ be a vector in $\mathcal{H}$ and let L_1 and L_2 be two self-adjoint, commuting operators on $\mathcal{H}$ such that $I+L_1$ is invertible and L_2 is non negative. Let moreover $f:\mathcal{H}\to\mathbb{C}$ be the Fourier transform of a complex bounded variation measure μ_f on $\mathcal{H}$:*

$$f(x)=\hat{\mu}_f(x),\qquad f(x)=\int_{\mathcal{H}} e^{i\langle x,k\rangle}\mu_f(dk).$$

Then the infinite dimensional normalized oscillatory integral (with complex phase)

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I+L)x\rangle}e^{\langle y,x\rangle}f(x)dx$$

is well defined and it is given by

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I+L)x\rangle}e^{\langle y,x\rangle}f(x)dx=\int_{\mathcal{H}} e^{\frac{-i\hbar}{2}\langle k-iy,(I+L)^{-1}(k-iy)\rangle}\mu_f(dk) \tag{3.8}$$

(L being the operator on the complexification $\mathcal{H}^{\mathbb{C}}$ of the real Hilbert space $\mathcal{H}$ given by $L=L_1+iL_2$).

Proof. As in theorem 11, the result can be proved by computing the finite dimensional approximations of both sides of equation (3.8). □

In the following section we shall see how the latter results can be applied to the computation of the solution of a stochastic Schrödinger equation.

3.2 Belavkin equation

In the traditional formulation of quantum mechanics the continuous time evolution described by the Schrödinger equation (1) is valid if the quantum system is "undisturbed". On the other hand we should not forget that all the informations we can have on the state of a quantum particle are the result of some measurement process. When the particle interacts with the measuring apparatus, its time evolution is no longer continuous: the state of the system after the measurement is the result of a random and discontinuous change, the so-called "collapse of the wave function", which cannot be described by the ordinary Schrödinger equation. Quoting Dirac [54], after the introduction of the Planck constant $\hbar$ the concept of "large" and "small" are no longer relative: it is "microscopic"[1] one object such that the influence on the measuring apparatus on it cannot be neglected. Let us recall the main features of the traditional quantum description of the measurement of an observables O. Any observable A is represented by a self-adjoint operator on the Hilbert space $\mathcal{H}$, whose unitary vectors represents the states of the system. Let us consider for simplicity the case A is bounded and its spectrum is discrete. Let $\{a_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ and $\{\psi_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$ be the corresponding eigenvalues and eigenvectors. According to the traditional mathematical formulation by von Neumann the consequences of the measurement are:

1. the *decoherence* of the state of the quantum system: because of the interaction with the measuring apparatus the initial pure state ψ becomes a mixed state, described by the density operator $\rho^{prior}(t) = \sum_i w_i P_{\psi_i}$, where P_{ψ_i} denotes the projector operator onto the eigenspace which is spanned by the vector ψ_i and $w_i = |\langle\psi_i, \psi\rangle|^2$. Considering another observable B (represented by a bounded self-adjoint operator), its expectation value at time t, after the measurement of the observable A (but without the information of the result of the measurement of A), is given by
$$\mathbb{E}(B)_t^{prior} = \mathrm{Tr}[\rho^{prior}(t)B]$$
The existence of the trace is assumed. The transformation mapping ψ to the so-called "prior state" $\rho^{prior}(t)$ is named "prior dynamics" or non selective dynamics.

2. The so-called "collapse of the wave function": after the reading of the result of the measurement (i.e. the real number a_i) the state of the system is the corresponding eigenstate of the measured observable:
$$\rho(t)_{a_i}^{post} = P_{\psi_i}.$$

[1]It would be more correct the word "quantum" as there exist also macroscopic quantum systems, but they were unknown at Dirac's time.

The expectation value of another observable B of the system at time t (taking into account the information about the value of the measurement of A) is given by:

$$\mathbb{E}^{post}(B|A = a_i)_t = \mathrm{Tr}[\rho_{a_i}^{post}(t)B] = \langle\psi_i, B\psi_i\rangle$$

The transformation mapping the initial state ψ to one of the so-called "posterior states" $\rho_{a_i}^{post}(t)$ is called "posterior dynamics" or selective dynamics and depends on the result a_i of the measurement of A.

As it is suggested by the collapse of the wave function, the non selective dynamics maps pure states to mixed states, while the selective one maps pure states to pure states. The relation between the posterior state and the prior state is given by:

$$\rho^{prior}(t) = \sum_i P(A = a_i)\rho_{a_i}^{post}(t)$$

where $P(A = a_i)$ that the outcome of the measurement of A is the eigenvalue a_i and it is given by

$$P(A = a_i) = |\langle\psi_i, \psi\rangle|^2.$$

We remark that

$$\mathbb{E}(B)_t^{prior} = \sum_i \mathbb{E}^{post}(B|A = a_i)P(A = a_i), \tag{3.9}$$

Such a situation cannot be described by the traditional Schrödinger equation. There are several efforts to include the process of measurement into the traditional quantum theory and to deduce from its laws, instead of postulating both the process of decoherence (see point 1) and the collapse of the wave function (point 2). In particular the aim of the *quantum theory of measurement* is a description of the process of measurement taking into account the properties of the measuring apparatus, which is handled as a quantum system, and its interaction with the system submitted to the measurement [50]. Even if also this approach is not completely satisfactory (also in this case one has to postulate the collapse of the state of the compound system "measuring apparatus plus observed system") it is able to give a better description of the process of measurement.

An example of this approach is for instance the paper by Caldeira and Legget [39], where the Lindblad equation for the evolution of the density operator ρ, describing the process of decoherence (i.e. the prior dynamics) is heuristically derived:

$$\frac{\partial}{\partial t}\rho^{prior} = \frac{1}{i\hbar}[H, \rho^{prior}] - \frac{\eta k T}{\hbar^2}[x, [x, \rho^{prior}]]. \tag{3.10}$$

The authors show how equation (3.10) is a consequence of the interaction of the system with a ensemble of oscillators representing for instance the normal

modes of an electromagnetic field or of the vibrations of the atoms in a crystal. H is the Hamiltonian of the system, k is Boltzmann constant, T is the temperature of the crystal and η is a damping constant.
Another interesting result of the quantum theory of measurement is the so-called "Zeno effect", which seems to forbid a satisfactory description of continuous measurements. Indeed if a sequence of "ideal"[2] measurements of an observable A with discrete spectrum is performed and the time interval between two measurements is sufficiently small, then the observed system does not evolve. In other words a particle whose position is continuously monitored cannot move. This result is in apparent contrast with the experience: indeed in a bubble chamber repeated measurements of the position of microscopical particles are performed without "freezing" their state. For a detailed description of the quantum Zeno paradox see for instance [81, 42, 86].
In the physical and in the mathematical literature a class of stochastic Schrödinger equations giving a phenomenological description of this situation has been proposed by several authors, see for instance [33, 27, 28, 53, 79, 60]. We consider in particular Belavkin equation, a stochastic Schrödinger equation describing the selective dynamics of a $d-$dimensional particle submitted to the measurement of one of its (possible $M-$dimensional vector) observables, described by the self-adjoint operator R on $L^2(\mathbb{R}^d)$

$$\begin{cases} d\psi(t,x) = -\frac{i}{\hbar}H\psi(t,x)dt - \frac{\lambda}{2}R^2\psi(t,x)dt + \sqrt{\lambda}R\psi(t,x)dW(t) \\ \psi(0,x) = \psi_0(x) \qquad (t,x) \in [0,T] \times \mathbb{R}^d \end{cases} \tag{3.11}$$

where H is the quantum mechanical Hamiltonian, W is an $M-$dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $dW(t)$ is the Ito differential and $\lambda > 0$ is a coupling constant, which is proportional to the accuracy of the measurement. In the particular case of the description of the continuous measurement of position one has $R = x$, so that equation (3.11) assumes the following form:

$$\begin{cases} d\psi(t,x) = -\frac{i}{\hbar}H\psi(t,x)dt - \frac{\lambda}{2}x^2\psi(t,x)dt + \sqrt{\lambda}x\psi(t,x)dW(t) \\ \psi(0,x) = \psi_0(x) \qquad (t,x) \in [0,T] \times \mathbb{R}^d, \end{cases} \tag{3.12}$$

while in the case of momentum measurement, $(R = -i\hbar\nabla)$ one has:

$$\begin{cases} d\psi(t,x) = -\frac{i}{\hbar}H\psi(t,x)dt + \frac{\lambda\hbar^2}{2}\Delta\psi(t,x)dt - i\sqrt{\lambda}\hbar\nabla\psi(t,x)dW(t) \\ \psi(0,x) = \psi_0(x) \qquad (t,x) \in [0,T] \times \mathbb{R}^d. \end{cases} \tag{3.13}$$

[2]A measurement is called *ideal* if the correlation between the state of the measuring apparatus and the state of the system after the measurement is maximal

Belavkin derives equation (3.11) by modeling the measuring apparatus (but it is better to say "the informational enviroment") by means of a one-dimensional bosonic field and by assuming a particular form for the interaction Hamiltonian between the field and the system on which the measurement is performed. The resulting dynamics is such that there exists a family of mutually commuting Heisenberg operators of the compound system, denoted by $X(t)_{t\in[0,T]}$:

$$[X(t), X(s)] = 0, \qquad s, t \in [0, T],$$

(on a dense domain in $L^2(\mathbb{R}^d)$). In this description the oncept of trajectory of X is meaningful, even from a quantum mechanical point of view. Moreover the "non-demolition principle" is fulfilled: the measurement of any future Heisenberg operator $Z(t)$ of the system is compatible with the measurement of the trajectory of X up to time t, that is

$$[Z(t), X(s)] = 0, \qquad s < t$$

(on a dense domain in $L^2(\mathbb{R}^d)$). The measured observable R is connected to the operator X by the following relation

$$X(t) = R(t) + \lambda(B_t + B_t^+), \tag{3.14}$$

(where $(B_t + B_t^+)$ is a quantum Brownian motion [66]). Equation (3.14) shows how the measurement of $X(t)$ gives some (indirect and not precise) informations on the value of R, overcoming the problems of quantum Zeno paradox. Indeed we are dealing with "unsharp" in spite of "ideal" measurements.
The solution ψ of Belavkin equation is a stochastic process, whose expectation values have an interesting physical meaning. Let $\omega(s), s \in [0, t]$ be a continuous path (from $[0, t]$ into $\mathbb{R}^M$), I a Borel set in the Banach space $C([0, t], \mathbb{R}^M)$ endowed with the sup norm, let P be the Wiener measure on $C([0, t], \mathbb{R}^M)$. The probability that the observed trajectory of X up to time t belongs to the subset I is given by the following Wiener integral:

$$\mathbb{P}(X(s) = \omega(s)_{s\in[0,t]} \in I) = \int_I |\psi(t, \omega)|^2 P(d\omega).$$

Moreover if we measure at time t another observable of the system, denoted with Z, then its expected value, conditioned to the information that the observed trajectory of X up to time t belongs to the Borel set I, is given by:

$$\mathbb{E}(Z(t)|X(s) = \omega(s)_{s\in[0,t]} \in I) = \int_I \frac{\langle\psi(t, \omega), Z\psi(t, \omega)\rangle}{|\psi(t, \omega)|^2} P(d\omega).$$

(where $\psi(t, \omega) \neq 0$ is assumed).
In other words $\psi(t, \omega)$ represents the posterior state and Belavkin equation

describes the selective dynamics of the system. The non selective dynamics can be obtained by means of the following generalization of formula (3.9) to the continuous case:

$$\rho^{prior}(t) = \int_{C([0,t],\mathbb{R}^M)} \rho^{post}_\omega(t)\mathbb{P}(X \in d\omega) = \int_{C([0,t],\mathbb{R}^d)} P_{\psi(t,\omega)}P(d\omega) \qquad (3.15)$$

By means of Ito formula one can verify that the prior state $\rho^{prior}(t)$ satisfies Lindblad equation:

$$\frac{\partial}{\partial t}\rho^{prior}(t) = \frac{1}{i\hbar}[H, \rho^{prior}(t)] - \frac{\lambda}{2}[R, [R, \rho^{prior}(t)]].$$

Analogously to the traditional Schrödinger equation, one can look for a path integral representation for the solution of Belavkin equation. In fact M.B. Mensky [79] proposed an heuristic formula for the selective dynamics of a particle whose position is continuously observed. According to Mensky the state of the particle at time t if the observed trajectory is the path $\omega(s)_{s\in[0,t]}$ is given by the "restricted path integrals"

$$\psi(t, x, [a]) = \text{``} \int_{\{\gamma(t)=x\}} e^{\frac{i}{\hbar}S_t(\gamma)} e^{-\lambda\int_0^t(\gamma(s)-\omega(s))^2 ds}\phi(\gamma(0))D\gamma \text{''} \qquad (3.16)$$

One can see that, as an effect of the correction term $e^{-\lambda\int_0^t(\gamma(s)-\omega(s))^2 ds}$ due to the measurement, the paths γ giving the main contribution to the integral (3.16) are those closer to the observed trajectory ω. In fact by means of the infinite dimensional oscillatory integrals described in the previous section, it is possible to prove a Feynman path integral representation of the solution of Belavkin equation and give a rigorous mathematical meaning to Mensky's heuristic formula. Indeed in the particular case of position measurement we shall prove that the solution of equation (3.12) can be represented by

$$\psi(t, x) = \widetilde{\int} e^{\frac{i}{2\hbar}\int_0^t|\dot\gamma(s)|^2ds - \lambda\int_0^t|\gamma(s)+x|^2ds} e^{-\frac{i}{\hbar}\int_0^t V(\gamma(s)+x)ds}$$
$$e^{\int_0^t\sqrt{\lambda}(\gamma(s)+x)\cdot dW(s)}\psi_o(\gamma(0) + x)d\gamma. \qquad (3.17)$$

In the case of Belavkin equation describing momentum measurement the stochastic term plays the role of a complex random potential depending on the momentum of the particle. In this case one has to use the phase space Feynman path integrals described in chapter 2. More precisely by means of a infinite dimensional oscillatory integral with complex phase on the space of paths in phase space one can give a rigorous mathematical meaning to the following heuristic expression:

$$\psi(t, x) = \widetilde{\int} e^{\frac{i}{\hbar}(\int_0^t(\dot q(s)p(s) - \frac{1}{2m}p(s)^2)ds - \lambda\int_0^t p(s)^2ds} e^{-\frac{i}{\hbar}\int_0^t V(q(s)+x)ds}$$
$$\cdot e^{\sqrt{\lambda}\int_0^t p(s)\cdot dW(s)}\psi_0(\gamma(0) + x)dqdp. \qquad (3.18)$$

3.3 Position measurement

In this section we consider Belavkin equation describing the posterior dynamics of a quantum particle, whose position is continuously observed:

$$\begin{cases} d\psi = -\frac{i}{\hbar}H\psi dt - \frac{\lambda|x|^2}{2}\psi dt + \sqrt{\lambda}x\psi\, dW(t) \\ \psi(0,x) = \psi_0(x) \qquad\qquad t \geq 0,\ x \in \mathbb{R}^d \end{cases} \tag{3.19}$$

where W is an d-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $dW(t)$ denotes the Ito stochastic differential; for each $\omega \in \Omega$, $\psi(\omega) \in C([0,T], \mathcal{H})$, $\mathcal{H} = L_2(R^d)$, and $\lambda > 0$ is a coupling constant. We denote the $\mathbb{R}^d$ norm with $|\ |$ and the scalar product with $a \cdot b = \sum_{i=1}^d a_i b_i$. Equation (3.19) can also be written in the Stratonovich equivalent form:

$$\begin{cases} d\psi = -\frac{i}{\hbar}H\psi dt - \lambda|x|^2\psi dt + \sqrt{\lambda}x\psi \circ dW(t) \\ \psi(0,x) = \psi_0(x) \qquad\qquad t \geq 0,\ x \in \mathbb{R}^d \end{cases} \tag{3.20}$$

The existence and uniqueness of a strong solution of equation (3.19) is proved in [60]. We shall prove that it can be represented by an infinite dimensional oscillatory integral on a suitable Hilbert space. We recall the definition of strong solution in the case of a Schrödinger equation.

Definition 7. *A strong solution for the stochastic equation* (3.20) *is a predictable process with values in* $\mathcal{H} = L^2(\mathbb{R}^d)$*, such that*
$\psi(t) \in D(-i/\hbar H - \lambda|x|^2)$ $\mathbf{P}$*-a.s.*
$\mathbf{P}\Big(\int_0^T (\|\psi(t)\|^2 + \|(-i/\hbar H - \lambda|x|^2)\psi\|^2)\, dt < \infty\Big) = 1$
$\mathbf{P}\Big(\int_0^T \||x|\psi(t)\, dt\|^2 < \infty\Big) = 1$ *and*
$\mathbf{P}$ *a.s. for all* $t \in [0,T]$*:*

$$\begin{cases} d\psi = -\frac{i}{\hbar}H\psi dt - \lambda|x|^2\psi dt + \sqrt{\lambda}x \cdot \psi \circ dW(t) \qquad t \geq 0,\ x \in \mathbb{R}^d \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{3.21}$$

Let us consider the Cameron Martin space H_t introduced in section 1.3 and let $H_t^{\mathbb{C}}$ be its complexification. Let $L : H_t^{\mathbb{C}} \to H_t^{\mathbb{C}}$ the operator on $H_t^{\mathbb{C}}$ defined by

$$\langle \gamma_1, L\gamma_2 \rangle = -a^2 \int_0^t \gamma_1(s) \cdot \gamma_2(s) ds;$$

where $a^2 = -2i\lambda\hbar$. The $j-$th component of $L\gamma$,$L\gamma = (L\gamma_1, \ldots, L\gamma_d)$, is given by

$$(L\gamma)_j(s) = 2i\lambda\hbar \int_s^t ds' \int_0^{s'} \gamma_j(s'')ds'' \qquad j = 1, \ldots, d \tag{3.22}$$

one can verify (see [57] for more details) that $iL : H \to H$ is self-adjoint with respect to the H_t-inner product, it is trace-class and its Fredholm determinant is given by:

$$\det(I+L) = \cos(at).$$

Moreover $(I+L)$ is invertible and its inverse is given by:

$$[(I+L)^{-1}\gamma]_j(s) = \gamma_j(s) - a\int_s^t \sin[a(s'-s)]\gamma_j(s')ds' +$$

$$+\sin[a(t-s)]\int_0^t [\cos at]^{-1} a\cos(as')\gamma_j(s')ds' \qquad j=1,\dots,d.$$

Let us introduce moreover the vector $l \in H_t$ defined by

$$\langle l,\gamma\rangle = -\sqrt{\lambda}\int_0^t \omega(s)\cdot\dot{\gamma}(s)ds = \sqrt{\lambda}\int_0^t \gamma(s)\cdot dW(s), \tag{3.23}$$

$$l(s) = \sqrt{\lambda}\int_s^t \omega(\tau)d\tau.$$

Given these results, it is possible to apply the theory of the first section and prove that, under suitable assumptions on the potential V and the initial wave function ψ_0, the heuristic expression (3.17) can be realized as the infinite dimensional oscillatory integral with complex phase on the Cameron-Martin space H_t:

$$C(t,x,\omega)\widetilde{\int_{H_t}} e^{\frac{i}{2\hbar}\langle\gamma,(I+L)\gamma\rangle} e^{\langle l,\gamma\rangle} e^{-2\lambda x\cdot\int_0^t \gamma(s)ds} e^{-\frac{i}{\hbar}\int_0^t V(\gamma(s)+x)ds}\psi_0(\gamma(0)+x)d\gamma \tag{3.24}$$

where $C(t,x,\omega) = e^{-\lambda|x|^2+\sqrt{\lambda}x\cdot\omega(t)}$ is a constant depending on t, $x \in \mathbb{R}^d$, $\omega \in \Omega$. Indeed the integrand $\exp(\frac{i}{2\hbar}\Phi)$ in (3.17), where $\Phi(\gamma) \equiv \int_0^t |\dot{\gamma}(s)|^2 ds + 2i\hbar\lambda\int_0^t |\gamma(s)+x|^2 ds - 2i\hbar\int_0^t \sqrt{\lambda}(\gamma(s)+x)\cdot dW(s)$ can be rigorously defined as the functional on the Cameron Martin space H_t given by $\Phi(\gamma) = \langle\gamma,(I+L)\gamma\rangle - 2i\hbar\langle l,\gamma\rangle - 2\hbar\int_0^t a^2 x\cdot\gamma(s)ds - a^2|x|^2 t - 2i\hbar\sqrt{\lambda}x\cdot\omega(t)$, where L is the operator (3.22) and l is the vector (3.23).
By means of theorem 11 one can compute the integral (3.24) in terms of an absolutely convergent integral on H_t. Moreover it is possible to prove it represents the solution of Belavkin equation (3.20) (see [9]).

Theorem 13. *Let V and ψ_0 be Fourier transforms of complex bounded variation measures on $\mathbb{R}^d$. Then there exist a (strong) solution to the Stratonovich stochastic differential equation (3.20) and it is given by the infinite dimensional oscillatory integral with complex phase* (3.24).

Remark 7. *The result can be extended to general initial vectors $\psi_0 \in L^2(\mathbb{R}^d)$, using the fact that $\mathcal{F}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$.*

Proof. The proof in divided into 3 steps: in the first two we consider the case $V \equiv 0$. First of all we deal with an approximated problem and we find a representation for its solution via a infinite dimensional oscillatory integral, then we show that the sequence of approximated solutions converges in a suitable sense to the solution of problem (3.20). In the final step we introduce the potential V and show that the right hand side of (3.24) is in fact the solution of the equation (3.20).

1. **The solution of the approximated problem.** We approximate the trajectory $t \to \omega(t)$ of the Wiener process by a sequence of smooth curves. More precisely we consider the sequence of functions [3]

$$n \int_{t-\frac{1}{n}}^{t} \omega(s) d\, s \equiv \omega_n(t), \qquad n \in \mathbb{N}.$$

We have $\omega_n \to \omega$ uniformly on $[0, T]$, indeed

$$\sup_{s \in [0,T]} |W_n(s) - W(s)| \to 0 \quad \text{as } n \to \infty \qquad \mathbb{P} \text{ a.s.}$$

Let us consider the sequence of approximated problems:

$$\begin{cases} d\psi_n = -\frac{i}{\hbar} H \psi_n dt - \lambda |x|^2 \psi_n dt + \sqrt{\lambda} x \cdot \psi_n dW_n(t) \\ \psi_n(0, x) = \psi_0(x) \end{cases} \tag{3.25}$$

where $dW_n(t)$ is an ordinary differential, i.e. $dW_n(t) = \dot{\omega}_n(t) dt$, and we can also write:

$$\begin{cases} \dot{\psi}_n = -\frac{i}{\hbar} H \psi_n - \lambda |x|^2 \psi_n + \sqrt{\lambda} x \cdot \psi_n \dot{\omega}_n(t) \\ \psi_n(0, x) = \psi_0(x) \end{cases} \tag{3.26}$$

which can be recognized as a family of Schrödinger equations, with a complex potential, labeled by the random parameter $\omega \in \Omega$.

Now we compute a representation of the solution of (3.26) by means of an infinite dimensional oscillatory integral with complex phase , under suitable assumptions on the (real) potential V and on the initial datum $\psi_n(0, x, \omega) = \psi_0(x)$.

We can write equation (3.26) in the following form:

$$\begin{cases} \dot{\psi}_n = -\frac{i}{\hbar} \left(\frac{-\hbar^2 \Delta}{2m} - i\lambda\hbar |x|^2\right) \psi_n - \frac{i}{\hbar} V \psi_n + \sqrt{\lambda} x \cdot \psi_n \dot{\omega}_n(t) \\ \psi_n(0, x) = \psi_0(x) \end{cases} \tag{3.27}$$

so that we can recognize in it the Schrödinger equation for an anharmonic oscillator with a complex potential, i.e.

$$\begin{cases} \dot{\psi}_n = -\frac{i}{\hbar} \left(\frac{-\hbar^2 \Delta}{2m} + \frac{a^2}{2} |x|^2\right) \psi_n - \frac{i}{\hbar} U \psi_n \\ \psi_n(0, x) = \psi_0(x) \end{cases} \tag{3.28}$$

[3]Here we denote, as usual, the trajectory of the Wiener process $W(t)$ as $\omega(t)$.

where $a^2 = -2i\lambda\hbar$ and $U = U(t,x,\omega) = V(x) + i\hbar\sqrt{\lambda}x \cdot \dot{\omega}_n(t)$.
We introduce the sequence of vectors $l_n \in H$ defined by

$$\langle l_n, \gamma \rangle = \sqrt{\lambda}\int_0^t \gamma(s) \cdot \dot{\omega}_n(s) ds = -\sqrt{\lambda}\int_o^t \omega_n(s) \cdot \dot{\gamma}(s) ds,$$

which is given by

$$l_n(s) = \sqrt{\lambda}\int_s^t \omega_n(\tau) d\tau. \tag{3.29}$$

First of all let us consider equation (3.20) with H replaced by the free Hamiltonian $H = -\hbar^2\Delta/2$. The following result holds:

Lemma 3. *Let $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$. Then the solution of the Cauchy problem:*

$$\begin{cases} \dot{\psi}_n(t,x) = \frac{i\hbar}{2}\Delta\psi_n(t,x) - \lambda|x|^2\psi_n(t,x) + \sqrt{\lambda}x \cdot \dot{\omega}_n(t)\psi_n(t,x) \\ \psi_n(0,x) = \psi_0(x), \qquad x \in \mathbb{R}^d \end{cases} \tag{3.30}$$

is given by :

$$\psi_n(t,x) = \widetilde{\int_{H_t}} e^{\frac{i}{2\hbar}\int_0^t |\dot{\gamma}(s)|^2 ds - \lambda\int_0^t |\gamma(s)+x|^2 ds} e^{\sqrt{\lambda}\int_0^t (\gamma(s)+x)\cdot\dot{\omega}_n(s) ds} \psi_0(\gamma(0)+x) d\gamma \tag{3.31}$$

(where the right hand side is interpreted as the infinite dimensional oscillatory integral of $\psi_0(\gamma(0)+x)e^{\langle l_n,\gamma\rangle}$ with complex quadratic phase function $\langle \gamma, (I+L)\gamma\rangle/\hbar$, with H_t the Cameron-Martin space, l_n the vector defined by (3.29) and L the operator defined by (3.22).)

Proof. Formula (3.31) can be realized as

$$\widetilde{\int_{H_t}} e^{\frac{i}{2\hbar}\int_0^t |\dot{\gamma}(s)|^2 ds - \lambda\int_0^t |\gamma(s)+x|^2 ds} e^{\sqrt{\lambda}\int_0^t (\gamma(s)+x)\cdot\dot{\omega}_n(s) ds} \psi_o(\gamma(0)+x) d\gamma =$$

$$= e^{\frac{-ia^2|x|^2 t}{2\hbar} + \sqrt{\lambda}x\cdot\omega_n(t)} \widetilde{\int_{H_t}} e^{\frac{i}{2\hbar}\langle\gamma,(I+L)\gamma\rangle} e^{\langle l_n,\gamma\rangle} \int_{\mathbb{R}^d} e^{i\alpha\cdot x} e^{i\langle b(\alpha,x),\gamma\rangle} \tilde{\psi}_0(\alpha) d\alpha d\gamma$$

where $b(\alpha,x) \in H_t$, precisely:

$$b(\alpha,x)(s) = \alpha(t-s) - \frac{xa^2}{2\hbar}(t^2 - s^2),$$

One can directly verify that the function $f(\gamma) \equiv \int_{\mathbb{R}^d} e^{i\alpha\cdot x} e^{i\langle b(\alpha,x),\gamma\rangle} \tilde{\psi}_0(\alpha) d\alpha$ is the Fourier transform of a measure $\mu \in \mathcal{M}(H)$, that is:

$$\mu(d\gamma) = \int_{\mathbb{R}^d} e^{i\alpha\cdot x} \tilde{\psi}_0(\alpha) \delta_{b(\alpha,x)}(d\gamma) d\alpha$$

so we can apply theorem 2 and have:

$$\psi_n = e^{\frac{-ia^2|x|^2t}{2\hbar}+\sqrt{\lambda}\cdot\omega_n(t)} \int_{\mathbb{R}^d} e^{i\alpha\cdot x} \det(I+L)^{-1/2} e^{\frac{-i\hbar}{2}\langle b(\alpha,x)-il_n,(I+L)^{-1}(b(\alpha,x)-il_n)\rangle}\tilde{\psi}_0(\alpha)d\alpha$$

By simple calculations we get the final result:

$$\psi_n(t,x) = \int_{\mathbb{R}^d} G_n(t,x,y)\psi_0(y)dy$$

where $G_n(t,x,y)$ is given by:

$$\begin{aligned} G_n(t,x,y) &\equiv \frac{1}{\sqrt{2\pi i\hbar}}\sqrt{\frac{a}{\sin(at)}}e^{\sqrt{\lambda}x\cdot\omega_n(t)-\frac{\sqrt{\lambda}ax}{\sin(at)}\cdot\int_0^t\omega_n(s)\cos(as)ds} \\ & e^{\frac{i\hbar\lambda}{2}\int_0^t|\omega_n(s)|^2ds}e^{\frac{i\hbar\lambda}{2}(-a\int_0^t\omega_n(s)\cdot\int_s^t\omega_n(s')\sin[a(s'-s)]ds'ds)} \\ & \cdot e^{\frac{i\hbar\lambda}{2}(-a\int_0^t\sin(as)\omega_n(s)ds\cdot\int_0^t\cos(as)\omega_n(s)ds-a\cot(at)|\int_0^t\cos(as)\omega_n(s)ds|^2)} \\ & e^{\frac{i}{2\hbar}(\cot(at)(|x|^2+|y|^2)-\frac{2x\cdot y}{\sin(at)})}\cdot e^{a\sqrt{\lambda}y\cdot(\cot(at)\int_0^t\cos(as)\omega_n(s)ds+\int_0^t\sin(as)\omega_n(s)ds)} \end{aligned} \tag{3.32}$$

which is, as one can easily directly verify, the fundamental solution to the approximate Cauchy problem (3.25). □

2. **The convergence of the sequence of approximated solutions.** We will prove the following result:

Lemma 4. *The following equation*

$$\begin{cases} d\psi = -\frac{i}{\hbar}H\psi dt - \lambda|x|^2\psi dt + \sqrt{\lambda}x\cdot\psi\circ dW(t) & t>0 \\ \psi(0,x)=\psi_0(x), \quad \psi_0\in S(\mathbb{R}^d) \end{cases} \tag{3.33}$$

has a unique strong solution given by the Feynman path integral

$$\psi(t,x) = \widetilde{\int} e^{\frac{i}{2\hbar}\int_0^t|\dot\gamma(s)|^2ds-\lambda\int_0^t|\gamma(s)+x|^2ds}e^{\sqrt{\lambda}\int_0^t(\gamma(s)+x)\cdot dW(s)}\psi_0(\gamma(0)+x)d\gamma$$

rigorously realized as the infinite dimensional oscillatory integral with complex phase on H_t

$$e^{-\lambda|x|^2+\sqrt{\lambda}x\cdot\omega(t)}\widetilde{\int_{H_t}} e^{\frac{i}{2\hbar}\langle\gamma,(I+L)\gamma\rangle}e^{\langle l,\gamma\rangle}e^{-2\lambda x\cdot\int_0^t\gamma(s)ds}\psi_0(\gamma(0)+x)d\gamma$$

Moreover it can be represented by the process

$$\psi(t,x) = \int_{\mathbb{R}^d} G(t,x,y)\psi_0(y)dy$$

where

$$G(t,x,y)=\frac{1}{\sqrt{2\pi i\hbar}}\sqrt{\frac{a}{\sin(at)}}e^{\sqrt{\lambda}x\cdot\omega(t)-\frac{\sqrt{\lambda}ax}{\sin(at)}\cdot\int_0^t\cos(as)\omega(s)ds}$$

$$e^{\frac{i\hbar\lambda}{2}(-a\int_0^t\omega(s)\cdot\int_s^t\omega(s')\sin[a(s'-s)]ds'ds)}$$

$$\cdot e^{\frac{i\hbar\lambda}{2}(-a\int_0^t\sin(as)\omega(s)ds\cdot\int_0^t\cos(as)\omega(s)ds-a\cot(at)|\int_0^t\cos(as)\omega(s)ds|^2)}$$

$$e^{\frac{i}{2\hbar}\left(\cot(at)(|x|^2+|y|^2)-\frac{2x\cdot y}{\sin(at)}\right)}e^{a\sqrt{\lambda}y\cdot\frac{1}{\sin(at)}(\int_0^t\cos[a(s-t)]\omega(s)ds)}$$

Proof. As first we consider the sequence $\psi_n(t,x)=\int_{\mathbb{R}^d}G_n(t,x,y)\psi_0(y)dy$. Using the dominated convergence theorem we have that :

$$\mathbf{P}\Big(\lim_{n\to\infty}\int_{\mathbb{R}^d}|\psi_n(t,x)-\tilde{\psi}(t,x)|^2dx\to 0\Big)=1 \tag{3.34}$$

with $\tilde{\psi}(t,x)=\int_{\mathbb{R}}G(t,x,y)\psi_0(y)dy$, as:

$$\lim_{n\to\infty}|G_n(t,x,y)-G(t,x,y)|\to 0$$

for all $t\in[0,T]$ and $x,y\in\mathbb{R}^d$. Moreover, one can see by a direct computation that $a=\sqrt{-2i\hbar\lambda}$ can be chosen is such a way that:

$$|\int_{\mathbb{R}^d}G_n(t,x,y)\psi_0(y)dy|^2\leq C(t)e^{P(t,x)}\|\psi_0(y)\|^2, \tag{3.35}$$

where $P(t,x)$ is a second order polynomial with negative leading coefficient and $C(t)$ and $P(t,x)$ are continuous functions of the variable $t\in[0,T]$. Applying the Itô formula to the limit process $\tilde{\psi}(t)$ we see that it verifies equation (3.33) for every (t,x,y). Since the kernel $G(t,x,y)$ is $\mathcal{F}_t$ adapted by construction it follows that the solution is predictable. By direct computation and using estimates analogous to (3.35) one can verify that $\tilde{\psi}$ is a strong solution. On the other hand every $\psi_n(t,x)$ is equal to

$$\widetilde{\int_{H_t}}e^{\frac{i}{2\hbar}\int_0^t|\dot{\gamma}(s)|^2ds-\lambda\int_0^t|\gamma(s)+x|^2ds}e^{\sqrt{\lambda}\int_0^t(\gamma(s)+x)\cdot\dot{\omega}_n(s)ds}\psi_0(\gamma(0)+x)d\gamma=$$

$$=e^{\frac{-ia^2|x|^2t}{2\hbar}+\sqrt{\lambda}x\cdot\omega_n(t)}\widetilde{\int_{H_t}}e^{\frac{i}{2\hbar}\langle\gamma,(I+L)\gamma\rangle}e^{\langle l_n,\gamma\rangle}e^{-i\int_0^t a^2x\cdot\gamma(s)ds}\psi_0(\gamma(0)+x)d\gamma$$

$$=e^{\frac{-ia^2|x|^2t}{2\hbar}+\sqrt{\lambda}x\cdot\omega_n(t)}\det(I+L)^{-1/2}\int_{H_t}e^{\frac{-i\hbar}{2}\langle\gamma-il_n,(I+L)^{-1}(\gamma-il_n)\rangle}\mu(d\gamma)$$

where $\mu(d\gamma)$ is the measure on H whose Fourier transform is the function $\gamma\to e^{-i\int_0^t a^2x\cdot\gamma(s)ds}\psi_0(\gamma(0)+x)$.

We have $||l_n - l||^2_H \to 0$ as $n \to \infty$, where $l(s) = \sqrt{\lambda}\int_s^t \omega(r)dr$. Therefore, by the Lebesgue's dominated convergence theorem, we have that, for every $x \in \mathbb{R}^d$:

$$\lim_{n\to\infty} e^{\frac{-ia^2|x|^2 t}{2\hbar}+\sqrt{\lambda}x\cdot\omega_n(t)} \det(I+L)^{-1/2} \int_{H_t} e^{\frac{-i\hbar}{2}\langle\gamma-il_n,(I+L)^{-1}(\gamma-il_n)\rangle}\mu(d\gamma)$$

$$= e^{\frac{-ia^2|x|^2 t}{2\hbar}+\sqrt{\lambda}x\omega(t)} \det(I+L)^{-1/2} \int_{H_t} e^{\frac{-i\hbar}{2}\langle\gamma-il,(I+L)^{-1}(\gamma-il)\rangle}\mu(d\gamma) \tag{3.36}$$

Therefore, taking into account the uniqueness of the pointwise limit, we have shown that:

$$\psi(t,x) = \int_{\mathbb{R}} G(t,x,y)\psi_0(y)dy =$$
$$\widetilde{\int_{H_t}} e^{\frac{i}{2\hbar}\int_0^t |\dot\gamma(s)|^2 ds - \lambda\int_0^t |\gamma(s)+x|^2 ds} e^{\int_0^t (\gamma(s)+x)\cdot dW(s)}\psi_0(\gamma(0)+x)d\gamma. \tag{3.37}$$

□

Remark 8. *The result can be extended by continuity to all $\psi_0 \in L^2(\mathbb{R}^d)$, using the density of $S(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$.*

3. **The proof of Feynman-Kac-Ito formula by means of Dyson expansion**

In this subsection we generalize our previous results to the case $H = -\hbar^2\Delta/2 + V$ and complete the proof of theorem 13. We follow here the technique of Elworthy and Truman [57].

We set for $t > 0$, $x \in \mathbb{R}^d$:

$$\Theta(t,0)\psi_0(x) = \widetilde{\int_{H_t}} e^{\frac{i}{2\hbar}\int_0^t |\dot\gamma(s)|^2 ds - \lambda\int_0^t |\gamma(s)+x|^2 ds} e^{-\frac{i}{\hbar}\int_0^t V(\gamma(s)+x)ds}$$
$$\cdot e^{\sqrt{\lambda}\int_0^t (\gamma(s)+x)\cdot dW(s)}\psi_0(\gamma(0)+x)d\gamma \tag{3.38}$$

and

$$\Theta_0(t,0)\psi_0(x) = \widetilde{\int_{H_t}} e^{\frac{i}{2\hbar}\int_0^t |\dot\gamma(s)|^2 ds - \lambda\int_0^t |\gamma(s)+x|^2 ds} e^{\sqrt{\lambda}\int_0^t (\gamma(s)+x)\cdot dW(s)}\psi_0(\gamma(0)+x)d\gamma. \tag{3.39}$$

Then we have:

$$\Theta(t,0)\psi_0(x) = e^{\frac{-ia^2|x|^2 t}{2\hbar}+\sqrt{\lambda}x\cdot\omega(t)} \widetilde{\int_{H_t}} e^{\frac{i}{2\hbar}\langle\gamma,(I+L)\gamma\rangle} e^{\langle l,\gamma\rangle} e^{-i\int_0^t a^2 x\cdot\gamma(s)ds}$$
$$\cdot e^{-i\int_0^t V(x+\gamma(s))ds}\psi_0(\gamma(0)+x)d\gamma \tag{3.40}$$

Let $\mu_0(\psi)$ be the measure on H_t such that its Fourier transform evaluated in $\gamma \in H_t$ is $\psi_0(\gamma(0)+x)$.

For $0 \le u \le t$ let $\mu_u(V,x)$, $\nu_u^t(V,x)$ and $\eta_u^t(x)$ be the measures on H_t, whose Fourier transforms when evaluated at $\gamma \in H_t$ are respectively $V(x+\gamma(u))$, $\exp\Big(-i\int_u^t V(x+\gamma(s))ds\Big)$, and $\exp\Big(-i\int_u^t a^2 x\gamma(s)ds\Big)$. We shall often write $\mu_u \equiv \mu_u(V,x)$, $\nu_u^t \equiv \nu_u^t(V,x)$ and $\eta_u^t \equiv \eta_u^t(x)$ If $\{\mu_u : a \le u \le b\}$ is a family in $\mathcal{M}(H_t)$, we shall let $\int_a^b \mu_u du$ denote the measure on H_t given by :

$$f \to \int_a^b \int_{H_t} f(\gamma)\mu_u(d\gamma)du$$

whenever it exists.
Then, since for any continuous path γ

$$\begin{aligned} &\exp\Big(-\frac{i}{\hbar}\int_0^t V(\gamma(s))ds\Big) = \\ &1 - \frac{i}{\hbar}\int_0^t V(\gamma(u))\exp\Big(-\frac{i}{\hbar}\int_u^t V(\gamma(s))ds\Big)du, \end{aligned} \tag{3.41}$$

we have

$$\nu_0^t = \delta_0 - \frac{i}{\hbar}\int_0^t (\mu_u * \nu_u^t)du \tag{3.42}$$

where δ_0 is the Dirac measure at $0 \in H_t$.
By the Cameron-Martin formula:

$$\begin{aligned} \Theta(t,0)\psi_0(x) = e^{\frac{-ia^2|x|^2t}{2\hbar}+\sqrt{\lambda}x\cdot\omega(t)} \det(I+L)^{-1/2} \\ \cdot \int_{H_t} e^{\frac{-i\hbar}{2}\langle \alpha - il, (I+L)^{-1}(\alpha - il)\rangle}(\eta_0^t * \nu_0^t * \mu_0(\psi))(d\alpha) \end{aligned} \tag{3.43}$$

Applying to this equality (3.42) we obtain:

$$\begin{aligned} &\Theta(t,0)\psi_0(x) = \\ &e^{\frac{-ia^2|x|^2t}{2\hbar}+\sqrt{\lambda}x\cdot\omega(t)} \det(I+L)^{-1/2}\int_{H_t} e^{\frac{-i\hbar}{2}\langle \alpha - il, (I+L)^{-1}(\alpha - il)\rangle}(\eta_0^t * \mu_0(\psi))(d\alpha) + \\ &-\frac{i}{\hbar}\int_0^t e^{\frac{-ia^2|x|^2t}{2\hbar}+\sqrt{\lambda}x\cdot\omega(t)} \det(I+L)^{-1/2} \\ &\qquad \cdot \int_{H_t} e^{\frac{-i\hbar}{2}\langle \alpha - il, (I+L)^{-1}(\alpha - il)\rangle}(\eta_0^t * \mu_u(V,x) * \nu_u^t * \mu_0(\psi))(d\alpha)du \\ &= \Theta_0(t,0)\psi_0(x) - \frac{i}{\hbar}\int_0^t \widetilde{\int_{H_t}} e^{\frac{i}{2\hbar}\int_0^t |\dot\gamma(s)|^2ds - \lambda\int_0^t |\gamma(s)+x|^2 ds} e^{-\frac{i}{\hbar}\int_u^t V(\gamma(s)+x)ds} \\ &\qquad e^{\sqrt{\lambda}\int_0^t (\gamma(s)+x)\cdot dW(s)} V(\gamma(u)+x)\psi_0(\gamma(0)+x)d\gamma du \end{aligned}$$

By the Fubini theorem for oscillatory integrals (see [12, 4]), we get that

$$\widetilde{\int_{H_t}} e^{\frac{i}{2\hbar}\int_0^t |\dot\gamma(s)|^2 ds-\lambda\int_0^t|\gamma(s)+x|^2ds} e^{-\frac{i}{\hbar}\int_u^t V(\gamma(s)+x)ds} e^{\sqrt{\lambda}\int_0^t(\gamma(s)+x)\cdot dW(s)} V(\gamma(u)+x)\cdot$$

$$\psi_0(\gamma(0)+x)d\gamma = \widetilde{\int_{H_{u,t}}} e^{\frac{i}{2\hbar}\int_u^t |\dot\gamma_2(s)|^2ds-\lambda\int_u^t|\gamma_2(s)+x|^2ds} e^{-\frac{i}{\hbar}\int_u^{tu} V(\gamma_2(s)+x)ds}\cdot$$

$$e^{\sqrt{\lambda}\int_u^t(\gamma_2(s)+x)\cdot dW(s)} V(\gamma_2(u)+x)\widetilde{\int_{H_{0,u}}} e^{\frac{i}{2\hbar}\int_0^u|\dot\gamma_1(s)|^2ds-\lambda\int_0^u|\gamma_1(s)+\gamma_2(u)+x|^2ds}\cdot$$

$$e^{\sqrt{\lambda}\int_0^u(\gamma_1(s)+\gamma_2(u)+x)\cdot dW(s)}\psi_0(\gamma_1(0)+\gamma_2(u)+x)d\gamma_1 d\gamma_2.$$

Here $\gamma_1 \in H_{0,u}$ and $\gamma_2 \in H_{u,t}$ are the integration variables. We denote by $H_{r,s}$ the Cameron-Martin space of paths $\gamma : [r,s] \to \mathbb{R}^d$.
Finally we have:

$$\Theta(t,0)\psi_0(x) = \Theta_0(t,0)\psi_0(x) - i\int_0^t \Theta(t,u)(V\Theta_0(u,0)\psi_0)(x)du \tag{3.44}$$

Now the iterative solution of the latter integral equation is the Dyson series for $\Theta(t,0)$, which coincides with the corresponding power series expansion of the solution of the stochastic Schrödinger equation, which converges strongly in $L^2(\mathbb{R}^d)$. The equality holds pointwise. On the other hand, following [60], it is possible to prove that the problem (3.33) has a strong solution that verifies (3.44) in the L^2 sense, therefore $\Theta(t,0)\psi_0$ coincides with the solution $\psi(t)$. This concludes the proof of theorem 13.

□

3.4 Momentum measurement

In this section we study Belavkin's equation describing the continuous measurement of the momentum p of a $d-$dimensional quantum particle:

$$\begin{cases} d\psi(t,x) = -\frac{i}{\hbar}H\psi(t,x)dt + \frac{\lambda\hbar^2}{2}\Delta\psi(t,x)dt - i\sqrt{\lambda}\hbar\nabla\psi(t,x)dW(t) \\ \psi(0,x) = \psi_0(x) \qquad (t,x)\in[0,T]\times\mathbb{R}^d \end{cases} \tag{3.45}$$

Our main interest is to give a rigorous definition of the solution as a Feynman path integral defined on the phase space. Once we have defined a Feynman path integral as a candidate for the solution of (3.45), we still have to prove that it solves effectively the problem (3.45). When the evolution of the free particle (i.e. for $V=0$) is considered, equation (3.45) reduces to the following:

$$\begin{cases} d\psi(t,x) = (\frac{i\hbar}{2m}\Delta + \frac{\lambda\hbar^2}{2}\Delta)\psi(t,x)dt - i\sqrt{\lambda}\hbar\nabla\psi(t,x)dW(t) \\ \psi(0,x) = \psi_0(x) \qquad (t,x)\in[0,t]\times\mathbb{R}^d \end{cases} \tag{3.46}$$

In this situation we derive from our rigorously defined infinite dimensional oscillatory integral an expression for the solution as a finite dimensional integral involving the initial data and a Green kernel and prove directly that our Feynman path integral represents the strong solution for the problem (3.46). In the more general situation of problem (3.45) we use an analytic result based on the method of stochastic characteristics to show that our Feynman path integral is in fact the solution to the Belavkin equation (3.45).
In the first and second subsections we provide the analytic tools to guarantee that the infinite dimensional oscillatory integral we shall define in the third subsection gives indeed a solution of problem (3.45).

3.4.1 Existence and uniqueness results

In this subsection we are interested in finding a unique strong solution for problem (3.45). Let us first introduce the framework in which we will consider the problem.
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $W(t)$ a d- dimensional standard Brownian motion, we will denote by $\mathcal{F}_t$ its natural filtration completed with the null sets of $\mathcal{F}$. Let $L^2(\mathbb{R}^d)$ be the complex Hilbert space of square integrable functions endowed with its natural inner product $\langle f, g\rangle = \int_{\mathbb{R}^d} \overline{f}(x)g(x)dx$, we will denote by $|\cdot|$ the corresponding norm induced by the sesquilinear form. We denote by A the realization of the operator $\frac{i}{\hbar}\frac{\hbar^2}{2m}\Delta + \frac{\lambda\hbar^2}{2}\Delta$ in the space $\mathcal{H} = L^2(\mathbb{R}^d)$, with domain $D(A) = \{f \in L^2(\mathbb{R}^d) : \Delta f \in L^2(\mathbb{R}^d)\} \subset L^2(\mathbb{R}^d)$. It is easy to prove the following property:

Proposition 1. *The operator A is closed on $D(A)$, is dissipative and generates a C_0-semigroup e^{tA} in $\mathcal{H}$. Moreover*

$$e^{tA} = e^{(\frac{i\hbar}{2m}\Delta)t} e^{(\frac{\lambda\hbar^2}{2}\Delta)t} \tag{3.47}$$

Proof. The first assertion is a straightforward application of the Lumer Phillips Theorem (see [85]), the operator A being dissipative and with dense domain in $L^2(\mathbb{R}^d)$. Identity (3.47) is guaranteed by the Trotter product formula (see [85]) and the fact that Δ and $i\Delta$ are generators of C_0-contractive semigroups and commute. □

The domain $D(A)$, endowed with the graph norm is equivalent to the complex Hilbert space $H^2(\mathbb{R}^d)$ of functions with all the first and second partial derivatives, defined in distributional sense, in $L^2(\mathbb{R}^d)$. The scalar product in $H^2(\mathbb{R}^d)$ is given in its natural way (Sobolev space). We will denote with B the realization of $-i\sqrt{\lambda}\hbar\nabla\cdot$ in $L^2(\mathbb{R}^d)$, with domain $D(B) = \{f \in L^2(\mathbb{R}^d) : (-\Delta)^{\frac{1}{2}} f \in L^2(\mathbb{R}^d)\}$. The graph norm induced by the operator B is equivalent to the usual norm of the Sobolev space $H^1(\mathbb{R}^d)$.

Finally we denote by $L^2_W([0,T];H^2(\mathbb{R}^d))$ the space of $L^2(\mathbb{R}^d)$ valued processes which are predictable and belong to $L^2([0,T];L^2(\Omega;H^2(\mathbb{R}^d))$([4]). Similarly the space of $L^2(\mathbb{R}^d)$ valued processes that are predictable and belong to $C([0,T];L^2(\Omega;L^2(\mathbb{R}^d))$ is denoted by $C_W([0,T];L^2(\mathbb{R}^d))$. The two spaces are endowed respectively with the following norms:

$$|u|^2_{L^2_W([0,T];H^2(\mathbb{R}^d))} \doteq \int_0^T \mathbb{E}|u(t)|^2_{H^2(\mathbb{R}^d)}\,dt$$

and

$$|u|^2_{C_W([0,T];L^2(\mathbb{R}^d))} \doteq \sup_{t\in[0,T]} \mathbb{E}|u(t)|^2_{L^2(\mathbb{R}^d)}$$

We recall the definition of a strong solution for problem (3.46), see also [48] for a more general definition:

Definition 8. *Given an initial data $\psi_0 \in H^2(\mathbb{R}^d)$, we define a solution ψ for problem* (3.46) *as a process $\psi \in C_W([0,T];L^2(\mathbb{R}^d)) \cap L^2_W([0,T];H^2(\mathbb{R}^d))$, that verifies the following equation:*

$$\begin{cases} d\psi(t) = A\psi(t)dt + B\psi(t)dW(t) \quad \mathbb{P}-a.s. \\ \psi(0) = \psi_0 \end{cases}$$

We can prove the following:

Proposition 2. *Problem* (3.46) *has a unique strong solution that is represented in mild form as follows:*

$$\psi(t) = e^{tA}\psi_0 + \int_0^t e^{(t-s)A}B\psi(s)\,dW(s) \tag{3.48}$$

Proof. We first notice that A admits a spectral decomposition: indeed it can be diagonalized by means of Fourier transform

$$\tilde{\psi}(k) = \frac{1}{(2\pi\hbar)^{d/2}}\int_{\mathbb{R}^d} e^{-ikx/\hbar}\psi(x)dx$$

$$\widetilde{A\psi}(k) = -(\frac{i}{2m\hbar} + \frac{\lambda}{2})k^2\tilde{\psi}(k),$$

therefore it is possible to rearrange the proof of [47][Theorem 4.3.5, pag.79] to prove the result.

[4]Note that $L^2([0,T];L^2(\Omega;H^2(\mathbb{R}^d)) \simeq L^2([0,T]\times\Omega;H^2(\mathbb{R}^d))$ by the Fubini-Tonelli theorem ($\simeq$ means isomorphism between Hilbert spaces).

One has that there exist positive constants $K(\hbar,\lambda,m)$ and $C(T)$ such that:

$$\int_0^T \mathbb{E}|(-A)\int_0^t e^{(t-s)A}B\psi(s)\,dW(s)|^2 dt \le \frac{K(\hbar,\lambda,m)}{2}\int_0^T \mathbb{E}|(-A)\psi(s)|^2\,ds \tag{3.49}$$

$$\sup_{t\in[0,T]} \mathbb{E}|\int_0^t e^{(t-s)A}B\psi(s)\,dW(s)|^2 \le C\mathbb{E}\int_0^T |e^{(t-s)A}B\psi(s)|^2\,ds \tag{3.50}$$
$$\le C(T)\sup_{t\in[0,T]}\mathbb{E}|\psi(t)|^2$$

We will show only the first inequality, as the second follows in a similar way. Setting $\bar{A} =: 1/(\frac{i}{2m\hbar}+\frac{\lambda}{2})A$, we have that the graph norm of $D(A)$ is equivalent to the graph norm of $D(\bar{A})$, therefore there exists a positive constant $C(\hbar,\lambda,m)$ such that:

$$\int_0^T \mathbb{E}|(-A)\int_0^t e^{(t-s)A}B\psi(s)\,dW(s)|^2 \tag{3.51}$$
$$\le \frac{C(\hbar,\lambda,m)}{2}\int_0^T \mathbb{E}|(-\bar{A})\int_0^t e^{(t-s)A}B\psi(s)\,dW(s)|^2$$

Then for any $\varepsilon > 0$ we have:

$$\int_0^T \mathbb{E}|(-\bar{A})(I-\epsilon\bar{A})^{-1}\int_0^t e^{(t-s)\bar{A}}B\psi(s)\,dW(s)|^2\,dt$$
$$= \int_0^T \mathbb{E}\int_0^t |(-\bar{A})^{1/2}(I-\epsilon\bar{A})^{-1}e^{(t-s)\bar{A}}(-\bar{A})^{1/2}B\psi(s)|^2\,ds\,dt$$
$$= \int_0^T\int_{\mathbb{R}^d} k^2\mathbb{E}\int_0^t \frac{e^{-k^2(t-s)}}{(1+\epsilon k^2)^2}|\widetilde{(-\bar{A})^{1/2}B\psi}(s,k)|^2\,dk\,ds\,dt$$
$$= \frac{1}{2}\int_0^T\int_{\mathbb{R}^d}\mathbb{E}\frac{1-e^{-k^2(T-s)}}{(1+\epsilon k^2)^2}|\widetilde{(-\bar{A})^{1/2}B\psi}(s,k)|^2\,dk\,ds$$
$$\le \frac{1}{2}\mathbb{E}\int_0^T |(-\bar{A})^{1/2}B\psi(s)|^2\,ds \le \frac{C'(\hbar,\lambda,m)}{2}\mathbb{E}\int_0^T |(-A)\psi(s)|^2\,ds$$

Now, letting $\epsilon\to 0$ and recalling that (3.51) holds, we deduce (3.49) from the inequality. These are the two key estimates needed to prove, using a fixed point technique in the spaces $L^2_W([0,T];H^2(\mathbb{R}^d))$ and $C_W([0,T];L^2(\mathbb{R}^d))$, the existence of the mild representation for the solution. But then, having the regularity implied by the definition of the spaces $L^2_W([0,T];H^2(\mathbb{R}^d))$ and $C_W([0,T];L^2(\mathbb{R}^d))$, it is possible to apply the Itô formula to ψ written in the form (3.48), obtaining that the mild solution is in fact a strong solution. □

In the next subsection we are going to prove another characterization of the strong solution of (3.46): we will show that there exists a strict relation between the solution of (3.46) and the solution of a classical Schrödinger equation. To this purpose we will consider Eq.(3.46) in the momentum representation:

$$\tilde{\psi}(k) = \frac{1}{(2\pi\hbar)^{d/2}} \int_{\mathbb{R}^d} e^{-ikx/\hbar}\psi(x)dx$$

We need this second characterization in order to identify the solution with the Feynman path integral rigorously defined in subsection 3.4.3.

3.4.2 Solution by the stochastic characteristics method

Applying to both sides of (3.46) the Fourier transform we obtain the following equation :

$$d\tilde{\psi}(k) = -(\frac{i}{\hbar}H + \frac{\lambda y^2}{2})\tilde{\psi}(k)dt + \sqrt{\lambda}y\tilde{\psi}(k)dW(t), \tag{3.52}$$

where H is the Hamiltonian of the free particle, which in momentum representation is simply the multiplication operator:

$$H\tilde{\psi}(k) = \frac{k^2}{2m}\tilde{\psi}(k).$$

The corresponding Cauchy problem assumes then the following form

$$\begin{cases} d\tilde{\psi}(k) = -(\frac{i}{\hbar}\frac{k^2}{2m} + \frac{\lambda k^2}{2})\tilde{\psi}(k)dt + \sqrt{\lambda}k\tilde{\psi}(k)dW(t) \\ \tilde{\psi}(0,k) = \tilde{\psi}_0(k) \end{cases} \tag{3.53}$$

In this section we show that problem (3.53) is equivalent in a suitable sense, to a deterministic Schrödinger equation expressed in momentum coordinates. The main tool is a simple application of the stochastic characteristics method that allows to transform the stochastic partial differential equation into a family of deterministic equations. Let us denote by ρ the function defined by $\rho(t,k) = \exp(-\sqrt{\lambda}kW_t + \lambda k^2 t)$.
We can prove the following:

Proposition 3. *The strong solution of* (3.53) *has the following representation:*

$$\tilde{\psi}(t,k) = e^{\sqrt{\lambda}kW(t)-(\lambda k^2+\frac{i}{\hbar}\frac{k^2}{2m})t}\tilde{\psi}_0(k) \tag{3.54}$$

Proof. The proof is divided in two steps: in the first we will prove that (3.54) solves problem (3.53) and it is a strong solution, in the second we prove that

this solution is the unique strong solution.
first step:
Let us consider the following problem:

$$\begin{cases} d\phi(k) = -\frac{i}{\hbar}\frac{k^2}{2m}\phi(k)dt \\ \\ \phi(0,k) = \tilde{\psi}_0(k) \end{cases} \tag{3.55}$$

It is well known that if $\tilde{\psi}_0 \in H^2(\mathbb{R}^d)$, then the solution $\phi(t,k) = e^{-\frac{i}{\hbar}\frac{k^2 t}{2m}}\tilde{\psi}_0(k)$ of problem (3.55) belongs to $L^2(0,T;H^2(\mathbb{R}^d)) \cap C([0,T];L^2(\mathbb{R}^d))$. Our intention is to prove that the function $\rho^{-1}(t)\phi(t)$ that corresponds to (3.54), is actually a solution to problem (3.53). We apply, first formally, the Itô formula to the function $\rho^{-1}(t)\phi(t)$:

$$\rho^{-1}(t)\phi(t) = \tilde{\psi}_0 - \int_0^t \frac{i}{\hbar}\Big[\frac{k^2}{2m} + \frac{\lambda k^2}{2}\Big]\rho^{-1}(s)\phi(s)\,ds \tag{3.56}$$
$$+ \int_0^t \sqrt{\lambda}k\rho^{-1}(s)\phi(s)\,dW(s).$$

Setting $\tilde{\psi}(t) = \rho^{-1}(t)\phi(t)$ we have that ψ is a solution to equation (3.53). This procedure becomes rigorous as long as we can give a meaning to ρ^{-1}. Notice that for each fixed t the multiplication operator $\sqrt{\lambda}kW(t) - \lambda k^2 t$ is the generator of a C_0 semigroup, being self adjoint and having the leading term dissipative, therefore $\rho^{-1}(t)$ can be regarded as the semigroup $e^{(\sqrt{\lambda}kW(t)-\lambda k^2 t)s}$ evaluated at $s = 1$, see also [60]. Let us take an element $\varphi \in D(\Delta)$, then the vector $\rho^{-1}(t)\varphi$, which in momentum representation is given by $e^{(\sqrt{\lambda}kW(t)-\lambda k^2 t)}\tilde{\varphi}(k)$ is still in $D(\Delta)$ thanks to the properties of commutativity of the generator with the semigroup. Moreover for every fixed $k \in \mathbb{R}^d$ it is possible to evaluate the Itô differential of the process $e^{(\sqrt{\lambda}kW(t)-\lambda k^2 t)}\tilde{\varphi}(k)$. It is easy to prove using the Fubini Theorem and thanks to the spectral decomposition of $\rho^{-1}(t)e^{\frac{i}{\hbar}H}$, the following estimate:

$$\mathbb{E}\int_0^T |k^2 e^{\sqrt{\lambda}kW(t)-(\lambda k^2+\frac{i}{\hbar}k^2)t}\tilde{\psi}_0(k)|^2_{L^2(\mathbb{R}^d)}\,dt \tag{3.57}$$
$$= \int_0^T \{\int_{\mathbb{R}^d} e^{-2\lambda k^2 t}k^4\tilde{\psi}_0^2(k)\mathbb{E}[e^{2\sqrt{\lambda}kW(t)}]\,dk\}\,dt \le T\int_{\mathbb{R}^d} k^4\tilde{\psi}_0^2(k)\,dk$$

This implies that the identity (3.56) can be understood in the space $L^2(\mathbb{R}^d)$.

So far we have obtained a solution $\tilde{\psi}(t) = \rho(t)^{-1}\phi(t)$ for equation (3.53) and the regularity for $\tilde{\psi}$ is directly inherited from ϕ, by the special expression for $\tilde{\psi}$: $\tilde{\psi} \in L^2_W(0,T;H^2(\mathbb{R}^d)) \cap C_W(0,T;L^2(\mathbb{R}^d))$.

Remark 1. *The form* (3.54) *of the solution shows that it is no longer unitary pathwise.*

second step: Uniqueness of the solution. We will find an a priori estimate for the solution that ensures the uniqueness of the solution. Since the problem is linear if one finds a continuous dependence on the initial data one gets immediately that the solution is unique. Now let us consider the solution $\tilde{\psi}$ splitted in real part and imaginary part, $\tilde{\psi} = \psi_1 + i\psi_2$, and consider $d\langle\tilde{\psi}(t), \tilde{\psi}(t)\rangle = d(|\psi_1(t)|^2 + |\psi_2(t)|^2)$. The equations solved by the real and the imaginary parts are respectively:

$$
\begin{aligned}
d\psi_1(t) &= \tfrac{1}{\hbar}\tfrac{k^2}{2m}\psi_2(t)\,dt - \tfrac{\lambda k^2}{2}\psi_1(t)\,dt + \sqrt{\lambda}k\psi_1(t)\,dW(t) \\
d\psi_2(t) &= \tfrac{-1}{\hbar}\tfrac{k^2}{2m}\psi_1(t)\,dt - \tfrac{\lambda k^2}{2}\psi_2(t)\,dt + \sqrt{\lambda}k\psi_2(t)\,dW(t)
\end{aligned}
\tag{3.58}
$$

Now we can apply the Itô formula to $|\psi_1(t)|^2$ and $|\psi_2(t)|^2$:

$$
\begin{aligned}
&|\psi_1(t)|^2 = |Re\tilde{\psi}_0|^2 + 2\tfrac{1}{\hbar}\textstyle\int_0^t\langle\tfrac{k^2}{2m}\psi_2(s), \psi_1(s)\rangle\,ds - 2\int_0^t\langle\tfrac{\lambda k^2}{2}\psi_1(s), \psi_1(s)\rangle\,ds \\
&+2\textstyle\int_0^t\langle\sqrt{\lambda}k\psi_1(s), \psi_1(s)\rangle\,dW(s) + \lambda\int_0^t|k\psi_1(s))|^2\,ds.
\end{aligned}
$$

and

$$
\begin{aligned}
&|\psi_2(t)|^2 = |Im\tilde{\psi}_0|^2 - 2\tfrac{1}{\hbar}\textstyle\int_0^t\langle\tfrac{k^2}{2m}\psi_1(s), \psi_2(s)\rangle\,ds - 2\int_0^t\langle\tfrac{\lambda k^2}{2}\psi_2(s), \psi_2(s)\rangle\,ds \\
&+2\textstyle\int_0^t\langle\sqrt{\lambda}k\psi_2(s), \psi_2(s)\rangle\,dW(s) + \int_0^t\lambda|k\psi_2(s))|^2\,ds.
\end{aligned}
$$

thus:

$$
\begin{aligned}
&|\psi_1(t)|^2 + |\psi_2(t)|^2 = |\tilde{\psi}_0|^2 - \textstyle\int_0^t\lambda(|k\psi_1(s)|^2 + |k\psi_2(s)|^2)\,ds \\
&+2\textstyle\int_0^t\sqrt{\lambda}(|\sqrt{k}\psi_1(s)|^2 + |\sqrt{k}\psi_2(s)|^2)\,dW(s) \\
&+\textstyle\int_0^t\lambda(|k\psi_1(s))|^2 + |k\psi_2(s))|^2)\,ds.
\end{aligned}
\tag{3.59}
$$

We recall that $\tilde{\psi} \in L^2_W(0, T; H^2(\mathbb{R}^d))$, then the stochastic integral in (3.59) is a martingale. Thus passing to the expected value in (3.59) we get that:

$$
E(|\tilde{\psi}(t)|^2) = |\tilde{\psi}_0|^2 \tag{3.60}
$$

Moreover one reads from identity (3.59) that $|\tilde{\psi}(t)|^2$ is a martingale with respect to the filtration $\mathcal{F}_t$ and that the solution is unique in the class of strong solutions. □

Let us denote by $\{\Psi(t),\ t \in [0, T]\}$ the family of random operators defined by $\Psi(t)\tilde{\psi}_0 = \tilde{\psi}(t)$

Remark 2. *Thanks to estimate* (3.60) *we can extend each operator* $\Psi(t)$ *to the whole* $L^2(\mathbb{R}^d)$*,we will denote its extension again with* $\Psi(t)$*.*

Let Φ be the isometry form $L^2(\mathbb{R}^d)$ in the momentum representation to $L(\mathbb{R}^d)$ in the coordinate representation, we define:

$$\Theta(t,0) \doteq \Phi \circ \Psi(t) \circ \Phi^{-1} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \tag{3.61}$$

for each $t \in [0,T]$. We are now ready to prove the following:

Proposition 4. *Problem* (3.46) *has a unique strong solution with the following representation formula:*

$$\psi(t,x) = (\Theta(t,0)\psi_0)(x), \qquad t \geq 0, x \in \mathbb{R}^d \tag{3.62}$$

Proof. $(\Theta(t,0)\psi_0)(x)$ is a strong solution of problem (3.46) thanks to the property of the Fourier transforms. Therefore it has to coincide with the mild representation found in proposition 2. □

Remark 3. *It is clearly possible to consider Eq.*(3.46) *starting at time* s*, then having that the solution is unique, we can define the random evolution operator* $\Theta(t,s) : D(A) \to L^2(\Omega; L^2(\mathbb{R}^d))$ *that associate any initial data* f *with the solution at time* t *of* (3.46) *starting at* s *in* ψ_0*.*

Now let us consider the following Cauchy problem:

$$\begin{cases} d\psi(s,x) & = [-\frac{i}{\hbar}(-\frac{\hbar^2}{2m}\Delta + V(x)) + \frac{\lambda\hbar^2}{2}\Delta]\psi(s,x)dt \\ & \quad -i\sqrt{\lambda}\hbar\nabla \cdot \psi(s,x)dW(t) \\ & \\ \psi(0,x) & = \psi_0(x) \qquad\qquad (s,x) \in [0,t] \times \mathbb{R}^d \end{cases} \tag{3.63}$$

One has:

Theorem 14. *For given* $V, V_{x_i}, V_{x_i,x_j} \in L^\infty(\mathbb{R}^d)$([5]) *and* $\psi_0 \in H^2(\mathbb{R}^d)$*, the problem* (3.63) *has a unique strong solution that satisfies the following integral equation:*

$$\psi(t,x) = \Theta(t,0)\psi_0(x) - \frac{i}{\hbar}\int_0^t \Theta(t,s)V(x)\psi(s,x)\,ds \tag{3.64}$$

[5] With $L^\infty(\mathbb{R}^d)$ we denote, as usual, the space of integrable functions almost everywhere bounded. V_{x_i}, V_{x_i,x_j} denote the partial derivatives of V with respect to x_i, respectively x_i, x_j.

Proof. Let us define

$$\Gamma(\psi)(t) = e^{tA}\psi_0 - \frac{i}{\hbar}\int_0^t e^{(t-s)A} V\psi(s)\,ds + \int_0^t e^{(t-s)A} B\psi(s)\,dW(s). \quad (3.65)$$

Following again [47] or [48] we show that there exist a fix point of Γ in $L^2_W([0,T];H^2(\mathbb{R}^d))$ and in $C_W([0,T];L^2(\mathbb{R}^d))$. Then an application of the Itô formula will complete the proof of the existence of a unique strong solution.

On the other hand it is possible to give a meaning to the expression (3.64) again by a fixed point argument in the same spaces thanks to estimate (3.57) and to the boundedness of V. Now we have to show that the integral equation (3.64) coincides with the mild representation, see also [37].
We have:

$$\psi(t,x) = \Theta(t,0)\psi_0(x) - \frac{i}{\hbar}\int_0^t \Theta(t,s)V(x)\psi(s,x)\,ds \quad (3.66)$$

Let us remind that $\Theta(t,0)\psi_0(x)$ is the mild solution of the "free" problem (3.46), therefore:

$$\Theta(t,0)\psi_0(x) = e^{tA}\psi_0 + \int_0^t e^{(t-s)A} B\Theta(s,0)\psi_0\,dW(s) \quad (3.67)$$

Also $\Theta(t,s)V(x)\psi(s,x)$ is the solution at time t of the "free" problem (3.46) started at time s in the state $V(x)\psi(s,x)$, therefore:

$$\Theta(t,s)(V\psi(s)) = e^{(t-s)A}V\psi(s) + \int_s^t e^{(t-r)A}B\Theta(r,s)V\psi(s)\,dW(r) \quad (3.68)$$

Thus substituting (3.67) and (3.68) respectively in the first (respectively second) term on the right side of (3.66) we obtain:

$$\begin{aligned}\psi(t,x) &= \Theta(t,0)\psi_0(x) - \frac{i}{\hbar}\int_0^t \Theta(t,s)V(x)\psi(s,x)\,ds \\ &= e^{tA}\psi_0 + \int_0^t e^{(t-s)A}B\Theta(s,0)\psi_0\,dW(s) \\ &- \frac{i}{\hbar}\int_0^t e^{(t-s)A}V\psi(s) - \frac{i}{\hbar}\int_s^t e^{(t-r)A}B\Theta(r,s)V\psi(s)\,dW(r)]\,ds\end{aligned}$$

Thanks to the stochastic Fubini Theorem, see [48], we can interchange the order of the integration in the last term and we get

$$\begin{aligned}&\int_0^t e^{(t-s)A}B\Theta(s,0)\psi_0\,dW(s) - \frac{i}{\hbar}\int_s^t e^{(t-r)A}B\Theta(r,s)V\psi(s)\,dW(r)\,ds \\ &= \int_0^t e^{(t-s)A}B[\Theta(s,0)\psi_0 - \frac{i}{\hbar}\int_0^s \Theta(s,r)V\psi(r)\,dr]\,dW(s) \\ &= \int_0^t e^{(t-s)A}B\psi(s,x)\,dW(s)\end{aligned}$$

This gives that $\psi(t,x)$ defined in (3.66) corresponds the fixed point of Γ defined in (3.65) of (3.63), which concludes the proof. □

3.4.3 Solution by means of phase space Feynman path integrals

In this section we are going to prove that, under suitable assumptions on the potential V and on the initial data ψ_0 the solution to problem (3.63) can be given by means of an infinite dimensional oscillatory integral: a rigorously defined "Feynman path integral" on the space of paths in phase space. More precisely we are going to give a meaning to the following heuristic expression and to prove it represents the solution to the problem (3.63):

$$\psi(t,x) = const \int \exp\Big(\frac{i}{\hbar}S(q+x,p) - \lambda \int_0^t p^2(s)ds\Big) \cdot \exp\Big(\sqrt{\lambda}\int_0^t p(s)dW(s)\Big)\psi_0(q(0)+x)dqdp \quad (3.69)$$

where the integral is meant to be taken on an infinite dimensional space of paths $(q(s),p(s))_{s\in[0,t]}$ in the phase space, such that $q(t)=0$. The functional $S(q,p)$ is the classical action of the system evaluated along the path (q,p):

$$S(q,p) = \int_0^t [p(s)\dot{q}(s) - H(p(s),q(s))]ds.$$

Expression (3.69) does not make sense as it stands: indeed neither the normalization constant in front of the integral, nor the infinite dimensional Lebesgue measure $dqdp$ on the space of paths are well defined. The aim of this section is twofold: first of all by means of the theory of chapter 2 and section 3.1 we realize the Feynman path integral (3.69) as an infinite dimensional oscillatory integral with complex phase on a suitable Hlbert space; secondly we show that, under suitable hypothesis on the potential V and on the initial data ψ_0, the so defined (3.69) gives a representation of the solution of the Cauchy problem (3.63) in the sense of theorem 14.

Let us consider again the Hilbert space $\mathcal{H}_t \times \mathcal{L}_t$ introduced in chapter 2, namely the space of paths in the $d-$dimensional phase space $(q(s),p(s))_{s\in[0,t]}$ (where the path $(q(s))_{s\in[0,t]}$ belongs to the Cameron-Martin space $\mathcal{H}_t$, while the path in the momentum space $(p(s))_{s\in[0,t]}$ belongs to $\mathcal{L}_t = L_2([0,t],\mathbb{R}^d)$), endowed with the natural inner product

$$\langle q,p;Q,P\rangle = \int_0^t \dot{q}(s)\dot{Q}(s)ds + \int_0^t p(s)P(s)ds.$$

Let us consider also the complexification of $\mathcal{H}_t \times \mathcal{L}_t$, denoted $(\mathcal{H}_t \times \mathcal{L}_t)^{\mathbb{C}}$. Let us introduce the following bilinear form:

$$[q,p;Q,P] = \int_0^t \dot{q}(s)P(s)ds + \int_0^t p(s)\dot{Q}(s)ds - (1/m - 2i\lambda\hbar)\int_0^t p(s)P(s)ds$$
$$= \langle q,p;A(Q,P)\rangle, \quad (3.70)$$

where A is the following operator:

$$A(Q,P)(s) = (\int_t^s P(u)du, \dot{Q}(s) - (1/m - 2i\lambda\hbar)P(s)). \qquad (3.71)$$

Note that the latter formula makes sense in $(\mathcal{H}_t \times \mathcal{L}_t)^{\mathbb{C}}$, so expression (3.70) can be recognized as the restriction on $\mathcal{H}_t \times \mathcal{L}_t$ of a quadratic form on $(\mathcal{H}_t \times \mathcal{L}_t)^{\mathbb{C}}$. $A(Q,P)$ is densely defined, e.g. on $C^1([0,t];\mathbb{C}^d) \times C^1([0,t];\mathbb{C}^d)$. Moreover $A(Q,P)$ is invertible with inverse given by

$$A^{-1}(Q,P)(s) = (\int_t^s P(u)du + (1/m - 2i\lambda\hbar)Q(s), \dot{Q}(s)) \qquad (3.72)$$

(on the range of A).
Let us also introduce the vector $l = (\bar{q}, \bar{p}) \in \mathcal{H}_t \times \mathcal{L}_t$.
Let $g : \mathcal{H}_t \times \mathcal{L}_t \to \mathbb{C}$ be the function on $\mathcal{H}_t \times \mathcal{L}_t$ which is the Fourier transform of a complex bounded variation measure μ_g on $\mathcal{H}_t \times \mathcal{L}_t$:

$$g(q,p) = \int_{\mathcal{H}_t \times \mathcal{L}_t} e^{i\langle q,p;Q,P\rangle} d\mu_g(Q,P).$$

Then by means of the theory of chapter 2 and section 3.1 one can define (see [9, 10]) the "complex normalized infinite dimensional oscillatory integral" on $\mathcal{H}_t \times \mathcal{L}_t$ of the function $e^{\langle l;\cdot\rangle}g(\cdot)$ with respect to the operator A:

$$\widetilde{\int}_{\mathcal{H}_t \times \mathcal{L}_t} e^{\frac{i}{2\hbar}\langle q,p;A(q,p)\rangle} e^{\langle l;q,p\rangle} g(q,p)dqdp.$$

By theorem 12 the integral can be computed interms of a well defined complex integral on $\mathcal{H}_t \times \mathcal{L}_t$:

$$\widetilde{\int}_{\mathcal{H}_t \times \mathcal{L}_t} e^{\frac{i}{2\hbar}\langle q,p;A(q,p)\rangle} e^{\langle l;q,p\rangle} g(q,p)dqdp = \int_{\mathcal{H}_t \times \mathcal{L}_t} e^{\frac{-i\hbar}{2}\langle (q,p)-il;A^{-1}((q,p)-il)\rangle} d\mu_g(q,p) \qquad (3.73)$$

Next we show that the Fresnel integral (3.69) can be defined as the limit of phase space Feynman path integrals (3.73) and that this limit is the strong solution of problem (3.53) found in the previous section.

Solution of Belavkin equation with a free hamiltonian

Let us consider first of all Belavkin equation with a free Hamiltonian $H = p^2/2m$ in its Stratonovich equivalent form:

$$\begin{cases} d\psi(t,x) = (\frac{i\hbar}{2m}\Delta + \lambda\hbar^2\Delta)\psi(t,x)dt - i\sqrt{\lambda}\hbar\nabla\psi(t,x)\circ dW(t) \\ \psi(0,x) = \psi_0(x) \qquad (t,x)\in[0,t]\times\mathbb{R}^d \end{cases} \tag{3.74}$$

We associate to (3.74) a sequence of approximated equations of the same type

$$\begin{cases} d\psi_n(t,x) = (\frac{i\hbar}{2m}\Delta + \lambda\hbar^2\Delta)\psi_n(t,x)dt - i\sqrt{\lambda}\hbar\nabla\psi_n(t,x)\circ dW_n(t) \\ \psi(0,x) = \psi_0(x) \qquad (t,x)\in[0,t]\times\mathbb{R}^d \end{cases} \tag{3.75}$$

where by W_n we mean a smooth approximation of the trajectories of the Brownian motion, say

$$W_n(t) = n\int_t^{t-\frac{1}{n}} W(s)ds, \tag{3.76}$$

so that $\dot{W}_n$ belongs to $\mathcal{L}_t$.

Proposition 5. *Let us suppose that the initial data ψ_0 is the Fourier transform of a finite complex Borel measure μ_0 on $\mathbb{R}^d$. Then the solution ψ_n of problem (3.75) has the following representation:*

$$\begin{aligned}\psi_n(t,x) &= \widetilde{\int}_{\mathcal{H}_t\times\mathcal{L}_t} e^{\frac{i}{2\hbar}\langle(q,p),A(q,p)\rangle} e^{\langle(q,p),(0,l_n)\rangle}\psi_0(q(0)+x)dqdp \\ &= \int_{\mathcal{H}_t\times\mathcal{L}_t} e^{\frac{-i\hbar}{2}\langle(q,p)-i(0,l_n),A^{-1}((q,p)-i(0,l_n))\rangle}\mu_0(dqdp)\end{aligned}$$

where l_n is the vector belonging to $\mathcal{L}_t$ given by $l_n = \sqrt{\lambda}\dot{W}_n$.
Moreover, if $\psi_0 \in S(\mathbb{R}^d)$ (the Schwartz test function space), the integrals can be explicitly computed:

$$\psi_n(t,x) = \frac{1}{(2\pi\hbar)^{d/2}}\int_{\mathbb{R}^d} e^{i\frac{x\cdot k}{\hbar}} e^{(-\frac{i}{\hbar}\frac{k^2}{2m}t - \lambda k^2 t + \sqrt{\lambda}kW_n(t))}\tilde{\psi}_0(k)dk$$

where $\tilde{\psi}_0$ is the Fourier transform of ψ_0, $t>0$, $x\in\mathbb{R}^d$.

Remark 4. *Heuristically $\psi_n(t,x)$ is given by*

$$const\int e^{\frac{i}{\hbar}S(q+x,p)-\lambda\int_0^t p^2(s)ds} e^{\sqrt{\lambda}\int_0^t p(s)\dot{W}_n(s)ds}\psi_0(q(0)+x)dqdp$$

Proof. (3.75) is a random family of ordinary Schrödinger equations (but with a complex potential depending on the momentum). Following [8] and [9] (see chapter 2 and section 3.1) the solution of (3.75) can be given by means of rigorusly defined phase space Feynman path integrals (3.73).
After the introduction of the vector $l_n \in \mathcal{L}_t$, $l_n = \sqrt{\lambda}\dot{W}_n$, the heuristic expression

$$const \int e^{\frac{i}{\hbar}S(q+x,p)-\lambda\int_0^t p^2(s)ds} e^{\sqrt{\lambda}\int_0^t p(s)\dot{W}_n(s)ds}\psi_0(q(0)+x)dqdp$$

can be interpreted as the following rigorously defined infinite dimensional oscillatory integral:

$$\widetilde{\int}_{\mathcal{H}_t\times\mathcal{L}_t} e^{\frac{i}{2\hbar}\langle (q,p),A(q,p)\rangle} e^{(q,p),(0,l_n)\rangle}\psi_0(q(0)+x)dqdp$$

which is equal to

$$\int_{\mathcal{H}_t\times\mathcal{L}_t} e^{\frac{-i\hbar}{2}\langle (q,p)-i(0,l_n),A^{-1}((q,p)-i(0,l_n))\rangle}\mu_0(dqdp)$$

where μ_0 is the complex bounded-variation measure on $\mathcal{H}_t\times\mathcal{L}_t$ whose Fourier transform is the function $(q,p)\to\psi_0(q(0)+x)$[6]. Let

$$\psi_0(x) = \frac{1}{(2\pi\hbar)^{d/2}}\int_{\mathbb{R}^d} e^{i\frac{xk}{\hbar}}\tilde{\psi}_0(k)dk,$$

then

$$\psi_0(q(0)+x) = \frac{1}{(2\pi\hbar)^{d/2}}\int_{\mathbb{R}^d} e^{i\frac{xk}{\hbar}} e^{i\frac{q(0)k}{\hbar}}\tilde{\psi}_0(k)dk =$$
$$= \frac{1}{(2\pi\hbar)^{d/2}}\int_{\mathbb{R}^d} e^{i\frac{xk}{\hbar}} e^{i\frac{\langle q,kG(0)\rangle}{\hbar}}\tilde{\psi}_0(k)dk$$

where $G(0)\in\mathcal{H}_t$ is such that $\langle q,G(0)\rangle_{\mathcal{H}_t} = q(0)$, that is $G(0)(s) = (t-s)$. With these notations we have:

$$\psi_0(q(0)+x) = \frac{1}{(2\pi\hbar)^{d/2}}\int_{\mathbb{R}^d} e^{i\frac{xk}{\hbar}}\int_{\mathcal{H}_t} e^{i\langle q,Q\rangle_{\mathcal{H}_t}}\delta_{kG(0)/\hbar}(dQ)\tilde{\psi}_0(k)dk$$
$$= \int_{\mathcal{H}_t} e^{i\langle q,Q\rangle_{\mathcal{H}_t}}\mu_{\psi_0}(dQ)$$

where

$$\mu_{\psi_0}(E) = \frac{1}{(2\pi\hbar)^{d/2}}\int_E e^{i\frac{xk}{\hbar}}\tilde{\psi}_0(k)\delta_{kG(0)/\hbar}(dQ)dk \qquad E\in\mathcal{B}(\mathcal{H}_t),$$

[6]Such a measure exists if the initial data ψ_0, as a function from $\mathbb{R}^d$ to $\mathbb{C}$ is the Fourier transform of a bounded variation measure on $\mathbb{R}^d$. This condition is fulfilled if for instance $\psi_0\in S(\mathbb{R}^d)$

so that

$$\mu_0(dqdp) = \delta_0(p)\mu_{\psi_0}(dq)$$

We have :

$$\begin{aligned}
\psi_n(t,x) &= \int_{\mathcal{H}_t\times\mathcal{L}_t} e^{\frac{-i\hbar}{2}\langle (q,p)-i(0,l_n), A^{-1}((q,p)-i(0,l_n))\rangle}\mu_0(dqdp) \qquad (3.77)\\
&= \frac{1}{(2\pi\hbar)^{d/2}}\int_{\mathbb{R}^d} e^{i\frac{xk}{\hbar}}e^{i\frac{\hbar}{2}\langle ((\frac{(kG(0)}{\hbar},0)-i(0,l_n), A^{-1}(\frac{(kG(0)}{\hbar},0)-i(0,l_n))\rangle}\tilde{\psi}_0 dk\\
&= \frac{1}{(2\pi\hbar)^{d/2}}\int_{\mathbb{R}^d} e^{i\frac{xk}{\hbar}}e^{(-\frac{i}{\hbar}\frac{k^2}{2m}t-\lambda k^2 t+\sqrt{\lambda}kW_n(t))}\tilde{\psi}_0 dk
\end{aligned}$$

□

The next step we shall undertake is the proof of the convergence of the solutions of the approximated problems to the solution of the Cauchy problem (3.74). Moreover we shall prove that this solution is given by a rigorously defined infinite dimensional oscillatory integral with complex phase. First of all let us state the following general result:

Theorem 15. *Let $f:\mathcal{H}_t\times\mathcal{L}_t\to\mathbb{C}$ be the Fourier transform of a complex bounded-variation measure μ_f on $\mathcal{H}_t\times\mathcal{L}_t$. Then the following process, defined as the phase space Feynman path integral*

$$\int e^{\frac{i}{\hbar}(\int_0^t p(s)\dot{q}(s)ds-\int_0^t p(s)^2ds)-\lambda\int_0^t p^2(s)ds}e^{\sqrt{\lambda}\int_0^t p(s)dW(s)}f(q,p)dqdp :=$$

$$\int_{\mathcal{H}_t\times\mathcal{L}_t} e^{-i\hbar(\int_0^t \dot{q}(s)p(s)ds+\frac{1}{2m}\int_0^t\dot{q}(s)^2ds)}e^{-\lambda\hbar^2\int_0^t\dot{q}(s)^2ds-\hbar\sqrt{\lambda}\int_0^t\dot{q}(s)dW(s)}\mu_f(dqdp)$$

is the limit in $L^2(\Omega,\mathbb{P})$ of the sequence of processes

$$\widetilde{\int}_{\mathcal{H}_t\times\mathcal{L}_t} e^{\frac{i}{2\hbar}\langle (q,p),A(q,p)\rangle}e^{\langle (q,p),(0,l_n)\rangle}f(q,p)dqdp$$

$$= \int_{\mathcal{H}_t\times\mathcal{L}_t} e^{\frac{-i\hbar}{2}\langle (q,p)-i(0,l_n),A^{-1}((q,p)-i(0,l_n))\rangle}\mu_f(dqdp)$$

$$= \int_{\mathcal{H}_t\times\mathcal{L}_t} e^{-i\hbar(\int_0^t\dot{q}(s)p(s)ds+\frac{1}{2m}\int_0^t\dot{q}(s)^2ds)}e^{-\lambda\hbar^2\int_0^t\dot{q}(s)^2ds-\hbar\sqrt{\lambda}\int_0^t\dot{q}(s)\dot{W}_n(s)ds}\mu_f(dqdp)$$

Proof.

$$\mathbb{E}\Big(\Big|\int_{\mathcal{H}_t\times\mathcal{L}_t} e^{-i\hbar(\int_0^t\dot{q}(s)p(s)ds+\frac{1}{2m}\int_0^t\dot{q}(s)^2ds)}e^{-\lambda\hbar^2\int_0^t\dot{q}(s)^2ds-\hbar\sqrt{\lambda}\int_0^t\dot{q}(s)dW(s)}\mu_f(dqdp)+$$

$$-\int_{\mathcal{H}_t\times\mathcal{L}_t} e^{-i\hbar(\int_0^t\dot{q}(s)p(s)ds+\frac{1}{2m}\int_0^t\dot{q}(s)^2ds)}e^{-\lambda\hbar^2\int_0^t\dot{q}(s)^2ds-\hbar\sqrt{\lambda}\int_0^t\dot{q}(s)\dot{W}_n(s)ds}\mu_f(dqdp)\Big|^2\Big)$$

$$\leq \mu_f(\mathcal{H}_t\times\mathcal{L}_t)\mathbb{E}\Big(\int_{\mathcal{H}_t\times\mathcal{L}_t} e^{-2\lambda\hbar^2\int_0^t \dot q(s)^2ds}|e^{-\hbar\sqrt{\lambda}\int_0^t \dot q(s)dW(s)}-e^{-\hbar\sqrt{\lambda}\int_0^t \dot q(s)\dot W_n(s)ds}|^2\mu_f(dqdp)\Big)$$

$$= \mu_f(\mathcal{H}_t\times\mathcal{L}_t)\int_{\mathcal{H}_t\times\mathcal{L}_t} e^{-2\lambda\hbar^2\int_0^t \dot q(s)^2ds}\mathbb{E}(|e^{-\hbar\sqrt{\lambda}\int_0^t \dot q(s)dW(s)}-e^{-\hbar\sqrt{\lambda}\int_0^t \dot q(s)\dot W_n(s)ds}|^2)\mu_f(dqdp)$$

$$= \mu_f(\mathcal{H}_t\times\mathcal{L}_t)\int_{\mathcal{H}_t\times\mathcal{L}_t} e^{-2\lambda\hbar^2\int_0^t \dot q(s)^2ds}\mathbb{E}(|e^{-2\hbar\sqrt{\lambda}\int_0^t \dot q(s)dW(s)}+$$

$$+e^{-2\hbar\sqrt{\lambda}\int_0^t \dot q(s)\dot W_n(s)ds}-2e^{-\hbar\sqrt{\lambda}(\int_0^t \dot q(s)dW(s)+\int_0^t \dot q(s)\dot W_n(s)ds)}|)\mu_f(dqdp)$$

$$= \mu_f(\mathcal{H}_t\times\mathcal{L}_t)\int_{\mathcal{H}_t\times\mathcal{L}_t} e^{-2\lambda\hbar^2\int_0^t \dot q(s)^2ds}\big(e^{2\hbar^2\lambda\int_0^t \dot q(s)^2ds}+e^{2\hbar^2\lambda\int_0^t \dot q_n(s)^2ds}+$$

$$-2e^{\frac{\hbar^2\lambda}{2}\int_0^t(\dot q(s)+\dot q_n(s))^2ds}\big)\mu_f(dqdp)$$

where, given $h\in L^2(0,t)$, we define h_n as

$$h_n(s)=n\int_s^{s+1/n}h(u)du=h*g_n$$

g_n being the mollifier in $L^1(0,t)$ given by $g_n(s)=n\chi_{[s,s+1/n]}$. Note that $\|g_n\|_{L^1(0,t)}=1$ and moreover the Young inequality holds:

$$\|h_n\|_{L^2(0,t)}\leq\|g_n\|_{L^1(0,t)}\|h\|_{L^2(0,t)}=\|h\|_{L^2(0,t)}.$$

Thanks to this inequality one can get easily the following uniform estimate:

$$e^{-2\lambda\hbar^2\int_0^t \dot q(s)^2ds}\big(e^{2\hbar^2\lambda\int_0^t \dot q(s)^2ds}+e^{2\hbar^2\lambda\int_0^t \dot q_n(s)^2ds}-2e^{\frac{\hbar^2\lambda}{2}\int_0^t(\dot q(s)+\dot q_n(s))^2ds}\big)\leq 4$$

and by the dominated convergence theorem we can pass to the limit under the integral. The conclusion follows from the convergence of $\dot q_n$ to $\dot q$ in $L^2(0,t)$, see for instance [25]. □

Proposition 6. *Let $\psi_0\in S(\mathbb{R}^d)$. Then, for each $t\geq 0$ and $x\in\mathbb{R}^d$ the solution $\psi_n(t,x)$ of the approximated problem (3.75) converges in $L^2(\Omega,\mathbb{P})$ to the process*

$$\frac{1}{(2\pi\hbar)^{d/2}}\int_{\mathbb{R}^d}e^{i\frac{xk}{\hbar}}e^{(-\frac{i}{\hbar}\frac{k^2}{2m}t-\lambda k^2t+\sqrt{\lambda}kW(t))}\tilde\psi_0(k)dk \tag{3.78}$$

which is the strong solution of (3.74).
Moreover it can be represented by a phase space Feynman path integral in the sense of [8] and chapter 2

$$\widetilde{\int}_{\mathcal{H}_t\times\mathcal{L}_t}e^{\frac{i}{\hbar}(\int_0^t\dot q(s)p(s)ds-\frac{1}{2m}\int_0^t p(s)^2ds)-\lambda\int_0^t p(s)^2ds}e^{\sqrt{\lambda}\int_0^t p(s)dW(s)}\psi_0(q(0)+x)dqdp \tag{3.79}$$

since, as $n \to \infty$, the following infinite-dimensional oscillatory integral on $\mathcal{H}_t \times \mathcal{L}_t$

$$\widetilde{\int}_{\mathcal{H}_t\times\mathcal{L}_t} e^{\frac{i}{2\hbar}\langle (q,p),A(q,p)\rangle} e^{\langle (q,p),(0,l_n)\rangle}\psi_0(q(0)+x)dqdp \tag{3.80}$$

$$= \int_{\mathcal{H}_t\times\mathcal{L}_t} e^{\frac{-i\hbar}{2}\langle (q,p)-i(0,l_n),A^{-1}((q,p)-i(0,l_n))\rangle}\mu_0(dqdp) \tag{3.81}$$

$$= \int_{\mathcal{H}_t\times\mathcal{L}_t} e^{-i\hbar(\int_0^t \dot q(s)p(s)ds+\frac{1}{2m}\int_0^t \dot q(s)^2ds)} e^{-\lambda\hbar^2\int_0^t \dot q(s)^2 ds-\hbar\sqrt{\lambda}\int_0^t \dot q(s)\dot W_n(s)ds}\mu_0(dqdp) \tag{3.82}$$

converges in $L^2(\Omega,\mathbb{P})$ to

$$\int_{\mathcal{H}_t\times\mathcal{L}_t} e^{-i\hbar(\int_0^t \dot q(s)p(s)ds+\frac{1}{2m}\int_0^t \dot q(s)^2ds)} e^{-\lambda\hbar^2\int_0^t \dot q(s)^2 ds-\hbar\sqrt{\lambda}\int_0^t \dot q(s)dW(s)}\mu_0(dqdp) \tag{3.83}$$

$$= \widetilde{\int}_{\mathcal{H}_t\times\mathcal{L}_t} e^{\frac{i}{\hbar}(\int_0^t \dot q(s)p(s)ds-\frac{1}{2m}\int_0^t p(s)^2ds)-\lambda\int_0^t p(s)^2ds} e^{\sqrt{\lambda}\int_0^t p(s)dW(s)}\psi_0(q(0)+x)dqdp$$

Proof. By direct application of the Itô formula one can check that (3.78) is the strong solution of the Cauchy problem (3.74). Moreover one can verify by a direct calculation that the infinite dimensional integral (3.83) is equal to the finite dimensional integral (3.78).
The convergence in $L^2(\Omega,\mathbb{P})$ of the sequence of processes (3.82) to the process (3.83) follows from theorem 15. □

Remark 5. *Heuristically the solution (3.79) can be written as*

$$const\int e^{\frac{i}{\hbar}S(q+x,p)-\lambda\int_0^t p^2(s)ds} e^{\sqrt{\lambda}\int_0^t p(s)dW(s)}\psi_0(q(0)+x)dqdp. \tag{3.84}$$

The introduction of the potential

Now we generalize the previous results to a more general class of quantum mechanical Hamiltonians $H=-\hbar^2\Delta/2m+V(x)$. We consider the Belavkin equation (3.63) in its Stratonovich equivalent form

$$\begin{cases} d\psi(s,x) &= [-\frac{i}{\hbar}(-\frac{\hbar^2}{2m}\Delta+V(x))+\lambda\hbar^2\Delta]\psi(s,x)dt \\ & \quad -i\sqrt{\lambda}\hbar\nabla\cdot\psi(s,x)\circ dW(s) \\ \psi(0,x) &= \psi_0(x) \qquad (s,x)\in[0,t]\times\mathbb{R}^d \end{cases} \tag{3.85}$$

and the sequence of approximated Cauchy problems

$$\begin{cases} d\psi_n(s,x) &= [-\frac{i}{\hbar}(-\frac{\hbar^2}{2m}\Delta+V(x))+\lambda\hbar^2\Delta]\psi_n(s,x)dt \\ & \quad -i\sqrt{\lambda}\hbar\nabla\cdot\psi_n(s,x)\circ dW_n(s) \\ \psi_n(0,x) &= \psi_0(x) \qquad (s,x)\in[0,t]\times\mathbb{R}^d \end{cases} \tag{3.86}$$

Proposition 7. *Let $V : \mathbb{R}^d \to \mathbb{R}$ be the Fourier transform of a finite complex Borel measure on $\mathbb{R}^d$ and let $\psi_0 \in S(\mathbb{R}^d)$. Then the solution to the Cauchy problem (3.63) is given by equation (3.69).*

Proof. Let us set

$$\psi(t,x) = \widetilde{\int} e^{\frac{i}{\hbar}(\int_0^t (\dot{q}(s)p(s) - \frac{1}{2m}p(s)^2)ds - \lambda\int_0^t p(s)^2 ds} e^{-\frac{i}{\hbar}\int_0^t V(q(s)+x)ds} \cdot e^{\sqrt{\lambda}\int_0^t p(s)\cdot dW(s)} \psi_0(\gamma(0)+x)dqdp, \quad (3.87)$$

$$\Theta(t,0)\psi_0(x) = \widetilde{\int} e^{\frac{i}{\hbar}(\int_0^t (\dot{q}(s)p(s) - \frac{1}{2m}p(s)^2)ds - \lambda\int_0^t p(s)^2 ds} e^{\sqrt{\lambda}\int_0^t p(s)\cdot dW(s)} \cdot \psi_0(\gamma(0)+x)dqdp, \quad (3.88)$$

$$\psi_n(t,x) = \widetilde{\int} e^{\frac{i}{\hbar}(\int_0^t (\dot{q}(s)p(s) - \frac{1}{2m}p(s)^2)ds - \lambda\int_0^t p(s)^2 ds} e^{-\frac{i}{\hbar}\int_0^t V(q(s)+x)ds} \cdot e^{\sqrt{\lambda}\int_0^t p(s)\cdot \dot{W}_n(s)ds} \psi_0(\gamma(0)+x)dqdp, \quad (3.89)$$

$$\Theta_n(t,0)\psi_0(x) = \widetilde{\int} e^{\frac{i}{\hbar}(\int_0^t (\dot{q}(s)p(s) - \frac{1}{2m}p(s)^2)ds - \lambda\int_0^t p(s)^2 ds} e^{\sqrt{\lambda}\int_0^t p(s)\cdot \dot{W}_n(s)ds} \cdot \psi_0(\gamma(0)+x)dqdp, \quad (3.90)$$

So far we have proved that (3.90) and (3.88) are the solutions of the Cauchy problems (3.75) and (3.74) respectively. We are going to prove that (3.89) is the solution of (3.86) and that it converges in $L^2(\Omega,\mathbb{P})$ to (3.87), which is a representation of the solution of (3.85).
Let $\mu_0(\psi)$ be the measure on $\mathcal{H}_t \times \mathcal{L}_t$ such that its Fourier transform evaluated in $\gamma \in H$ is $\psi_0(q(0)+x)$.
For $0 \leq u \leq t$ let $\mu_u(V_0,x)$ and $\nu_0^u(V_0,x)$ be the measures on $\mathcal{H}_t \times \mathcal{L}_t$, whose Fourier transforms when evaluated at $(q,p) \in \mathcal{H}_t \times \mathcal{L}_t$ are respectively $V_0(x+q(u))$, and $\exp\Big(-\frac{i}{\hbar}\int_0^u V_0(x+q(s))ds\Big)$. We shall use the short notation $\mu_u \equiv \mu_u(V_0,x)$ and $\nu_0^u \equiv \nu_0^u(V_0,x)$. If $\{\mu_u : a \leq u \leq b\}$ is a family in $\mathcal{M}(\mathcal{H}_t \times \mathcal{L}_t)$, we shall let $\int_a^b \mu_u du$ denote the measure on $\mathcal{H}_t \times \mathcal{L}_t$ given by :

$$f \to \int_a^b \int_{\mathcal{H}_t \times \mathcal{L}_t} f(q,p)\mu_u(dqdp)du$$

whenever it exists.
Then, since for any continuous path $q \in \mathcal{H}_t$

$$\begin{aligned}&\exp\Big(-\frac{i}{\hbar}\int_0^t V_0(q(s))ds\Big) = \\ &1-\frac{i}{\hbar}\int_0^t V_0(q(u))\exp\Big(-\frac{i}{\hbar}\int_0^u V_0(q(s))ds\Big)du,\end{aligned} \tag{3.91}$$

we have

$$\nu_0^t = \delta_0 - \frac{i}{\hbar}\int_0^t (\mu_u * \nu_0^u)du \tag{3.92}$$

where δ_0 is the Dirac measure at $0 \in H$.
By the formula (3.73) we have:

$$\begin{aligned}\psi_n(t,x) = \widetilde{\int}_{\mathcal{H}_t\times\mathcal{L}_t} & e^{\frac{i}{2\hbar}\langle (q,p),A(q,p)\rangle} e^{\langle (q,p),(0,l_n)\rangle} e^{-\frac{i}{\hbar}\int_0^t V(q(s)+x)ds} \\ & \cdot \psi_0(\gamma(0)+x)dqdp \end{aligned} \tag{3.93}$$

$$= \int_{\mathcal{H}_t\times\mathcal{L}_t} e^{\frac{-i\hbar}{2}\langle (q,p)-i(0,l_n),A^{-1}((q,p)-i(0,l_n))\rangle}(\nu_0^t * \mu_0(\psi))(dqdp)$$

$$\begin{aligned}&= \int_{\mathcal{H}_t\times\mathcal{L}_t} e^{\frac{-i\hbar}{2}\langle (q,p)-i(0,l_n),A^{-1}((q,p)-i(0,l_n))\rangle}\mu_0(\psi)(dqdp) \\ &\quad -\frac{i}{\hbar}\int_0^t\int_{\mathcal{H}_t\times\mathcal{L}_t} e^{\frac{-i\hbar}{2}\langle (q,p)-i(0,l_n),A^{-1}((q,p)-i(0,l_n))\rangle}(\mu_u * \nu_0^u * \mu_0(\psi))(dqdp)du \\ &\quad = \Theta_n(t,0)\psi_0(x) - \frac{i}{\hbar}\int_0^t\int_{\mathcal{H}_t\times\mathcal{L}_t} e^{\frac{i}{2\hbar}\langle (q,p),A(q,p)\rangle}e^{\langle (q,p),(0,l_n)\rangle}e^{-\frac{i}{\hbar}\int_0^u V(q(s)+x)ds} \\ &\qquad V(q(u)+x)\psi_0(q(0)+x)dqdpdu\end{aligned} \tag{3.94}$$

As $\mathcal{H}_t \times \mathcal{L}_t = (\mathcal{H}_{[0,u]} \times \mathcal{L}_{[0,u]}) \oplus (\mathcal{H}_{[u,t]} \times \mathcal{L}_{[u,t]})$, where by $\mathcal{H}_{[r,s]}$ we denote the Cameron Martin space of path $\gamma : [r,s] \mapsto \mathbb{R}^d$ and by $\mathcal{L}_{[r,s]}$ the space $L^2[r,s]$, by setting $(q,p) = (q_1,p_1,q_2,p_2)$ where $(q_1,p_1) \in \mathcal{H}_{[0,u]} \times \mathcal{L}_{[0,u]}$ and $(q_2,p_2) \in \mathcal{H}_{[u,t]} \times \mathcal{L}_{[u,t]}$, $q_1(s) = q(s) - q(u), s \in [0,u]$, $q_2(s) = q(s), s \in [u,t]$, by Fubini theorem for Feynman path integrals [12] we have

$$\begin{aligned}&\int_{\mathcal{H}_t\times\mathcal{L}_t} e^{\frac{i}{2\hbar}\langle (q,p),A(q,p)\rangle}e^{\langle (q,p),(0,l_n)\rangle}e^{-\frac{i}{\hbar}\int_0^u V(q(s)+x)ds}V(q(u)+x)\psi_0(q(0)+x)dqdp \\ &= \int_{\mathcal{H}_{[u,t]}\times\mathcal{L}_{[u,t]}} e^{\frac{i}{2\hbar}\langle (q_2,p_2),A(q_2,p_2)\rangle}e^{\langle (q_2,p_2),(0,l_{n,2})\rangle}V(q_2(u)+x)\int_{\mathcal{H}_{[0,u]}\times\mathcal{L}_{[0,u]}} e^{\frac{i}{2\hbar}\langle (q_1,p_1),A(q_1,p_1)\rangle} \\ &\ e^{\langle (q_1,p_1),(0,l_{n,1})\rangle}e^{-\frac{i}{\hbar}\int_0^u V(q_1(s)+q_2(u)+x)ds}\psi_0(q_1(0)+q_2(u)+x)dq_1dp_1dq_2dp_2\end{aligned} \tag{3.95}$$

so that the expression (3.94) assumes the following form:

$$\psi_n(t,x) = \Theta_n(t,o)\psi_0(x) - \frac{i}{\hbar}\int_0^t \Theta_n(t,u)V(x)\psi_n(u,x)du \tag{3.96}$$

By Lebesgue's dominated convergence theorem and by theorem 15 the latter expression converges as $n \to \infty$ to

$$\psi(t,x) = \Theta(t,0)\psi_0(x) - \frac{i}{\hbar}\int_0^t \Theta(t,u)V(x)\psi(u,x)du \tag{3.97}$$

Now the iterative solutions of the integral equations (3.96) and (3.97) are convergent Dyson series for ψ_n and ψ respectively, which by theorem 14 coincide with the corresponding power series expansions of the solution of the stochastic Schrödinger equations (3.86) and (3.85).

□

Chapter 4

Generalized Fresnel integrals

In this chapter we focus on the finite dimensional oscillatory integrals and generalize the results of section 1.1 by including more general phase functions Φ.
Our aim is in fact the definition and the study of oscillatory integrals of the form

$$\int_{\mathbb{R}^N} e^{\frac{i}{\hbar}\Phi(x)} f(x)dx, \tag{4.1}$$

where the phase function Φ is a smooth function bounded at infinity by a a polynomial $P(x)$ on $\mathbb{R}^n$, $Im(\hbar) \leq 0$, $\hbar \neq 0$, and f is a suitable real-valued smooth function. The results we obtain will be generalized to the infinite dimensional case in the next chapter and applied to an extension of the class of phase functions for which the Feynman path integral had been defined before. Moreover we are interested in discussing the asymptotic behavior of the above integrals when the parameter $\hbar$ goes to 0. If the phase function Φ is quadratic, then the above integral reduces to a Fresnel integral (see section 1.1), while if $\Phi(x) = x^3$ one has Airy integrals. We also mention that the problem of definition and study of integrals of the form (4.1) but with $\hbar \in \mathbb{C}$, $Im(\hbar) < 0$ and Φ lower bounded has also been discussed. The convergence of the integral in this case is a simple matter, so the analysis has concentrated on a "perturbation theoretical" computation of the integral, like in [30, 31], resp. on a Laplace method for handling the $\hbar \to 0$ asymptotics, see, e.g. [17, 23, 4, 87] (the latter method has some relations with the stationary phase method).
In section 4.1 we introduce the notations, recall some known results and prove the existence of the oscillatory integral (4.1). In section 4.2 we prove that when f belongs to a suitable class of functions, this generalized Fresnel integral can be explicitly computed by means of an absolutely convergent Lebesgue integral. We prove a representation formula of the Parseval type (theorem 18) (similar to the one which was exploited in [12] in the case of quadratic phase functions), as well as a formula (corollary 1 to theorem 18) giving the integral

in terms of analytically continued absolutely convergent integrals. Even if our main interest came from the case $\hbar \in \mathbb{R} \setminus \{0\}$, both formulas are valid for all $\hbar \in \mathbb{C}$ with $Im(\hbar) \leq 0$, $\hbar \neq 0$. In the last section we consider the integral (4.1) in the particular case $P(x) = A_{2M}(x, \dots, x)$, where A_{2M} is a completely symmetric strictly positive covariant tensor of order $2M$ on $\mathbb{R}^N$, compute its detailed asymptotic power series expansion (in powers of $\hbar^{1/2M}$, for $Im(\hbar) \leq 0$, $\hbar \neq 0$) in the limit of "strong oscillations", i.e. $\hbar \to 0$. In particular we find explicit assumptions on the integrand f which are sufficient for having convergent, resp. Borel summable, expansions.

4.1 Definition of the generalized Fresnel integral

Let us consider a finite dimensional real Hilbert space $\mathcal{H}$, $dim(\mathcal{H}) = N$, and let us identify it with $\mathbb{R}^N$. We will denote its elements by $x \in \mathbb{R}^N$, $x = (x_1, \dots, x_N)$. We recall the definition of *oscillatory integrals* given by Hörmander [64, 65] (see section 1.1) and propose a related, more general definition of *oscillatory integral in the* Σ*-sense.*

Definition 9. *Let* Φ *be a continuous real-valued function on* $\mathbb{R}^N$*. The oscillatory integral on* $\mathbb{R}^N$*, with* $\hbar \in \mathbb{R} \setminus \{0\}$*,*

$$\int_{\mathbb{R}^N} e^{\frac{i}{\hbar}\Phi(x)} f(x)dx,$$

is well defined if for each test function $\phi \in \mathcal{S}(\mathbb{R}^N)$*, such that* $\phi(0) = 1$*, the limit of the sequence of absolutely convergent integrals*

$$\lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^N} e^{\frac{i}{\hbar}\Phi(x)} \phi(\epsilon x) f(x)dx,$$

exists and is independent on ϕ*. In this case the limit is denoted by*

$$\int_{\mathbb{R}^N} e^{\frac{i}{\hbar}\Phi(x)} f(x)dx.$$

If the same holds only for ϕ *such that* $\phi(0) = 1$ *and* $\phi \in \Sigma$*, for some subset* Σ *of* $\mathcal{S}(\mathbb{R}^N)$*, we say that the oscillatory integral exists in the* Σ*-sense and we shall denote it by the same symbol.*

Let us consider the space $\mathcal{M}(\mathbb{R}^N)$ of complex bounded variation measures on $\mathbb{R}^N$ endowed with the total variation norm and the space $\mathcal{F}(\mathbb{R}^N)$ of functions $f : \mathbb{R}^N \to \mathbb{C}$ which are the Fourier transforms of complex bounded variation measures $\mu_f \in \mathcal{M}(\mathbb{R}^N)$. We recall that if there exists a self-adjoint linear isomorphism $Q : \mathbb{R}^N \to \mathbb{R}^N$ such that the phase function Φ is given by $\Phi(x) = \langle x, Qx \rangle$ and $f \in \mathcal{F}(\mathbb{R}^N)$, then the oscillatory integral $\int_{\mathbb{R}^N} e^{\frac{i}{\hbar}\langle x,Qx\rangle} f(x)dx$ can be explicitly computed by means of the following Parseval-type formula (see section 1.1, theorem 1):

$$\int_{\mathbb{R}^N} e^{\frac{i}{2\hbar}\langle x,Qx\rangle} f(x)dx = \\ = (2\pi i\hbar)^{N/2} e^{\frac{-\pi i}{2} Ind(Q)} |\det(Q)|^{-1/2} \int_{\mathbb{R}^N} e^{-\frac{i\hbar}{2}\langle x,Q^{-1}x\rangle} \mu_f(dx,) \quad (4.2)$$

where $Ind(Q)$ is the number of negative eigenvalues of the operator Q, counted with their multiplicity.

In the following we shall generalize this result to more general phase functions Φ, in particular those given by an even polynomial $P(x)$ in the variables $x_1, \dots, x_N$:

$$P(x) = A_{2M}(x, \dots, x) + A_{2M-1}(x, \dots, x) + \dots + A_1(x) + A_0, \quad (4.3)$$

where A_k are k_{th}-order covariant tensors on $\mathbb{R}^N$:

$$A_k : \underbrace{\mathbb{R}^N \times \mathbb{R}^N \times \dots \times \mathbb{R}^N}_{k-times} \to \mathbb{R}$$

and the leading term, namely $A_{2M}(x, \dots, x)$, is a $2M_{th}$-order completely symmetric positive covariant tensor on $\mathbb{R}^N$. First of all, following the methods used by Hörmander [64, 65], we prove the existence of the following generalized Fresnel integral:

$$\int_{\mathbb{R}^N} e^{\frac{i}{\hbar}\Phi(x)} f(x)dx \quad (4.4)$$

for suitable Φ. We recall the definition of *symbols* (see [64]).

Definition 10. *A C^∞ map $f : \mathbb{R}^N \to \mathbb{C}$ belongs to the space of symbols $S^n_\lambda(\mathbb{R}^N)$, where n, λ are two real numbers and $0 < \lambda \leq 1$, if for each $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{Z}^N$ there exists a constant $C_\alpha \in \mathbb{R}$ such that*

$$\left| \frac{d^{\alpha_1}}{dx_1^{\alpha_1}} \cdots \frac{d^{\alpha_N}}{dx_1^{\alpha_N}} f \right| \leq C_\alpha (1 + |x|)^{n - \lambda|\alpha|}, \qquad |x| \to \infty, \quad (4.5)$$

where $|\alpha| = |\alpha_1| + |\alpha_2| + \dots + |\alpha_N|$.

One can prove that S^n_λ is a Fréchet space under the topology defined by taking as seminorms $|f|_\alpha$ the best constants C_α in (4.5) (see [64]). The space increases as n increases and λ decreases. If $f \in S^n_\lambda$ and $g \in S^m_\lambda$, then $fg \in S^{n+m}_\lambda$. We denote $\bigcup_n S^n_\lambda$ by S^∞_λ. We shall see that S^∞_λ is included in the class

for which the generalized Fresnel integral (4.4) is well defined.
We say that a point $x = x_c \in \mathbb{R}^N$ is a critical point of the phase function $\Phi : \mathbb{R}^N \to \mathbb{R}$, $\Phi \in C^1$, if $\Phi'(x_c) = 0$. Let $\mathcal{C}(\Phi)$ be the set of critical points of Φ. In fact we have:

Theorem 16. *Let Φ be a real-valued C^2 function on $\mathbb{R}^N$ with the critical set $\mathcal{C}(\Phi)$ being finite. Let us assume that for each $N \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $\frac{|x|^{N+1}}{|\nabla\Phi(x)|^k}$ is bounded for $|x| \to \infty$. Let $f \in S^n_\lambda$, with $n, \lambda \in \mathbb{R}$, $0 < \lambda \leq 1$. Then the generalized Fresnel integral (4.4) exists for each $\hbar \in \mathbb{R} \setminus \{0\}$.*

Proof. We follow the method of Hörmander [64], see also [57, 13, 4].

Let us suppose that the phase function $\Phi(x)$ has l stationary points $c_1, \ldots, c_l$, that is

$$\nabla\Phi(c_i) = 0, \qquad i = 1, \ldots, l.$$

Let us choose a suitable partition of unity $1 = \sum_{i=0}^l \chi_i$, where χ_i, $i = 1, \ldots, l$, are $C_0^\infty(\mathbb{R}^N)$ functions constant equal to 1 in a open ball centered in the stationary point c_i respectively and $\chi_0 = 1 - \sum_{i=1}^l \chi_i$. Each of the integrals $I_i(f) \equiv \int_{\mathbb{R}^N} e^{\frac{i}{\hbar}\Phi(x)}\chi_i(x)f(x)dx$, $i = 1, \ldots, l$, is well defined in Lebesgue sense since $f\chi_i \in C_0(\mathbb{R}^N)$. Let $I_0 \equiv \int_{\mathbb{R}^N} e^{\frac{i}{\hbar}\Phi(x)}\chi_0(x)f(x)dx$. To see that I_0 is a well defined oscillatory integral let us introduce the operator L^+ with domain $D(L^+)$ in $L^2(\mathbb{R}^N)$ given by

$$L^+g(x) = -i\hbar\frac{\chi_0(x)}{|\nabla\Phi(x)|^2}\nabla\Phi(x)\nabla g(x)$$

$$g \in D(L^+) \equiv \Big\{g \in L^2(\mathbb{R}^N), \ \Big| \ \frac{\chi_0(x)}{|\nabla\Phi(x)|^2}\nabla\Phi(x)\nabla g(x) \in L^2(\mathbb{R}^N)\Big\}$$

while its adjoint in $L^2(\mathbb{R}^N)$ is given by

$$Lf(x) = i\hbar\frac{\chi_0(x)}{|\nabla\Phi(x)|^2}\nabla\Phi(x)\nabla f(x) + i\hbar \text{ div } \Big(\frac{\chi_0(x)}{|\nabla\Phi(x)|^2}\nabla\Phi(x)\Big)f(x)$$

for $f \in L^2(\mathbb{R}^N) \cap C^\infty$ such that

$$\Big|\frac{f(x)g(x)|x|^N}{|\nabla\Phi(x)|^2}\nabla\Phi(x)\cdot x\Big| \to 0 \text{ as } |x| \to \infty, \forall g \in D(L^+).$$

Let us choose $\psi \in \mathcal{S}(\mathbb{R}^N)$, such that $\psi(0) = 1$. It is easy to see that if $f \in S^n_\lambda$ then f_ϵ, defined as $f_\epsilon(x) := \psi(\epsilon x)f(x)$, belongs to $S^{n+1}_\lambda \cap \mathcal{S}(\mathbb{R}^N)$, for any $\epsilon > 0$. By iterated application of the Stokes formula, we have:

$$\begin{aligned}\int_{\mathbb{R}^N} e^{\frac{i}{\hbar}\Phi(x)}\psi(\epsilon x)f(x)\chi_0(x)dx &= \int_{\mathbb{R}^N} L^+(e^{\frac{i}{\hbar}\Phi(x)})\psi(\epsilon x)f(x)dx \\ &= \int_{\mathbb{R}^N} e^{\frac{i}{\hbar}\Phi(x)}Lf_\epsilon(x)dx = \int_{\mathbb{R}^N} e^{\frac{i}{\hbar}\Phi(x)}L^k f_\epsilon(x)dx. \quad (4.6)\end{aligned}$$

Now for k sufficiently large the last integral is absolutely convergent and we can pass to the limit $\epsilon \to 0$ by the Lebesgue dominated convergence theorem. Considering $\sum_{i=0}^{l} I_i(f)$ we have, by the existence result proved for I_0 and the additivity property of oscillatory integrals, that $\int_{\mathbb{R}^N} e^{\frac{i}{\hbar}\Phi(x)} f(x)dx$ is well defined and equal to $\sum_{i=0}^{l} I_i(f)$. □

Remark 9. *If $\mathcal{C}(\Phi)$ has countably many non accumulating points $\{x_c^i\}_{i\in\mathbb{N}}$, the same method yields $\int_{\mathbb{R}^N} e^{\frac{i}{\hbar}\Phi(x)} f(x)dx = \sum_{i=0}^{\infty} I_i(f)$ provided this sum converges.*
There are partial extensions of the above construction in the case of critical points which form a submanifold in $\mathbb{R}^N$ [56], or are degenerate [24], see also [44].

Remark 10. *In particular we have proved the existence for $f \in S_\lambda^\infty$, $0 < \lambda \leq 1$, of the oscillatory integrals $\int e^{ix^M} f(x)dx$, with M arbitrary. For $M = 2$ one has the Fresnel integral of [13], for $M = 3$ one has Airy integrals [65].*

Remark 11. *If Φ is of the form* (5.3)*, then the generalized Fresnel integral* (4.4) *also exists, even in Lebesgue sense, for $\hbar \in \mathbb{C}$ with $Im(\hbar) < 0$, as an analytic function in $\hbar$, as easily seen by the fact that the integrand is bounded by $|f| \exp(\frac{Im(\hbar)}{|\hbar|^2}\Phi)$.*

4.2 Generalized Parseval equality and analytic continuation

In this section we prove that, for a suitable class of functions $f : \mathbb{R}^N \to \mathbb{C}$, the generalized Fresnel integral (4.4) can be explicitly computed by means of a generalization of formula (4.2).

Lemma 5. *Let $P : \mathbb{R}^N \to \mathbb{R}$ be given by* (5.3)*. Then the Fourier transform of the distribution $e^{\frac{i}{\hbar}P(x)}$:*

$$\tilde{F}(k) = \int_{\mathbb{R}^N} e^{ik\cdot x} e^{\frac{i}{\hbar}P(x)} dx, \qquad \hbar \in \mathbb{R} \setminus \{0\} \tag{4.7}$$

is an entire bounded function and admits the following representation:

$$\tilde{F}(k) = e^{iN\pi/4M} \int_{\mathbb{R}^N} e^{ie^{i\pi/4M} k\cdot x} e^{\frac{i}{\hbar}P(e^{i\pi/4M}x)} dx, \qquad \hbar > 0 \tag{4.8}$$

or

$$\tilde{F}(k) = e^{-iN\pi/4M} \int_{\mathbb{R}^N} e^{ie^{-i\pi/4M} k\cdot x} e^{\frac{i}{\hbar}P(e^{-i\pi/4M}x)} dx, \qquad \hbar < 0 \tag{4.9}$$

Remark 12. *The integral on the r.h.s. of* (4.8) *is absolutely convergent as*

$$e^{\frac{i}{\hbar}P(e^{i\pi/4M}x)} = e^{-\frac{1}{\hbar}A_{2M}(x,\ldots,x)}e^{\frac{i}{\hbar}(A_{2M-1}(e^{i\pi/4M}x,\ldots,e^{i\pi/4M}x)+\ldots+A_1(xe^{i\pi/4M})+A_0)}.$$

A similar calculation shows the absolute convergence of the integral on the r.h.s. of (4.9).

Proof. (of lemma 5) Formulas (4.8) and (4.9) can be proved by using the analyticity of $e^{kz+\frac{i}{\hbar}P(z)}$, $z \in \mathbb{C}$, and a change of integration contour (see appendix A for more details). Representations (4.8) and (4.9) show the analyticity properties of $\tilde{F}(k)$, $k \in \mathbb{C}$. By a study of the asymptotic behavior of $\tilde{F}(k)$ as $|k| \to \infty$ we conclude that $\tilde{F}$ is always bounded (see appendix A for more details). □

Remark 13. *A representation similar to* (4.8) *holds also in the more general case* $\hbar \in \mathbb{C}$, $Im(\hbar) < 0$, $\hbar \neq 0$. *By setting* $\hbar \equiv |\hbar|e^{i\phi}$, $\phi \in [-\pi, 0]$ *one has:*

$$\tilde{F}(k) = \int_{\mathbb{R}^N} e^{ik\cdot x}e^{\frac{i}{\hbar}P(x)}dx =$$
$$= e^{iN(\pi/4M+\phi/2M)} \int_{\mathbb{R}^N} e^{ie^{i(\pi/4M+\phi/2M)}k\cdot x}e^{\frac{i}{\hbar}P(e^{i(\pi/4M+\phi/2M)}x)}dx \quad (4.10)$$

(see appendix A for more details).

By mimicking the proof of equation (4.8) (appendix A) one can prove in the case $\hbar > 0$ the following result (a similar one holds also in the case $\hbar < 0$):

Theorem 17. *Let us denote by* Λ *the subset of the complex plane*

$$\Lambda = \{\xi \in \mathbb{C} \mid 0 < \arg(\xi) < \pi/4M\} \subset \mathbb{C}, \quad (4.11)$$

and let $\bar{\Lambda}$ *be its closure. Let* $f : \mathbb{R}^N \to \mathbb{C}$ *be a Borel function defined for all* y *of the form* $y = \lambda x$, *where* $\lambda \in \bar{\Lambda}$ *and* $x \in \mathbb{R}^N$, *with the following properties:*

1. *the function* $\lambda \mapsto f(\lambda x)$ *is analytic in* Λ *and continuous in* $\bar{\Lambda}$ *for each* $x \in \mathbb{R}^N$, $|x| = 1$,

2. *for all* $x \in \mathbb{R}^N$ *and all* $\theta \in (0, \pi/4M)$

$$|f(e^{i\theta}x)| \leq AG(x),$$

where $A \in \mathbb{R}$ *and* $G : \mathbb{R}^N \to \mathbb{R}$ *is a positive function satisfying bound (a) or (b) respectively:*

(a) if P is as in the general case defined by (5.3)

$$G(x) \leq e^{B|x|^{2M-1}}, \quad B > 0$$

(b) if P is homogeneous, i.e. $P(x) = A_{2M}(x, \ldots, x)$,

$$G(x) \leq e^{\frac{\sin(2M\theta)}{\hbar} A_{2M}(x,x,\ldots,x)} g(|x|),$$

where $g(t) = O(t^{-(N+\delta)})$, $\delta > 0$, *as* $t \to \infty$.

Then the limit of regularized integrals:

$$\lim_{\epsilon \downarrow 0} \int e^{\frac{i}{\hbar}P(xe^{i\epsilon})} f(xe^{i\epsilon}) dx, \qquad 0 < \epsilon < \pi/4M, \quad \hbar > 0$$

is given by:

$$e^{iN\pi/4M} \int_{\mathbb{R}^N} e^{\frac{i}{\hbar}P(e^{i\pi/4M}x)} f(e^{i\pi/4M}x) dx \tag{4.12}$$

The latter integral is absolutely convergent and it is understood in Lebesgue sense.

The class of functions satisfying conditions (1) and (2) in theorem 17 includes for instance the polynomials of any degree and the exponentials. In the case $f \in S^n_\lambda$ for some n, λ, one is tempted to interpret expression (4.12) as an explicit formula for the evaluation of the generalized Fresnel integral $\int e^{\frac{i}{\hbar}P(x)} f(x) dx$, $\hbar > 0$, whose existence is assured by theorem 16. This is, however, not necessarily true for all $f \in S^\infty_\lambda$ satisfying (1) and (2). Indeed the definition 9 of oscillatory integral requires that the limit of the sequence of regularized integrals exists and is independent on the regularization. The identity

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^N} e^{\frac{i}{\hbar}P(x)} f(x) \psi(\epsilon x) dx = e^{iN\pi/4M} \int_{\mathbb{R}^N} e^{\frac{i}{\hbar}P(e^{i\pi/4M}x)} f(e^{i\pi/4Mx}) dx, \quad \hbar > 0$$

can be proved only by choosing regularizing functions ψ with $\psi(0) = 1$ and ψ in the class Σ consisting of all $\psi \in \mathcal{S}$ which satisfy (1) and are such that $|\psi(e^{i\theta}x)|$ is bounded as $|x| \to \infty$ for each $\theta \in (0, \pi/4M)$. In fact we will prove that expression (4.12) coincides with the oscillatory integral (4.4), i.e. one can take $\Sigma = \mathcal{S}(\mathbb{R}^N)$, by imposing stronger assumptions on the function f. First of all we show that the representation (4.8) for the Fourier transform of $e^{\frac{i}{\hbar}P(x)}$ allows a generalization of equation (4.2). Let us denote by $\bar{D} \subset \mathbb{C}$ the lower semiplane in the complex plane

$$\bar{D} \equiv \{z \in \mathbb{C} \mid Im(z) \leq 0\} \tag{4.13}$$

Theorem 18. *Let $f \in \mathcal{F}(\mathbb{R}^N)$, $f = \hat{\mu}_f$. Then the generalized Fresnel integral*

$$I(f) \equiv \int_{\mathbb{R}^N} e^{\frac{i}{\hbar}P(x)} f(x)dx, \qquad \hbar \in \bar{D} \setminus \{0\}$$

is well defined and it is given by the formula of Parseval's type:

$$\int_{\mathbb{R}^N} e^{\frac{i}{\hbar}P(x)} f(x)dx = \int_{\mathbb{R}^N} \tilde{F}(k)\mu_f(dk), \tag{4.14}$$

where $\tilde{F}(k)$ is given by (4.10) (see lemma 5 and remark 13)

$$\tilde{F}(k) = \int_{\mathbb{R}^N} e^{ikx} e^{\frac{i}{\hbar}P(x)} dx$$

The integral on the r.h.s. of (4.14) is absolutely convergent (hence it can be understood in Lebesgue sense).

Proof. Let us choose a test function $\psi \in \mathcal{S}(\mathbb{R}^N)$, such that $\psi(0) = 1$ and let us compute the limit

$$I(f) \equiv \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^N} e^{\frac{i}{\hbar}P(x)} \psi(\epsilon x) f(x)dx$$

By hypothesis $f(x) = \int_{\mathbb{R}^N} e^{ikx}\mu_f(dk)$, $x \in \mathbb{R}^N$, and substituting in the previous expression we get :

$$I(f) = \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^N} e^{\frac{i}{\hbar}P(x)} \psi(\epsilon x) \Big(\int_{\mathbb{R}^N} e^{ikx} \mu_f(dk) \Big) dx.$$

By Fubini theorem (which applies for any $\epsilon > 0$ since the integrand is bounded by $|\psi(\epsilon x)|$ which is dx-integrable, and μ_f is a bounded measure) the r.h.s. is

$$= \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} e^{\frac{i}{\hbar}P(x)} \psi(\epsilon x) e^{ikx} dx \Big) \mu_f(dk)$$
$$= \frac{1}{(2\pi)^N} \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \tilde{F}(k - \alpha\epsilon) \tilde{\psi}(\alpha) d\alpha \mu_f(dk) \tag{4.15}$$

(here we have used the fact that the integral with respect to x is the Fourier transform of $e^{\frac{iP(x)}{\hbar}} \psi(\epsilon x)$ and the inverse Fourier transform of a product is a convolution). Now we can pass to the limit using the Lebesgue bounded convergence theorem and get the desired result:

$$\lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^N} e^{\frac{i}{\hbar}P(x)} \psi(\epsilon x) f(x)dx = \int_{\mathbb{R}^N} \tilde{F}(k)\mu_f(dk),$$

where we have used that $\int \tilde{\psi}(\alpha)d\alpha = (2\pi)^N \psi(0)$ and lemma 5, which assures the boundedness of $\tilde{F}(k)$. □

Corollary 1. *Let* $\hbar = |\hbar| e^{i\phi}$, $\phi \in [-\pi, 0]$, $\hbar \neq 0$, $f \in \mathcal{F}(\mathbb{R}^N)$, $f = \hat{\mu}_f$ *be such that* $\forall x \in \mathbb{R}^N$

$$\int_{\mathbb{R}^N} e^{-kx\sin(\pi/4M+\phi/2M)}|\mu_f|(dk) \leq AG(x), \tag{4.16}$$

where $A \in \mathbb{R}$ *and* $G : \mathbb{R}^N \to \mathbb{R}$ *is a positive function satisfying bound (1) or (2) respectively:*

1. *if* P *is defined by* (5.3),

$$G(x) \leq e^{B|x|^{2M-1}}, \quad B > 0$$

2. *if* P *is homogeneous, i.e.* $P(x) = A_{2M}(x, \ldots, x)$*:*

$$G(x) \leq e^{\frac{1}{\hbar}A_{2M}(x,x,\ldots,x)} g(|x|),$$

where $g(t) = O(t^{-(N+\delta)})$, $\delta > 0$, *as* $t \to \infty$.

Then f *extends to an analytic function on* $\mathbb{C}^N$ *and its generalized Fresnel integral* (4.4) *is well defined and it is given by*

$$\int_{\mathbb{R}^N} e^{\frac{i}{\hbar}P(x)} f(x)dx = e^{iN(\pi/4M+\phi/2M)} \int_{\mathbb{R}^N} e^{\frac{i}{\hbar}P(e^{i(\pi/4M+\phi/2M)}x)} f(e^{i(\pi/4M+\phi/2M)}x)dx$$

Proof. By bound (4.16) it follows that the Laplace transform $f^L : \mathbb{C}^N \to \mathbb{C}$, $f^L(z) = \int_{\mathbb{R}^N} e^{kz}\mu_f(dk)$, of μ_f is a well defined entire function such that, for $x \in \mathbb{R}^N$, $f^L(ix) = f(x)$. By theorem 18 the generalized Fresnel integral can be computed by means of the Parseval type equality

$$\begin{aligned}\int_{\mathbb{R}^N} e^{\frac{i}{\hbar}P(x)} f(x)dx &= \int_{\mathbb{R}^N} \tilde{F}(k)\mu_f(dk) = \\ &= e^{iN(\pi/4M+\phi/2M)} \int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} e^{ikxe^{i(\pi/4M+\phi/2M)}} e^{\frac{i}{\hbar}P(e^{i(\pi/4M+\phi/2M)}x)} dx \Big) \mu_f(dk)\end{aligned}$$

By Fubini theorem, which applies given the assumptions on the measure μ_f, this is equal to

$$\begin{aligned}&e^{iN(\pi/4M+\phi/2M)} \int_{\mathbb{R}^N} e^{\frac{i}{\hbar}P(e^{i(\pi/4M+\phi/2M)}x)} \int_{\mathbb{R}^N} e^{ikxe^{i(\pi/4M+\phi/2M)}} \mu_f(dk)dx = \\ &= e^{iN(\pi/4M+\phi/2M)} \int_{\mathbb{R}^N} e^{\frac{i}{\hbar}P(e^{i(\pi/4M+\phi/2M)}x)} f^L(ie^{i(\pi/4M+\phi/2M)}x)dx \\ &= e^{iN(\pi/4M+\phi/2M)} \int_{\mathbb{R}^N} e^{\frac{i}{\hbar}P(e^{i(\pi/4M+\phi/2M)}x)} f(e^{i(\pi/4M+\phi/2M)}x)dx\end{aligned}$$

and the conclusion follows. □

4.3 Asymptotic expansion

In this section we study the asymptotic expansion of the generalized Fresnel integrals (4.4) in the particular case where the phase function $\Phi(x)$ is homogeneous and strictly positive:

$$\phi(x) = A_{2M}(x, \dots, x),$$

where $A_{2M} : \mathbb{R}^N \times \mathbb{R}^N \times \cdots \times \mathbb{R}^N \to \mathbb{R}$ is a completely symmetric strictly positive $2M_{th}-$order covariant tensor on $\mathbb{R}^N$. Under suitable assumptions on the function f, we prove either the convergence or the Borel summability of the asymptotic expansion. In the general case one would have to consider the type of degeneracy of the phase function, cf. [24, 56, 13, 4]. We leave the investigation of the corresponding expansions in our setting for a further publication.

Let us assume first of all $N = 1$ and study the asymptotic behavior of the integral:

$$\int_{-\infty}^{\infty} e^{i\frac{x^{2M}}{\hbar}} f(x)dx, \qquad \hbar \in \bar{D} \setminus \{0\}$$

Theorem 19. *Let us consider a function $f \in \mathcal{F}(\mathbb{R})$, which is the Fourier transform of a bounded variation measure μ_f on the real line satisfying the following bounds for all $l \in \mathbb{N}$, $\rho \in \mathbb{R}^+$, $\hbar \in \bar{D} \setminus \{0\}$:*

1.
$$\int |k|^{2l} |e^{ik\hbar^{1/2M}\rho e^{i\pi/4M}} + e^{-ik\hbar^{1/2M}\rho e^{i\pi/4M}}||\mu_f|(dk) \le F(l)g(\rho)e^{c|x|^{2M-1}},$$

   where $c \in \mathbb{R}$, $F(l)$ is a constant depending on l, $g : \mathbb{R} \to \mathbb{R}$ is a smooth function of polynomial growth as $\rho \to +\infty$

2.
$$|\int k^{2l}(e^{ik\hbar^{1/2M}\rho e^{i\pi/4M}} + e^{-ik\hbar^{1/2M}\rho e^{i\pi/4M}})\mu_f(dk)| \le Ac^l C(l, M),$$

   where $A, c, C(l, M) \in \mathbb{R}$.

Then the generalized Fresnel integral

$$I(\hbar) \equiv \int_{\mathbb{R}} e^{\frac{i}{\hbar}x^{2M}} f(x)dx, \qquad \hbar \in \bar{D} \setminus \{0\}$$

(with $\bar{D}$ given by (4.13)) admits the following asymptotic expansion in powers of $\hbar^{1/M}$:

$$I(\hbar) = \frac{e^{i\frac{\pi}{4M}}\hbar^{1/2M}}{M} \sum_{j=0}^{n-1} \frac{e^{i\frac{j\pi}{2M}}}{2j!} \hbar^{j/M} \Gamma\Big(\frac{1+2j}{2M}\Big) f^{(2j)}(0) + \mathcal{R}_n(\hbar) \qquad (4.17)$$

with $|\mathcal{R}_n(\hbar)| \leq \frac{|\hbar|^{1/2M}}{2M} Ac^n |\hbar|^{n/M} \frac{C(n,M)}{2n!} \Gamma\Big(\frac{1+2n}{2M}\Big)$ *(where* $A, c, C(n, M)$ *are the constants in (2)). If the constant* $C(n, M)$ *satisfies the bound*

$$C(n, M) \leq (2n)!\Gamma\Big(\frac{1+2n}{2M}\Big)^{-1}, \qquad \forall n \in \mathbb{N} \tag{4.18}$$

then the series given by (4.17) *for* $n \to \infty$ *has a positive radius of convergence, while if*

$$C(n, M) \leq (2n)!\Gamma\Big(1 + \frac{n}{M}\Big)\Gamma\Big(\frac{1+2n}{2M}\Big)^{-1}, \qquad \forall n \in \mathbb{N} \tag{4.19}$$

then the expansion (4.17) *is Borel summable in the sense of, e.g., [84, 62] and determines* $I(\hbar)$ *uniquely.*
Moreover if $f \in \mathcal{F}(\mathbb{R})$ *instead of (1), (2) satisfies the following "moment condition":*

$$\int |\alpha|^l |\mu_f|(d\alpha) \leq C'(l, M)Ac^l, \qquad A, c \in \mathbb{R}. \tag{4.20}$$

for all $l \in \mathbb{N}$*, where* $C'(l, M) \sim \Gamma\Big(l\Big(1 - \frac{1}{2M}\Big)\Big)$ *as* $l \to \infty$ *(where* $\sim$ *means that the quotient of the two sides converges to 1 as* $l \to \infty$*), then the asymptotic expansion* (4.17) *has a finite radius of convergence.*

Proof. First of all we recall that the integral $\int e^{\frac{ix^{2M}}{\hbar}} f(x)dx$ is a well defined convergent integral also for all $\hbar \in \mathbb{C}$ with $Im(\hbar) < 0$, thanks to the exponential decay of $e^{\frac{i}{\hbar}x^{2M}}$ and to the boundedness of f (cf Remark 11). Moreover it is an analytic function of the variable $\hbar \in \mathbb{C}$ in the domain $Im(\hbar) < 0$ as one can directly verify the Cauchy-Riemann conditions.
Let us compute the asymptotic expansion of this integral, considered as a function of $\hbar \in \mathbb{C}$, valid for $\hbar \in \bar{D} \setminus \{0\}$.
By formula(4.14) we have

$$\int_{\mathbb{R}} e^{\frac{i}{\hbar}x^{2M}} f(x)dx = \hbar^{1/2M} \int \tilde{F}_{2M}(\hbar^{1/2M}k)\mu_f(dk), \tag{4.21}$$

where, if $\hbar = |\hbar|e^{i\phi}$, $\phi \in [-\pi, 0]$, $\hbar^{1/2M} = |\hbar|^{1/2M}e^{i\phi/2M}$ and $\tilde{F}_{2M}(k) = \int_{\mathbb{R}} e^{ikx}e^{ix^{2M}}dx$, which, for lemma 5, is equal to $\tilde{F}_{2M} = e^{i\frac{\pi}{4M}} \int_{\mathbb{R}} e^{ikxe^{i\frac{\pi}{4M}}} e^{-x^{2M}}dx$. Such a representation assures the analyticity of $\tilde{F}_{2M}$. We can now expand $\tilde{F}_{2M}(\hbar^{1/2M}k)$ in a convergent power series in $\hbar^{1/2M}k$ around $\hbar = 0$:

$$F_{2M}(\hbar^{1/2M}k) = \sum_{n=0}^{\infty} \frac{F_{2M}^{(n)}(0)}{n!} \hbar^{n/2M}k^n.$$

The $n_{th}-$derivative of F_{2M} can be explicitly evaluated by means of the representation (5.6):

$$\tilde{F}^{(n)}_{2M}(0) = e^{i(n+1)\pi/4M}(i)^n(1+(-1)^n)\int_0^\infty \rho^n e^{-\rho^{2M}}d\rho$$

that is $F^{(n)}(0) = 0$ if n is odd, while if n is even we have

$$\tilde{F}^{(2j)}(0) = 2e^{i(2j+1)\pi/2M}(-1)^j\int_0^\infty \rho^{2j}e^{-\rho^{2M}}d\rho.$$

By means of a change of variables one can compute the latter integral explicitly:

$$\int_0^\infty \rho^{2j}e^{-\rho^{2M}}d\rho = \frac{1}{2M}\int_0^\infty e^{-t}t^{\frac{1+2j}{2M}-1}dt = \frac{1}{2M}\Gamma\Big(\frac{1+2j}{2M}\Big).$$

By substituting into (4.21) we get:

$$I(\hbar) = \frac{\hbar^{1/2M}}{M}\sum_{j=0}^{n-1}\frac{1}{2j!}e^{i(2j+1)\pi/2M}\Gamma\Big(\frac{1+2j}{2M}\Big)(-1)^j\hbar^{j/M}\int(k)^{2j}\mu_f(dk)+\mathcal{R}_n =$$

$$= \frac{\hbar^{1/2M}}{M}\sum_{j=0}^{n-1}\frac{1}{2j!}e^{i(2j+1)\pi/2M}\Gamma\Big(\frac{1+2j}{2M}\Big)\hbar^{j/M}f^{(2j)}(0)+\mathcal{R}_n \quad (4.22)$$

where

$$\mathcal{R}_n = \frac{\hbar^{1/2M}}{M}\int\sum_{j\geq n}\frac{1}{2j!}e^{i(2j+1)\pi/2M}\Gamma\Big(\frac{1+2j}{2M}\Big)(-1)^j\hbar^{j/M}(k)^{2j}\mu_f(dk).$$

If assumption (4.20) is satisfied, one can verify by means of Stirling's formula that the series (4.22) of powers of $\hbar^{1/M}$ has a finite radius of convergence.
In the more general case in which assumptions (1),(2) are satisfied, we can nevertheless prove a suitable estimate for $\mathcal{R}_n$, indeed:

$$\mathcal{R}_n = 2\hbar^{1/2M}e^{i\pi/2M}\int\sum_{j\geq n}(-1)^j\frac{1}{2j!}e^{ij\pi/M}\int_0^\infty \rho^{2j}e^{-\rho^{2M}}d\rho\hbar^{j/M}k^{2j}\mu_f(dk) =$$

$$= \hbar^{1/2M}e^{i\pi/2M}e^{in\pi/2M}\frac{1}{2n-1!}\hbar^{n/M}$$

$$\int k^{2n}\int_0^\infty\rho^{2n}\int_0^1(1-t)^{(2n-1)}(e^{ik\rho t\hbar^{1/2M}e^{i\pi/4M}}+e^{-ik\rho t\hbar^{1/2M}e^{i\pi/4M}})dte^{-\rho^{2M}}d\rho\mu_f(dk). \quad (4.23)$$

By Fubini theorem and assumptions (1) and (2) we get the uniform estimate in $\hbar$:

$$|\mathcal{R}_n| \leq \frac{|\hbar|^{1/M}}{2M}Ac^n\frac{C(n,M)}{2n!}\Gamma\Big(\frac{1+2n}{2M}\Big)|\hbar|^{n/M}.$$

If assumption (4.18) is satisfied, then the latter becomes

$$|\mathcal{R}_n| \leq \frac{|\hbar|^{1/M}}{2M} Ac^n |\hbar|^{n/M},$$

and the series has a positive radius of convergence, while if assumption (4.19) holds, we get the estimate

$$|\mathcal{R}_n| \leq \frac{|\hbar|^{1/M}}{2M} Ac^n \Gamma\Big(1+\frac{n}{M}\Big)|\hbar|^{n/M}.$$

This and the analyticity of $I(\hbar)$ in $\mathrm{Im}(\hbar) < 0$ by Nevanlinna theorem [84] assure the Borel summability of the power series expansion (4.17). □

These results can be easily generalized to the study of $N-$dimensional oscillatory integrals:

$$I_N(\hbar) \equiv \int_{\mathbb{R}^N} e^{\frac{i}{\hbar}A_{2M}(x,\dots,x)} f(x)dx, \qquad \hbar \in \bar{D}\setminus\{0\} \tag{4.24}$$

with A_{2M} a completely symmetric $2M_{th}$-order covariant tensor on $\mathbb{R}^N$ such that $A_{2M}(x,\dots,x) > 0$ unless $x = 0$.

Theorem 20. *Let $f \in \mathcal{F}(\mathbb{R}^N)$ be the Fourier transform of a bounded variation measure μ_f admitting moments of all orders.*
Let us suppose f satisfies the following conditions, for all $l \in \mathbb{N}$:

1. $$\int_{\mathbb{R}^N} |kx|^l e^{-kx}|\mu_f|(dk) \leq F(l)g(|x|)e^{c|x|^{2M-1}}, \qquad \forall x \in \mathbb{R}^N,$$

   where $c \in \mathbb{R}$, $F(l)$ is a positive constant depending on l, $g : \mathbb{R}^+ \to \mathbb{R}$ is a positive function with polynomial growth.

2. $$|\int_{\mathbb{R}^N} (ku)^l e^{ik\rho u\hbar^{1/2M}e^{i\pi/4M}}\mu_f(dk)| \leq Ac^l C(l,M,N)$$

   for all $u \in S_{N-1}$, $\rho \in \mathbb{R}^+$, $\hbar \in \bar{D}\setminus\{0\}$, where $A, c, C(l,M,N) \in \mathbb{R}$ (and S_{N-1} is the $(N-1)$-spherical hypersurface of radius 1 and centered at the origin).

then the oscillatory integral (4.24) admits (for $\hbar \in \bar{D}\setminus\{0\}$) the following asymptotic expansion in powers of $\hbar^{1/2M}$:

$$I_N(\hbar) = \hbar^{N/2M}\frac{e^{iN\pi/4M}}{2M}\sum_{l=0}^{n-1}\frac{(i)^l}{l!}(e^{i\pi/4M})^l\hbar^{l/2M}\Gamma\Big(\frac{l+N}{2M}\Big)$$
$$\int_{\mathbb{R}^N}\int_{S_{N-1}}(ku)^l P(u)^{-\frac{l+N}{2M}}d\Omega_{N-1}\mu_f(dk) + \mathcal{R}_n, \tag{4.25}$$

with $|\mathcal{R}_n| \leq A'|\hbar|^{n/2M}(c')^n \frac{C(n,M,N)}{n!}\Gamma\Big(\frac{n+N}{2M}\Big)$ *where* $A', c' \in \mathbb{R}$ *are suitable constants and* $C(n,M,N)$ *is the constant in (2). If* $C(n,M,N)$ *satisfies the following bound:*

$$C(n,M,N) \leq n!\Gamma\Big(\frac{n+N}{2M}\Big)^{-1} \tag{4.26}$$

then the series has a positive radius of convergence, while if

$$C(n,M,N) \leq n!\Gamma\Big(1+\frac{n}{2M}\Big)\Gamma\Big(\frac{n+N}{2M}\Big)^{-1} \tag{4.27}$$

then the expansion is Borel summable in the sense of, e.g. [84, 62] and determines $I(\hbar)$ *uniquely.*
Moreover if $f \in \mathcal{F}(\mathbb{R}^N)$ *instead of (1) and (2) satisfies the following moment condition:*

$$\int_{\mathbb{R}^N} |\alpha|^l |\mu_f|(d\alpha) \leq C'(l,M)Ac^l, \qquad A, c \in \mathbb{R}, \tag{4.28}$$

for all $l \in \mathbb{N}$, *where* $C'(l,M) \sim \Gamma\Big(l\Big(1-\frac{1}{2M}\Big)\Big)$ *as* $l \to \infty$, *then the asymptotic expansion has a finite radius of convergence.*

Proof. Let $\tilde{F}(k) \equiv \int_{\mathbb{R}^N} e^{ikx}e^{iA_{2M}(x,\ldots,x)}dx$, then by theorem 21 the oscillatory integral (4.24) is given by:

$$\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}A_{2M}(x,\ldots,x)}f(x)dx = \hbar^{N/2M}\int_{\mathbb{R}^N}\tilde{F}(\hbar^{1/2M}k)\mu_f(dk) \tag{4.29}$$

By lemma 5 $\tilde{F}$ is given by

$$\tilde{F}(\hbar^{1/2M}k) = e^{iN\pi/4M}\int_{\mathbb{R}^N} e^{i\hbar^{1/2M}kxe^{i\pi/4M}}e^{-A_{2M}(x,\ldots,x)}dx$$

Where, if $\hbar = |\hbar|e^{i\phi}$, $\phi \in [-\pi, 0]$, $\hbar^{1/2M} = |\hbar|^{1/2M}e^{i\phi/2M}$. By representing the latter absolutely convergent integral using polar coordinates in $\mathbb{R}^N$ we get:

$$\tilde{F}(\hbar^{1/2M}k) = e^{iN\pi/4M}\int_{S_{N-1}}\int_0^\infty e^{i\hbar^{1/2M}e^{i\pi/4M}\rho ku}e^{-\rho^{2M}A_{2M}(u,\ldots,u)}\rho^{N-1}d\rho d\Omega_{N-1}$$

where $d\Omega_{N-1}$ is the Riemann-Lebesgue measure on the $N-1$-dimensional spherical hypersurface S_{N-1}, $x = \rho u$, $\rho = |x|$, $u \in S_{N-1}$ is a unitary vector.

We can expand the latter integral in a power series of $\hbar^{1/2M}$:

$$\tilde{F}(\hbar^{1/2M}k) =$$

$$= e^{iN\pi/4M}\int_{S_{N-1}}\int_0^\infty \sum_{l=0}^\infty \frac{(i)^l}{l!}(e^{i\pi/4M})^l \hbar^{l/2M}\rho^l (ku)^l e^{-\rho^{2M}A_{2M}(u,\ldots,u)}\rho^{N-1}d\rho d\Omega_{N-1} =$$

$$= e^{iN\pi/4M}\sum_{l=0}^\infty \frac{(i)^l}{l!}(e^{i\pi/4M})^l \hbar^{l/2M}\int_{S_{N-1}}(ku)^l\int_0^\infty \rho^{l+N-1}e^{-\rho^{2M}A_{2M}(u,\ldots,u)}d\rho d\Omega_{N-1} =$$

$$= \frac{e^{iN\pi/4M}}{2M}\sum_{l=0}^\infty \frac{(i)^l}{l!}(e^{i\pi/4M})^l \hbar^{l/2M}\Gamma\Big(\frac{l+N}{2M}\Big)\int_{S_{N-1}}(ku)^l P(u)^{-\frac{l+N}{2M}}d\Omega_{N-1} \tag{4.30}$$

where $P(u)\equiv A_{2M}(u,\ldots,u)$ is a strictly positive continuous function on the compact set S_{N-1}, so that it admits an absolute minimum denoted by m. This gives

$$|\int_{S_{N-1}}(ku)^l P(u)^{-\frac{l+N}{2M}}d\Omega_{N-1}| \leq |k|^l m^{-\frac{l+N}{2M}}\Omega_{N-1}(S_{N-1}) = |k|^l m^{-\frac{l+N}{2M}}2\pi^{N/2}\Gamma\Big(\frac{N}{2}\Big)^{-1} \tag{4.31}$$

The latter inequality and the Stirling formula assure the absolute convergence of the series (4.30). We can now insert this formula into (4.29) and get:

$$\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}A_{2M}(x,\ldots,x)}f(x)dx =$$

$$= \hbar^{N/2M}\frac{e^{iN\pi/4M}}{2M}\sum_{l=0}^{n-1}\frac{(i)^l}{l!}(e^{i\pi/4M})^l \hbar^{l/2M}\Gamma\Big(\frac{l+N}{2M}\Big)$$
$$\int_{\mathbb{R}^N}\int_{S_{N-1}}(ku)^l P(u)^{-\frac{l+N}{2M}}d\Omega_{N-1}\mu_f(dk) + \mathcal{R}_n \tag{4.32}$$

By estimate (4.31) and Stirling's formula one can easily verify that if assumption (4.28) is satisfied, then the latter series in powers of $\hbar^{1/2M}$ has a strictly positive radius of convergence.
Equation (4.32) can also be written in the following form:

$$\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}A_{2M}(x,\ldots,x)}f(x)dx =$$

$$= \hbar^{N/2M}\frac{e^{iN\pi/4M}}{2M}\sum_{l=0}^{n-1}\frac{1}{l!}(e^{i\pi/4M})^l \hbar^{l/2M}\Gamma\Big(\frac{l+N}{2M}\Big)$$
$$\int_{S_{N-1}}P(u)^{-\frac{l+N}{2M}}\frac{\partial^l}{\partial u^l}f(0)d\Omega_{N-1} + \mathcal{R}_n \tag{4.33}$$

where $\frac{\partial^l}{\partial u^l} f(0)$ denotes the l_{th} partial derivative of f at 0 in the direction u, and

$$\mathcal{R}_n = \hbar^{N/2M} e^{iN\pi/4M} \int_{\mathbb{R}^N} \int_{S_{N-1}} \int_0^\infty \sum_{l=n}^\infty \frac{(i)^l}{l!} (e^{i\pi/4M})^l \hbar^{l/2M} \rho^l (ku)^l \\ e^{-\rho^{2M} A_{2M}(u,\dots,u)} \rho^{N-1} d\rho d\Omega_{N-1} \mu_f(dk). \quad (4.34)$$

In the more general case in which assumptions (1) and (2) are satisfied we can prove the asymptoticity of the expansion (4.32), indeed

$$\mathcal{R}_n = \hbar^{N/2M} e^{iN\pi/4M} \frac{(i)^n}{n-1!} (e^{i\pi/4M})^n \hbar^{n/2M} \int_{\mathbb{R}^N} \int_{S_{N-1}} \int_0^\infty \\ \int_0^1 (1-t)^{n-1} e^{iku\rho t \hbar^{1/2M} e^{i\pi/4M}} e^{-\rho^{2M} A_{2M}(u,\dots,u)} (ku)^n \rho^{n+N-1} dt d\rho d\Omega_{N-1} \mu_f(dk) \\ (4.35)$$

By assumptions (1), (2) and Fubini theorem the latter is bounded by

$$|\mathcal{R}_n| \leq \frac{A}{M} \pi^{N/2} \Gamma\Big(\frac{N}{2}\Big)^{-1} |\hbar|^{(n+N)/2M} c^n m^{-\frac{n+N}{2M}} \frac{C(n,M,N)}{n!} \Gamma\Big(\frac{n+N}{2M}\Big)$$

If assumption (4.26) is satisfied, then the latter becomes

$$|\mathcal{R}_n| \leq \frac{A}{M} \pi^{N/2} \Gamma\Big(\frac{N}{2}\Big)^{-1} |\hbar|^{(n+N)/2M} c^n m^{-\frac{n+N}{2M}}$$

and the series has a positive radius of convergence, while if assumption (4.27) holds, we get the estimate

$$|\mathcal{R}_n| \leq \frac{A}{M} \pi^{N/2} \Gamma\Big(\frac{N}{2}\Big)^{-1} |\hbar|^{(n+N)/2M} c^n m^{-\frac{n+N}{2M}} \Gamma\Big(1+\frac{n}{2M}\Big).$$

This and the analyticity of the $I_N(\hbar)$ in $\mathrm{Im}(\hbar) < 0$ (cf. Remark 11) by Nevanlinna theorem [84] assure the Borel summability of the power series expansion (4.17). □

Chapter 5

Feynman path integrals for polynomially growing potentials

In the first three chapters we have seen that the infinite dimensional oscillatory integrals are a powerful tool and can be used to give a rigorous mathematical meaning to a large class of "Feynman path integral representations". In section 1.3 we have seen the application to the the Schrödinger equation with an anharmonic oscillator potential

$$V(x) = \frac{1}{2}x \cdot \Omega^2 x + V_1(x), \tag{5.1}$$

where Ω^2 is a positive definite symmetric $d \times d$ matrix and V_1 is the Fourier transform of a complex bounded variation measure on $\mathbb{R}^d$. In chapter 2 the "phase space Feynman path integrals" give a representation of the solution of a Schrödinger equation in which the potential depends both on position and on momentum. In chapter 3 the solution of a class of stochastic Schrödinger equation is represented by an infinite dimensional oscillatory integral with complex phase.

The main problem of these techniques is the fact that the class of unbounded potentials for which a Feynman path integral representation for the solution of the corresponding Schrödinger equation exists is not very rich. Indeed the perturbation V_1 to the harmonic oscillator potential in equation (5.1) has to belong to the class of Fourier transforms of measures, so that is bounded. It is possible to deal with linear potentials $V_2(x) = Cx$ (see remark 1 in section 1.3) and extension to Laplace transforms of measures has been given in [6, 74], but even this approach does not cover the case of potentials which are polynomials of degree larger than two.

In this chapter we give a partial solution to this problem and develop a Feynman path integral representation for the solution of the Schrödinger equation for an anharmonic oscillator potential of the type

$$V(x) = \frac{1}{2}x \cdot \Omega^2 x + \lambda C(x,x,x,x), \tag{5.2}$$

where C is a completely symmetric positive fourth order covariant tensor on $\mathbb{R}^d$ and $\lambda \geq 0$ is a coupling constant. If $d = 1$, (5.2) reduces to $V(x) = \frac{1}{2}\Omega^2 x^2 + \lambda x^4$. In the first and in the second sections we extend the class of functions for which a generalized infinite dimensional oscillatory integral can be computed and prove a Parseval type equality. In addition we propose an analytic continuation formula which shows a direct connection between the infinite dimensional oscillatory integral and the Wiener integral. In the third section we consider the Schrödinger equation for a d- dimensional quantum particle under the action of the anharmonic oscillator potential (5.2), we give a functional integral representation for the solution of the corresponding Schrödinger equation and show that the so defined functional is analytic in the coupling constant $\lambda \in \mathbb{C}$ for $Im(\lambda) < 0$, continuous for $\lambda \in \mathbb{R}$ and coincides for $\lambda \leq 0$ with a well defined infinite dimensional oscillatory integral. We prove moreover the Borel summability of the asymptotic Dyson expansion (in powers of the coupling constant λ) for the scalar product $\langle \phi, e^{-i\frac{t}{\hbar}H}\psi_0\rangle$, where H is the quantum mechanical Hamiltonian $H = -\frac{\hbar^2}{2} + V$ and $\phi, \psi_0 \in L^2(\mathbb{R}^d)$ are suitable vectors.

5.1 A generalized oscillatory integral

In this section and in the following one, by means of the techniques of chapter 4, we shall generalize formulas (1.5) and (1.11) to a larger class of phase functions.
Let us deal first of all with the finite dimensional case, i.e. $dim(\mathcal{H}) = N$. Let $A : \mathcal{H} \times \mathcal{H} \times \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a completely symmetric and positive fourth order covariant tensor on $\mathcal{H}$. After the introduction of an orthonormal basis in $\mathcal{H}$, the elements $x \in \mathcal{H}$ can be identified with $N-$ple of real numbers, i.e. $x = (x_1, \ldots, x_N)$, and the action of the tensor A on the 4-ple (x,x,x,x) is represented by an homogeneous fourth order polynomial in the variables $x_1, \ldots, x_N$:

$$P(x) = A(x,x,x,x) = \sum_{j,k,l,m} a_{j,k,l,m} x_j x_k x_l x_m \tag{5.3}$$

with $a_{j,k,l,m} \in \mathbb{R}$.
We are going to define the following generalized Fresnel integral:

$$\widetilde{\int} e^{\frac{i}{2\hbar}x\cdot(I-B)x} e^{\frac{-i\lambda}{\hbar}P(x)} f(x)dx \tag{5.4}$$

where I, B are $N \times N$ matrices, I being the identity, $\lambda \in \mathbb{R}$, $f \in \mathcal{F}(\mathbb{R}^N)$ and $\hbar > 0$.

Lemma 6. *Let $P : \mathbb{R}^N \to \mathbb{R}$ be given by (5.3). Then the Fourier transform of the distribution* $\frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}P(x)}$*:*

$$\tilde{F}(k) = \int_{\mathbb{R}^N} e^{ik\cdot x} \frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}P(x)} d^N x \tag{5.5}$$

is a bounded complex-valued entire function on $\mathbb{R}^N$ admitting, if A is strictly positive, the following representations

$$\tilde{F}(k) = \begin{cases} e^{iN\pi/8} \int_{\mathbb{R}^N} e^{ie^{i\pi/8}k\cdot x} \frac{e^{\frac{ie^{i\pi/4}}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}} e^{\frac{\lambda}{\hbar}P(x)} d^N x & \lambda < 0 \\ e^{-iN\pi/8} \int_{\mathbb{R}^N} e^{ie^{-i\pi/8}k\cdot x} \frac{e^{\frac{ie^{-i\pi/4}}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}} e^{-\frac{\lambda}{\hbar}P(x)} d^N x & \lambda > 0 \end{cases} \tag{5.6}$$

Moreover, for general $A \geq 0$, if $\lambda \leq 0$ and $(I-B)$ is symmetric strictly positive then $\tilde{F}(k)$ can also be represented by

$$\tilde{F}(k) = \int_{\mathbb{R}^N} e^{ie^{i\pi/4}k\cdot x} \frac{e^{-\frac{1}{2\hbar}x\cdot(I-B)x}}{(2\pi\hbar)^{N/2}} e^{\frac{i\lambda}{\hbar}P(x)} d^N x = \mathbb{E}[e^{ie^{i\pi/4}k\cdot x} e^{\frac{i\lambda}{\hbar}P(x)} e^{\frac{1}{2\hbar}x\cdot Bx}] \tag{5.7}$$

where $\mathbb{E}$ denotes the expectation value with respect to the centered Gaussian measure on $\mathbb{R}^N$ with covariance operator $\hbar I$.

Proof. For the proof of the representation (5.6) and of the boundedness of $\tilde{F}$ see chapter 4, where a more general case is handled. From the representations (5.6) and (5.7) the analyticity of $\tilde{F}(k)$, $k \in \mathbb{C}^N$ follows immediately.
Let us here prove representation (5.7) in the particular case $B = 0$ and P of the special form $P(x) = \sum_{j=1}^N a_j x_j^4$, with $a_j \geq 0$. This is sufficient to show the main ideas of the proof, the general case is handled in appendix A.
In this case one has to study the following integral on the real line:

$$I_j(k_j) \equiv \int_{\mathbb{R}} e^{ik_j x_j} \frac{e^{\frac{i}{2\hbar}x_j^2}}{(2\pi i\hbar)^{1/2}} e^{\frac{-i\lambda}{\hbar}a_j x_j^4} dx_j, \qquad k = (k_1, \ldots, k_N), k_j \in \mathbb{R}$$

and then one has

$$\tilde{F}(k) = \prod_{j=1}^N I_j(k_j).$$

Moreover, as $e^{\frac{i}{2\hbar}x_j^2} e^{\frac{-i\lambda}{\hbar}a_j x_j^4}$ is an even function, we have $I_j(k_j) = I_{j,+}(k_j) + I_{j,+}(-k_j)$, with

$$I_{j,+}(k_j) = \int_0^\infty e^{ik_j x_j} \frac{e^{\frac{i}{2\hbar}x_j^2}}{(2\pi i\hbar)^{1/2}} e^{\frac{-i\lambda}{\hbar}a_j x_j^4} dx_j$$

In the following we will parametrize a complex number $z \in \mathbb{C}$ by means of its modulus ρ and its phase $\theta \in [0, 2\pi)$, i.e. $z = \rho e^{i\theta}$.
Since the integrand in $I_{j,+}(k_j)$ is oscillating, a priori it is not clear that $I_{j,+}(k)$ exists, even as an improper Riemann integral. For this reason we look at the corresponding integral in the upper halfplane of $\mathbb{C}$ with a "regularizing parameter" $0 < \epsilon < \pi/4$, which we send to zero at the end. For each $R > 0$ let us consider the closed path in the complex plane composed by three pieces: $\gamma_1, \gamma_2, \gamma_3$, where

$$\begin{array}{lcl} \gamma_1(R) = \{z \in \mathbb{C} & | & 0 \leq \rho \leq R, \quad \theta = \epsilon\} \\ \gamma_2(R) = \{z \in \mathbb{C} & | & \rho = R, \quad \epsilon \leq \theta \leq \pi/4\} \\ \gamma_3(R) = \{z \in \mathbb{C} & | & 0 \leq \rho \leq R, \quad \theta = \pi/4\} \end{array}$$

for some small $0 < \epsilon < \pi/4$. From the analyticity of $z_j \mapsto e^{ik_j z_j} e^{\frac{i}{2\hbar} z_j^2} e^{\frac{-i\lambda}{\hbar} a_j z_j^4}$, $k_j \in \mathbb{C}$, and the Cauchy theorem we have:

$$\int_{\gamma_1 \cup \gamma_2 \cup \gamma_3} e^{ik_j z_j} e^{\frac{i}{2\hbar} z_j^2} e^{\frac{-i\lambda}{\hbar} a_j z_j^4} dz_j = 0,$$

that is

$$\int_{\gamma_1} e^{ik_j z_j} e^{\frac{i}{2\hbar} z_j^2} e^{\frac{-i\lambda}{\hbar} a_j z_j^4} dz_j + iR \int_0^{\pi/4} e^{ik_j Re^{i\theta}} e^{\frac{i}{2\hbar} R^2 e^{2i\theta}} e^{\frac{-i\lambda}{\hbar} a_j R^4 e^{4i\theta}} e^{i\theta} d\theta + \\ - e^{i\pi/4} \int_0^R e^{ik_j \rho e^{i\pi/4}} e^{\frac{i}{2\hbar} \rho^2 e^{i\pi/2}} e^{i\frac{\lambda}{\hbar} a_j \rho^4} d\rho = 0 \quad (5.8)$$

Now we take the limit as $R \to +\infty$. The second integral converges to 0, as it is easy to verify by using the methods presented in appendix A. Hence we have:

$$\lim_{R \to \infty} \int_{\gamma_1} e^{ik_j z_j} e^{\frac{i}{2\hbar} z_j^2} e^{\frac{-i\lambda}{\hbar} a_j z_j^4} dz_j = e^{i\pi/4} \int_0^\infty e^{ik_j \rho e^{i\pi/4}} e^{\frac{i}{2\hbar} \rho^2 e^{i\pi/2}} e^{i\frac{\lambda}{\hbar} a_j \rho^4} d\rho.$$

The r.h.s. is independent of ϵ, hence the limit of the l.h.s. for $\epsilon \downarrow 0$ (ϵ entering in the definition of $\gamma_1(R)$) also exists and is equal to the r.h.s.
So we get :

$$I_{j,+}(k_j) = e^{i\pi/4} \int_0^\infty e^{ik_j \rho e^{i\pi/4}} \frac{1}{\sqrt{2\pi i\hbar}} e^{\frac{i}{2\hbar} \rho^2 e^{i\pi/2}} e^{\frac{-i\lambda}{\hbar} a_j \rho^4 e^{i\pi}} d\rho \\ = \int_0^\infty e^{ik_j \rho e^{i\pi/4}} \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{-\rho^2}{2\hbar}} e^{\frac{i\lambda}{\hbar} a_j \rho^4} d\rho \quad (5.9)$$

so that (with $k = (k_1, ..., K_N), \in \mathbb{R}^N$

$$\tilde{F}(k) = \prod_{j=1}^N (I_{j,+}(k_j) + I_{j,+}(-k_j)) = \int_{\mathbb{R}^N} e^{ie^{i\pi/4} k \cdot x} e^{i\frac{\lambda}{\hbar} P(x)} \frac{e^{-\frac{1}{2\hbar} x \cdot x}}{(2\pi\hbar)^{N/2}} d^n x = \\ = \mathbb{E}[e^{ie^{i\pi/4} k \cdot x} e^{i\frac{\lambda}{\hbar} P(x)}] \quad (5.10)$$

(where $\mathbb{E}$ is the expectation with respect to the Gaussian measure on $\mathbb{R}^N$ of mean zero and variance $\hbar^2 I$). $\square$

Remark 14. *A careful reading of this proof shows that the second part of the statement, that is representation* (5.7)*, is valid if and only if the degree of P is 4, but cannot be generalized to polynomial functions of higher (even) degree. In fact the proof is based on the analyticity of the integrand and on a deformation of the contour of integration into a region of the complex plane in which the real part of the leading term of the polynomial, that is of $Re(-i\lambda a z^4)$, is negative, where $\lambda < 0$, $a > 0$. By setting $z = \rho e^{i\theta}$ one can immediately verify that this condition is satisfied if and only if $0 \leq \theta \leq \pi/4$. By considering a polynomial of higher even degree $2M$ this condition becomes $0 \leq \theta \leq \pi/2M$ and if $M > 2$ the angle $\theta = \pi/4$ is no longer included. This angle is fundamental as the oscillatory function $\frac{e^{\frac{i}{2\hbar}z^2}}{(2\pi i\hbar)^{1/2}}$ evaluated in $z = \rho e^{i\pi/4}$ gives $e^{-i\pi/4}\frac{e^{\frac{-\rho^2}{2\hbar}}}{(2\pi\hbar)^{1/2}}$, that is the density of the normal distribution with mean zero and variance $\hbar^2$, multiplied by the factor $e^{-i\pi/4}$. These considerations also show the necessity of considering $\lambda \leq 0$.*

Remark 15. *We note that to have $\lambda = 0$ is equivalent to take $P = 0$. In this case by a deformation of the integration contour one has immediately:*

$$\int_{\mathbb{R}^N} e^{ik\cdot x}\frac{e^{\frac{i}{2\hbar}x\cdot x}}{(2\pi i\hbar)^{N/2}}d^N x =$$
$$= \int_{\mathbb{R}^N} e^{ik\cdot x e^{i\pi/4}}\frac{e^{\frac{-x\cdot x}{2\hbar}}}{(2\pi\hbar)^{N/2}}d^N x = \mathbb{E}[e^{ik\cdot x e^{i\pi/4}}] = e^{\frac{-i\hbar}{2}k\cdot k}. \quad (5.11)$$

We are going to apply these results to the definition of the generalized Fresnel integral (5.4).

Theorem 21. *("Parseval equality") Let $f \in \mathcal{F}(\mathbb{R}^N)$, $f = \hat{\mu}_f$. Then the generalized Fresnel integral*

$$I(f) \equiv \widetilde{\int_{\mathbb{R}^N}} e^{\frac{i}{2\hbar}x\cdot(I-B)x}e^{\frac{-i\lambda}{\hbar}P(x)}f(x)dx$$

is well defined and it is given by:

$$\widetilde{\int_{\mathbb{R}^N}} e^{\frac{i}{2\hbar}x\cdot(I-B)x}e^{\frac{-i\lambda}{\hbar}P(x)}f(x)dx = \int_{\mathbb{R}^N}\tilde{F}(k)\mu_f(dk), \quad (5.12)$$

where $\tilde{F}(k)$ is given by equation (5.6) *if A in* (5.3) *is strictly positive, or by equation* (5.7) *if $A \geq 0$, $\lambda \leq 0$ and $(I - B)$ is symmetric strictly positive. The integral on the r.h.s. of* (5.12) *is absolutely convergent (hence it can be understood in Lebesgue sense).*

Proof. Let us choose a test function $\psi \in \mathcal{S}(\mathbb{R}^N)$, such that $\psi(0) = 1$ and let us compute the limit

$$I(f) \equiv \lim_{\epsilon\downarrow 0} \int_{\mathbb{R}^N} \frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}P(x)} \psi(\epsilon x) f(x) dx$$

By hypothesis $f(x) = \hat{\mu}_f(x) = \int_{\mathbb{R}^N} e^{ikx}\mu_f(dk)$ and substituting into the previous expression we get :

$$I(f) = \lim_{\epsilon\downarrow 0} \int_{\mathbb{R}^N} \frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}P(x)} \psi(\epsilon x) \int_{\mathbb{R}^N} e^{ikx}\mu_f(dk) dx.$$

By Fubini theorem (which applies for any $\epsilon > 0$ since the integrand is bounded by $|\psi(\epsilon x)|$ which is dx-integrable, and μ_f is a bounded measure) the r.h.s. is

$$= \lim_{\epsilon\downarrow 0} \int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} \frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}P(x)} \psi(\epsilon x) e^{ikx} dx \Big) \mu_f(dk)$$
$$= \frac{1}{(2\pi)^N} \lim_{\epsilon\downarrow 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \tilde{F}(k - \alpha\epsilon) \tilde{\psi}(\alpha) d\alpha \mu_f(dk) \quad (5.13)$$

(here we have used the fact that the integral with respect to x is the Fourier transform of $\frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}P(x)} \psi(\epsilon x)$ and the inverse Fourier transform of a product is a convolution). Now we can pass to the limit using the Lebesgue bounded convergence theorem and get the desired result:

$$\lim_{\epsilon\downarrow 0} \int_{\mathbb{R}^N} \frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}P(x)} \psi(\epsilon x) f(x) dx = \int_{\mathbb{R}^N} \tilde{F}(k) \mu_f(dk),$$

where we have used that $\int \tilde{\psi}(\alpha) d\alpha = (2\pi)^N \psi(0)$ and lemma 6, which assures the boundedness of $\tilde{F}$. □

Corollary 2. *Let $(I - B)$ be symmetric and strictly positive, $\lambda \le 0$ and $f \in \mathcal{F}(\mathbb{R}^N)$, $f = \hat{\mu}_f$ such that $\forall x \in \mathbb{R}^N$ the integral $\int e^{-\frac{\sqrt{2}}{2}kx}|\mu_f|(dk)$ is convergent and the positive function $g : \mathbb{R}^n \to \mathbb{R}$, defined by $g(x) = e^{\frac{1}{2\hbar}x\cdot Bx} \int e^{-\frac{\sqrt{2}}{2}kx}|\mu_f|(dk)$ is summable with respect to the centered Gaussian measure on $\mathbb{R}^N$ with covariance $\hbar I$.*
Then f extends to an analytic function on $\mathbb{C}^N$ and the corresponding generalized Fresnel integral is well defined and it is given by

$$\widetilde{\int}_{\mathbb{R}^N} \frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}P(x)} f(x) dx = \mathbb{E}[e^{\frac{i\lambda}{\hbar}P(x)} e^{\frac{1}{2\hbar}x\cdot Bx} f(e^{i\pi/4}x)]. \quad (5.14)$$

Proof. By the assumption on the measure μ_f it follows that its Laplace transform $f^L : \mathbb{C}^N \to \mathbb{C}$, $f^L(z) = \int_{\mathbb{R}^N} e^{kz}\mu_f(dk)$, is a well defined entire function such that $f^L(ix) = f(x)$, $x \in \mathbb{R}^N$. By theorem 21 the generalized Fresnel integral can be computed by means of Parseval equality

$$\widetilde{\int}_{\mathbb{R}^N} \frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}P(x)} f(x)dx = \int_{\mathbb{R}^N} \tilde{F}(k)\mu_f(dk) =$$

$$= \int_{\mathbb{R}^N} \mathbb{E}[e^{ikxe^{i\pi/4}} e^{\frac{1}{2\hbar}x\cdot Bx} e^{\frac{i\lambda}{\hbar}P(x)}]\mu_f(dk)$$

By Fubini theorem, which applies given the assumptions on the measure μ_f, this is equal to

$$\mathbb{E}[e^{\frac{1}{2\hbar}x\cdot Bx} e^{\frac{i\lambda}{\hbar}P(x)} \int_{\mathbb{R}^N} e^{ikxe^{i\pi/4}} \mu_f(dk)] = \mathbb{E}[e^{\frac{1}{2\hbar}x\cdot Bx} e^{\frac{i\lambda}{\hbar}P(x)} f^L(ie^{i\pi/4}x)] =$$

$$= \mathbb{E}[e^{\frac{1}{2\hbar}x\cdot Bx} e^{\frac{i\lambda}{\hbar}P(x)} f(e^{i\pi/4}x)] \quad (5.15)$$

and the conclusion follows. □

Remark 16. *The latter theorem shows that, under suitable assumptions on the function f, the generalized Fresnel integral* (5.4) *can be explicitly computed by means of a Gaussian integral. By mimicking the proof of lemma 6 one can be tempted to generalize equation* (5.14) *to a larger class of functions, that are analytic in a suitable region of $\mathbb{C}^N$, but do not belong to $\mathcal{F}(\mathbb{R}^N)$ (see in chapter 4 the comment following theorem 17 for more details). In fact this is not possible, as the definition 9 of oscillatory integral requires that the limit of the sequence of regularized integrals exists and is independent of the regularization. Let us consider the subset of the complex plane*

$$\Lambda = \{\xi \in \mathbb{C} \mid 0 < \arg(\xi) < \pi/4\} \subset \mathbb{C}, \quad (5.16)$$

and let $\bar{\Lambda}$ be its closure. The identity

$$\lim_{\epsilon\to 0} \int \int_{\mathbb{R}^N} \frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}P(x)} f(x)\psi(\epsilon x)dx = \mathbb{E}[e^{\frac{1}{2\hbar}x\cdot Bx} e^{\frac{i\lambda}{\hbar}P(x)} f(e^{i\pi/4}x)]$$

(with $(I-B)$ symmetric strictly positive and $\lambda \leq 0$) can only be proved by choosing a regularizing function $\psi \in \mathcal{S}$, $\psi(0) = 1$, such that the function $z \mapsto \psi(zx)$ is analytic for $z \in \Lambda$ and continuous for $z \in \bar{\Lambda}$ for each $x \in \mathbb{R}^N$. Moreover one has to assume that $|\psi(e^{i\theta}x)|$ is bounded as $|x| \to \infty$ for each $\theta \in (0, \pi/4)$.

5.2 Infinite dimensional generalized oscillatory integrals

Let $\mathcal{H}$ be a real separable infinite dimensional Hilbert space, with inner product $\langle\ ,\ \rangle$ and norm $|\ |$. Let ν be the finitely additive cylinder measure on $\mathcal{H}$, defined by its characteristic functional $\hat{\nu}(x) = e^{-\frac{\hbar}{2}|x|^2}$. Let $\|\ \|$ be a "measurable" norm on $\mathcal{H}$, that is $\|\ \|$ is such that for every $\epsilon > 0$ there exist a finite-dimensional projection $P_\epsilon : \mathcal{H} \to \mathcal{H}$, such that for all $P \perp P_\epsilon$ one has

$$\nu(\{x \in \mathcal{H}|\ \|P(x)\| > \epsilon\}) < \epsilon,$$

where P and P_ϵ are called orthogonal ($P \perp P_\epsilon$) if their ranges are orthogonal in $(\mathcal{H}, \langle\ ,\ \rangle)$. One can easily verify that $\|\ \|$ is weaker than $|\ |$. Denoted by $\mathcal{B}$ the completion of $\mathcal{H}$ in the $\|\ \|$-norm and by i the continuous inclusion of $\mathcal{H}$ in $\mathcal{B}$, one can prove that $\mu \equiv \nu \circ i^{-1}$ is a countably additive Gaussian measure on the Borel subsets of $\mathcal{B}$. The triple $(i, \mathcal{H}, \mathcal{B})$ is called an *abstract Wiener space* [61, 75]. Given $y \in \mathcal{B}^*$ one can easily verify that the restriction of y to $\mathcal{H}$ is continuous on $\mathcal{H}$, so that one can identify $\mathcal{B}^*$ as a subset of $\mathcal{H}$. Moreover $\mathcal{B}^*$ is dense in $\mathcal{H}$ and we have the dense continuous inclusions $\mathcal{B}^* \subset \mathcal{H} \subset \mathcal{B}$. Each element $y \in \mathcal{B}^*$ can be regarded as a random variable $n(y)$ on $(\mathcal{B}, \mu)$. A direct computation shows that $n(y)$ is normally distributed, with covariance $|y|^2$. More generally, given $y_1, y_2 \in \mathcal{B}^*$, one has

$$\int_B n(y_1)n(y_2)d\mu = \langle y_1, y_2\rangle.$$

The latter result allows the extension to the map $n : \mathcal{H} \to L^2(\mathcal{B}, \mu)$, because $\mathcal{B}^*$ is dense in $\mathcal{H}$. Given an orthogonal projection P in $\mathcal{H}$, with

$$P(x) = \sum_{i=1}^{n} \langle e_i, x\rangle e_i$$

for some orthonormal $e_1, \ldots, e_n \in \mathcal{H}$, the stochastic extension $\tilde{P}$ of P on $\mathcal{B}$ is well defined by

$$\tilde{P}(\,\cdot\,) = \sum_{i=1}^{n} n(e_i)(\,\cdot\,)e_i.$$

Given a function $f : \mathcal{H} \to \mathcal{B}_1$, where $(\mathcal{B}_1, \|\ \|_{\mathcal{B}_1})$ is another real separable Banach space, the stochastic extension $\tilde{f}$ of f to $\mathcal{B}$ exists if the functions $f \circ \tilde{P} : \mathcal{B} \to \mathcal{B}_1$ converge to $\tilde{f}$ in probability with respect to μ as P converges strongly to the identity in $\mathcal{H}$. If $g : \mathcal{B} \to \mathcal{B}_1$ is continuous and $f := g|_{\mathcal{H}}$, then one can prove [61] that the stochastic extension of f is well defined and it is equal to g $\mu-$a.e. In this setting it is possible to extend the results of the

previous section to the infinite dimensional case.
Let $A : \mathcal{H}\times\mathcal{H}\times\mathcal{H}\times\mathcal{H} \to \mathbb{R}$ be a completely symmetric positive covariant tensor operator on $\mathcal{H}$ such that the map $V : \mathcal{H} \to \mathbb{R}^+$, $x \mapsto V(x) \equiv A(x,x,x,x)$ is continuous in the $\| \, \|$ norm. As a consequence V is continuous in the $| \, |$-norm, moreover it can be extended by continuity to a random variable $\bar{V}$ on $\mathcal{B}$, with $\bar{V}|_{\mathcal{H}} = V$. By the previous considerations, the stochastic extension $\tilde{V}$ of $V : \mathcal{H} \to \mathbb{R}$ exists and coincides with $\bar{V} : \mathcal{B} \to \mathbb{R}$ $\mu-$a.e. Moreover for any increasing sequence of $n-$dimensional projectors P_n in $\mathcal{H}$, the family of bounded random variables $e^{i\frac{\lambda}{\hbar}V\circ\tilde{P}_n(\,\cdot\,)} \equiv e^{i\frac{\lambda}{\hbar}V^n(\,\cdot\,)}$ converges $\mu-$a.e. to $e^{i\frac{\lambda}{\hbar}\bar{V}(\,\cdot\,)}$. In addition, for any $h \in \mathcal{H}$ the sequence of random variables

$$\sum_{i=1}^{n} h_i n(e_i), \qquad h_i = \langle e_i, h\rangle$$

converges in $L^2(\mathcal{B},\mu)$, and by subsequences a.e., to the random variable $n(h)$.
Let us consider a self-adjoint trace class operator $B : \mathcal{H} \to \mathcal{H}$. The quadratic form on $\mathcal{H} \times \mathcal{H}$:

$$x \in \mathcal{H} \mapsto \langle x, Bx\rangle$$

can be extended to a random variable on $\mathcal{B}$, denoted again by $\langle\,\cdot\,, B\,\cdot\,\rangle$. Indeed for each increasing sequence of finite dimensional projectors P_n converging strongly to the identity, $P_n(x) = \sum_{i=1}^n e_i\langle e_i, x\rangle$ ($\{e_i\}$ being a CONS in $\mathcal{H}$), the sequence of random variables

$$\omega \in \mathcal{B} \mapsto \sum_{i,j=1}^{n} \langle e_i, Be_j\rangle n(e_i)(\omega) n(e_j)(\omega)$$

is a Cauchy sequence in $L^1(\mathcal{B},\mu)$. By passing if necessary to a subsequence, it converges to $\langle\,\cdot\,, B\,\cdot\,\rangle$ $\mu-$a.e.
Let us assume that the largest eigenvalue of B is strictly less than 1 (or, in other words, that (I-B) is strictly positive). Then one can prove that the random variable $g(\,\cdot\,) := e^{\frac{1}{2\hbar}\langle\,\cdot\,, B\,\cdot\,\rangle}$ is μ-summable. Indeed by considering a CONS $\{e_i\}$ made of eigenvectors of the operator B, b_i being the corresponding eigenvalues, the sequence of random variables

$$g_n : \mathcal{B} \to \mathbb{C}, \qquad \omega \mapsto g_n(\omega) = e^{\frac{1}{2\hbar}\sum_{i=1}^n b_i([n(e_i)(\omega)]^2},$$

converges to $g(\omega)$ μ-a.e..
On the other hand one has

$$\int_{\mathcal{B}} g_n(\omega) d\mu(\omega) = \prod_{i=1}^{n} \int_{\mathbb{R}} \frac{e^{-\frac{1}{2\hbar}(1-b_i)x_i^2}}{\sqrt{2\pi\hbar}} dx_i = (\prod_{i=1}^{n}(1-b_i))^{-1/2}$$

so that $\int g_n d\mu$ converges, as $n \to \infty$, to $(\det(I-B))^{-1/2}$, where $\det(I-B)$ denotes the Fredholm determinant of $(I-B)$, which is well defined as B is trace class. Moreover $0 \le g_n \le g_{n+1}$ for each n . It follows that, as $n \to \infty$, $\int g_n d\mu \to \int g d\mu = (\det(I-B))^{-1/2}$. By an analogous reasoning one can prove that for any $y \in \mathcal{H}$, the sequence of random variables f_n:

$$\omega \mapsto f_n(\omega) = e^{\sum_{i=1}^n y_i n(e_i)(\omega)} e^{\frac{1}{2\hbar}\sum_{i=1}^n b_i([n(e_i)(\omega)]^2}$$

where $y_i = \langle y, e_i \rangle$, converges $\mu-$a.e. as n goes to ∞ to the random variable $f(\,\cdot\,) = e^{n(y)(\,\cdot\,)} e^{\frac{1}{2\hbar}\langle \,\cdot\,, B\,\cdot\,\rangle}$ and that

$$\int f_n d\mu \to \int f d\mu = (\det(I-B))^{-1/2} e^{\frac{\hbar}{2}\langle y,(I-B)^{-1}y\rangle}. \tag{5.17}$$

(see [75, 72]). The following result follows:

Lemma 7. *Let $B : \mathcal{H} \to \mathcal{H}$ be a self adjoint and trace class operator such that $I - B$ is strictly positive, let $k \in \mathcal{H}$ and $\lambda \le 0$. Then for any increasing sequence P_n of projectors onto n-dimensional subspaces of $\mathcal{H}$ such that $P_n \uparrow I$ strongly as $n \to \infty$, the following sequence of finite dimensional integrals:*

$$F_n(k) \equiv (2\pi i\hbar)^{-n/2} \int_{P_n\mathcal{H}} e^{i\langle P_n k, P_n x\rangle} e^{\frac{i}{2\hbar}\langle P_n x,(I-B)P_n x\rangle} e^{-i\frac{\lambda}{\hbar}V(P_n x)} d(P_n x)$$

converges, as $n \to \infty$, to the Gaussian integral on $\mathcal{B}$:

$$F(k) \equiv \mathbb{E}[e^{in(k)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle} e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)}] \tag{5.18}$$

($\mathbb{E}$ being the expectation with respect to μ on $\mathcal{B}$)

Proof. By lemma 6 one has

$$(2\pi i\hbar)^{-n/2} \int_{P_n\mathcal{H}} e^{i\langle P_n k, P_n x\rangle} e^{\frac{i}{2\hbar}\langle P_n x,(I-B)P_n x\rangle} e^{-i\frac{\lambda}{\hbar}V(P_n x)} d(P_n x) =$$
$$(2\pi\hbar)^{-n/2} \int_{P_n\mathcal{H}} e^{i\langle P_n k, P_n x\rangle e^{i\pi/4}} e^{-\frac{1}{2\hbar}\langle P_n x, P_n x\rangle} e^{\frac{1}{2\hbar}\langle P_n x, BP_n x\rangle} e^{i\frac{\lambda}{\hbar}V(P_n x)} d(P_n x) \tag{5.19}$$

Let us introduce an orthonormal base $\{e_i\}$ of $\mathcal{H}$ such that P_n is the projector onto the span of the first n vectors. Each element $P_n x \in P_n\mathcal{H}$ can be represented as an $n-$ple of real numbers $(x_1, \ldots, x_n)$, where $x_i = \langle x, e_i\rangle$. The latter integral can be written in the following form:

$$(2\pi\hbar)^{-n/2} \int_{\mathbb{R}^n} e^{i\sum_{i=1}^n k_i x_i e^{i\pi/4}} e^{-\frac{1}{2\hbar}\sum_{i=1}^n x_i^2} e^{\frac{1}{2\hbar}\sum_{i,j=1}^n B_{ij}x_i x_j}$$
$$e^{i\frac{\lambda}{\hbar}\sum_{ij,k,h=1}^n A_{ijkh}x_i x_j x_k x_h} dx_1 \ldots dx_n$$

where $B_{ij} = \langle e_i, Be_j\rangle$ and $A_{ijkh} = A(e_i, e_j, e_k, e_h)$.
On the other hand, this coincides with the Gaussian integral on $(\mathcal{B}, \mu)$:

$$\mathbb{E}[e^{i\sum_{i=1}^n k_i n(e_i)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\sum_{i,j=1}^n \langle e_i, Be_j\rangle n(e_i)(\omega)n(e_j)(\omega)} e^{\frac{\lambda}{\hbar}V\circ\tilde{P}_n(\omega)}]$$

By Lebesgue's dominated convergence theorem (which holds because of the assumption on the strict positivity of the operator $I - B$) this converges as $n \to \infty$ to

$$\mathbb{E}[e^{in(k)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\langle\omega, B\omega\rangle} e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)}].$$

and the conclusion follows. □

The above result allows to generalize theorem 21 to the infinite dimensional case.

Theorem 22. *Let B be self-adjoint trace class, $(I-B)$ strictly positive, $\lambda \leq 0$ and $f \in \mathcal{F}(\mathcal{H})$, $f \equiv \hat{\mu}_f$, and let us suppose that the bounded variation measure μ_f satisfies the following assumption*

$$\int_{\mathcal{H}} e^{\frac{\hbar}{4}\langle k, (I-B)^{-1}k\rangle} |\mu_f|(dk) < +\infty. \tag{5.20}$$

Then the infinite dimensional oscillatory integral

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x, (I-B)x\rangle} e^{-i\frac{\lambda}{\hbar}A(x,x,x,x)} f(x)dx \tag{5.21}$$

exists and is given by:

$$\int_{\mathcal{H}} \mathbb{E}[e^{in(k)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\langle\omega, B\omega\rangle} e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)}]\mu_f(dk)$$

Proof. By definition, choosing an increasing sequence of finite dimensional projectors P_n on $\mathcal{H}$, with $P_n \uparrow I$ strongly as $n \to \infty$, the oscillatory integral (5.21) is given by:

$$\lim_{n\to\infty} (2\pi i\hbar)^{-n/2} \int_{P_n\mathcal{H}} e^{\frac{i}{2\hbar}\langle P_n x, (I-B)P_n x\rangle} e^{-i\frac{\lambda}{\hbar}A(P_n x, P_n x, P_n x, P_n x)} f(P_n x)dP_n x. \tag{5.22}$$

Let $f^n : P_n\mathcal{H} \to \mathbb{C}$ be the function defined by $f^n(y) \equiv f(y)$, $y \in P_n\mathcal{H}$. One can easily verify that $f^n \in \mathcal{F}(P_n\mathcal{H})$, $f^n = \hat{\mu}_f^n$, where μ_f^n is the bounded variation measure on $P_n\mathcal{H}$ defined by $\mu_f^n(I) = \mu_f(P_n^{-1}I)$, I being a Borel subset of $P_n\mathcal{H}$, indeed:

$$\begin{aligned} f^n(y) = f(y) = \int_{\mathcal{H}} e^{i\langle y, k\rangle}\mu_f(dk) = \\ = \int_{\mathcal{H}} e^{i\langle P_n y, P_n k\rangle}\mu_f(dk) = \int_{P_n\mathcal{H}} e^{i\langle y, P_n k\rangle}\mu_f^n(dP_n k) \end{aligned} \tag{5.23}$$

where $y = P_n y$. By theorem 21 the limit (5.22) is equal to

$$\lim_{n\to\infty}\int_{P_n\mathcal{H}} G_n(P_n k)\mu_f^n(dP_n k), \tag{5.24}$$

where $G_n : P_n\mathcal{H} \to \mathbb{C}$ is given by:

$$G_n(P_n k) = (2\pi\hbar)^{-n/2}\int_{P_n\mathcal{H}} e^{i\langle P_n k, P_n x\rangle e^{i\pi/4}} e^{-\frac{1}{2\hbar}\langle P_n x,(I-B)P_n x\rangle} e^{i\frac{\lambda}{\hbar}A(P_n x,P_n x,P_n x,P_n x)} dP_n x$$

This, on the other hand (see the proof of lemma 7) is equal to

$$\mathbb{E}[e^{in(P_n k)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\sum_{i,j=1}^n B_{ij} n(e_i)(\omega)n(e_j)(\omega)} e^{i\frac{\lambda}{\hbar}V^n(\omega)}],$$

where $V^n = V \circ \tilde{P}_n$. By substituting the latter expression into (5.24) we have

$$\lim_{n\to\infty}\int_{P_n\mathcal{H}} \mathbb{E}[e^{in(P_n k)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\sum_{i,j=1}^n B_{ij} n(e_i)(\omega)n(e_j)(\omega)} e^{i\frac{\lambda}{\hbar}V^n(\omega)}]\mu_f^n(dP_n k) =$$
$$= \lim_{n\to\infty}\int_{\mathcal{H}} \mathbb{E}[e^{in(P_n k)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\sum_{i,j=1}^n B_{ij} n(e_i)(\omega)n(e_j)(\omega)} e^{i\frac{\lambda}{\hbar}V^n(\omega)}]\mu_f(dk) =$$
$$= \lim_{n\to\infty}\int_{\mathcal{H}} F_n(k)\mu_f(dk) \tag{5.25}$$

By lemma 7 and the dominated convergence theorem, applicable to the integral with respect to μ_f, due to assumption (5.20), we then get

$$\int_{\mathcal{H}} F(k)\mu_f(dk) = \int_{\mathcal{H}} \mathbb{E}[e^{in(k)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle} e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)}]\mu_f(dk)$$

and the conclusion follows. □

Corollary 2 can be generalized to the infinite dimensional case. Indeed due to the assumption (5.20) the function f on the real Hilbert space $\mathcal{H}$ can be extended to those vectors $y \in \mathcal{H}^{\mathbb{C}}$ in the complex Hilbert space $\mathcal{H}^{\mathbb{C}}$ of the form $y = zx$, $x \in \mathcal{H}$, $z \in \mathbb{C}$ as the integral

$$\int_{\mathcal{H}} e^{iz\langle x,k\rangle}\mu_f(dk)$$

is absolutely convergent. Moreover the latter can be uniquely extended to a random variable on $\mathcal{B}$, denoted again by f, by

$$f^z(\omega) \equiv f(z\omega) \equiv \int_{\mathcal{H}} e^{izn(k)(\omega)}\mu_f(dk), \quad \omega \in \mathcal{B}. \tag{5.26}$$

Moreover the random variable $e^{\frac{1}{2\hbar}\langle\,\cdot\,,B\,\cdot\,\rangle}f^z(\,\cdot\,)$ belongs to $L^1(\mathcal{B},\mu)$ if $Im(z)^2 \leq 1/2$.

Theorem 23. *Let $B : \mathcal{H} \to \mathcal{H}$ be self-adjoint trace class, $I-B$ strictly positive, $\lambda \leq 0$ and $f \in \mathcal{F}(\mathcal{H})$ be the Fourier transform of a bounded variation measure μ_f satisfying assumption* (5.20). *Then the infinite dimensional oscillatory integral* (5.21) *is well defined and it is given by:*

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I-B)x\rangle} e^{-i\frac{\lambda}{\hbar}A(x,x,x,x)} f(x)dx = \mathbb{E}[e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)} e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle} f(e^{i\pi/4}\omega)] \qquad (5.27)$$

Proof. By theorem 22 the infinite dimensional oscillatory integral (5.21) can be computed by means of the Parseval-type formula:

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I-B)x\rangle} e^{-i\frac{\lambda}{\hbar}A(x,x,x,x)} f(x)dx =$$

$$= \int_{\mathcal{H}} \mathbb{E}[e^{in(k)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle} e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)}]\mu_f(dk) \quad (5.28)$$

By Fubini theorem, which can be applied under the assumption (5.20), the integral on the r.h.s. of (5.28) is equal to

$$\mathbb{E}[e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)} e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle} \int_{\mathcal{H}} e^{in(k)(\omega)e^{i\pi/4}} \mu_f(dk)] =$$

$$= \mathbb{E}[e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)} e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle} f^{e^{i\pi/4}}(\omega)] = \mathbb{E}[e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)} e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle} f(e^{i\pi/4}\omega)]$$

The integral on the r.h.s. is absolutely convergent as $|e^{i\frac{\lambda}{\hbar}\bar{V}}| = 1$ and $e^{\frac{1}{2\hbar}\langle\,\cdot\,,B\,\cdot\,\rangle} f^{e^{i\pi/4}} \in L^1(\mathcal{B},\mu)$ as $Im(e^{i\pi/4}) = 1/\sqrt{2}$. □

Remark 17. *In the simpler case $\lambda = 0$, under the above assumptions on the function f and the operator B, the infinite dimensional oscillatory integral (given by* (5.27) *with $V = 0$) can also be explicitly computed by means of the absolutely convergent integrals:*

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I-B)x\rangle} f(x)dx = \frac{1}{\sqrt{\det(I-B)}} \int_{\mathcal{H}} e^{-\frac{i\hbar}{2}\langle k,(I-B)^{-1}k} \mu_f(dk) \qquad (5.29)$$

In fact, by means of different methods (see section 2), equation (5.29) *can be proved even without the assumption on the positivity of the operator $(I-B)$ (it suffices that $(I-B)$ be invertible).*

Remark 18. *So far we have proved, under suitable assumptions on the function $f : \mathcal{H} \to \mathbb{C}$ and the operator B, that, if $\lambda \leq 0$, the infinite dimensional generalized Fresnel integral* (5.21)

$$I^F(\lambda) \equiv \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I-B)x\rangle} e^{-i\frac{\lambda}{\hbar}A(x,x,x,x)} f(x)dx$$

on the Hilbert space $\mathcal{H}$ is exactly equal to a Gaussian integral on $\mathcal{B}$:

$$I^G(\lambda) \equiv \int_{\mathcal{H}} \mathbb{E}[e^{in(k)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle} e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)}]\mu_f(dk)$$

(theorem 22), and to

$$I^A(\lambda) \equiv \mathbb{E}[e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)} e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle} f(e^{i\pi/4}\omega)]$$

(theorem 23). One can easily verify that I^G and I^A are analytic functions of the complex variable λ in the region of the complex λ plane $\{Im(\lambda) > 0\}$, while they are continuous in $\{Im(\lambda) = 0\}$ and coincide with I^F in $\{Im(\lambda) = 0, Re(\lambda) \leq 0\}$.

5.3 Application to the Schrödinger equation

Let us consider the Schrödinger equation

$$i\hbar\frac{d}{dt}\psi = H\psi \tag{5.30}$$

on $L^2(\mathbb{R}^d)$ for an anharmonic oscillator Hamiltonian H of the following form:

$$H = -\frac{\hbar^2}{2}\Delta + \frac{1}{2}x\Omega^2 x + \lambda C(x,x,x,x), \tag{5.31}$$

where C is a completely symmetric positive fourth order covariant tensor on $\mathbb{R}^d$, Ω is a positive symmetric $d\times d$ matrix, $\lambda \geq 0$ a positive constant. It is well known, see [91], that H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$. By means of the results of the previous section we are going to give mathematical meaning to the "Feynman path integral" representation of the solution of equation (5.30):

$$\psi(t,x) = \text{“}\int_{\gamma(0)=x} e^{\frac{i}{\hbar}\int_0^t \frac{\dot{\gamma}(s)^2}{2}ds - \frac{i}{\hbar}\int_0^t [\frac{1}{2}\gamma(s)\Omega^2\gamma(s) + \lambda C(\gamma(s),\gamma(s),\gamma(s),\gamma(s))]ds}\psi_0(\gamma(t))D\gamma\text{”},$$

as the analytic continuation (in the parameter λ) of an infinite dimensional generalized oscillatory integral on a suitable Hilbert space.
Let us consider the Cameron-Martin space[1] H_t, that is the Hilbert space of absolutely continuous paths $\gamma : [0,t] \to \mathbb{R}^d$, with $\gamma(0) = 0$ and inner product $\langle\gamma_1,\gamma_2\rangle = \int_0^t \dot{\gamma}_1(s)\dot{\gamma}_2(s)ds$. The cylindrical Gaussian measure on H_t with covariance operator the identity extends to a σ-additive measure on the Wiener

[1]With an abuse of notation we call here Cameron-Martin space the space of paths γ belonging to the Sobolev space $H^{1,2}_{(t)}([0,t],\mathbb{R}^d)$ such that $\gamma(0) = 0$, while in the first three chapters with the same name we denoted the space of paths $\gamma \in H^{1,2}_{(t)}([0,t],\mathbb{R}^d)$ such that $\gamma(t) = 0$.

space $C_t = \{\omega \in C([0,t];\mathbb{R}^d) \mid \gamma(0)=0\}$: the Wiener measure W. (i, H_t, C_t) is an abstract Wiener space.
Let us consider moreover the Hilbert space $\mathcal{H} = \mathbb{R}^d \times H_t$, and the Banach space $\mathcal{B} = \mathbb{R}^d \times C_t$ endowed with the product measure $N(dx) \times W(d\omega)$, N being the Gaussian measure on $\mathbb{R}^d$ with covariance equal to the $d \times d$ identity matrix. $(i, \mathcal{H}, \mathcal{B})$ is an abstract Wiener space.
Let us consider two vectors $\phi, \psi_0 \in L^2(\mathbb{R}^d) \cap \mathcal{F}(\mathbb{R}^d)$. We are going to define the following infinite dimensional oscillatory integral on $\mathcal{H}$:

$$\text{“}\int_{\mathbb{R}^d \times H_t} \bar{\phi}(x) e^{\frac{i}{2\hbar}\int_0^t \dot{\gamma}(s)^2 ds} e^{-\frac{i}{2\hbar}\int_0^t [(\gamma(s)+x)\Omega^2(\gamma(s)+x) ds} \\ e^{\frac{i\lambda}{\hbar} C(\gamma(s)+x,\gamma(s)+x,\gamma(s)+x,\gamma(s)+x) ds} \psi_0(\gamma(t)+x) dx D\gamma\text{”} \quad (5.32)$$

Let us consider the operator $B : \mathcal{H} \to \mathcal{H}$ given by:

$$(x,\gamma) \longrightarrow (y,\eta) = B(x,\gamma),$$

$$y = t\Omega^2 x + \Omega^2 \int_0^t \gamma(s) ds, \qquad \eta(s) = \Omega^2 x (ts - \frac{s^2}{2}) - \int_0^s \int_t^u \Omega^2 \gamma(r) dr du \quad (5.33)$$

and the fourth order tensor operator A given by:

$$A((x_1,\gamma_1),(x_2,\gamma_2),(x_3,\gamma_3),(x_4,\gamma_4)) = \\ = \int_0^t C(\gamma_1(s)+x_1, \gamma_2(s)+x_2, \gamma_3(s)+x_3, \gamma_4(s)+x_4) ds. \quad (5.34)$$

Let us consider moreover the function $f : \mathcal{H} \to \mathbb{C}$ given by

$$f(x,\gamma) = (2\pi i\hbar)^{d/2} e^{-\frac{i}{2\hbar}|x|^2} \bar{\phi}(x) \psi_0(\gamma(t)+x) \quad (5.35)$$

With these notations expression (5.32) can be written in the following form:

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}(|x|^2+|\gamma|^2)} e^{-\frac{i}{2\hbar}\langle (x,\gamma), B(x,\gamma)\rangle} e^{-\frac{i\lambda}{\hbar} A((x,\gamma),(x,\gamma),(x,\gamma),(x,\gamma))} f(x,\gamma) dx d\gamma \quad (5.36)$$

Under suitable assumptions on Ω, λ the theory of the latter section applies, as we shall see below. In the following we will denote by Ω_i, $i = 1, \ldots, d$, the eigenvalues of the matrix Ω.

Theorem 24. *Let us assume that $\lambda \leq 0$, and that for each $i = 1, \ldots, d$ the following inequalities are satisfied*

$$\Omega_i t < \frac{\pi}{2}, \qquad 1 - \Omega_i \tan(\Omega_i t) > 0. \quad (5.37)$$

Let $\phi, \psi_0 \in L^2(\mathbb{R}^d) \cap \mathcal{F}(\mathbb{R}^d)$. Let μ_0 be the complex bounded variation measure on $\mathbb{R}^d$ such that $\hat{\mu}_0 = \psi_0$. Let μ_ϕ be the complex bounded variation measure *on* $\mathbb{R}^d$ *such that* $\hat{\mu}_\phi(x) = (2\pi i\hbar)^{d/2} e^{-\frac{i}{2\hbar}|x|^2}\bar{\phi}(x)$. *Assume in addition that the measures* μ_0, μ_ϕ *satisfy the following assumption:*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\frac{\hbar}{4}x\Omega^{-1}\tan(\Omega t)x} e^{(y+\cos(\Omega t)^{-1}x)(1-\Omega\tan(\Omega t))^{-1}(y+\cos(\Omega t)^{-1}x)} |\mu_0|(dx)|\mu_\phi|(dy) < \infty \quad (5.38)$$

Then the function $f : \mathcal{H} \to \mathbb{C}$, *given by* (5.35) *is the Fourier transform of a bounded variation measure* μ_f *on* $\mathcal{H}$ *satisfying*

$$\int_{\mathcal{H}} e^{\frac{\hbar}{4}\langle (y,\eta),(I-B)^{-1}(y,\eta)\rangle}|\mu_f|(dyd\eta) < \infty \quad (5.39)$$

(B being given by (5.33)*) and the infinite dimensional oscillatory integral* (5.36) *is well defined and is given by:*

$$\int_{\mathbb{R}^d\times H_t}\Big(\int_{\mathbb{R}^d\times C_t} e^{ie^{i\pi/4}(x\cdot y+\sqrt{\hbar}n(\gamma)(\omega))} e^{\frac{1}{2\hbar}\int_0^t(\sqrt{\hbar}\omega(s)+x)\Omega^2(\sqrt{\hbar}\omega(s)+x)ds} e^{i\frac{\lambda}{\hbar}\int_0^t C(\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x)ds} W(d\omega)\frac{e^{-\frac{|x|^2}{2\hbar}}}{(2\pi\hbar)^{d/2}}dx\Big)\mu_f(dyd\gamma). \quad (5.40)$$

This is also equal to

$$(i)^{d/2}\int_{\mathbb{R}^d\times C_t} e^{i\frac{\lambda}{\hbar}\int_0^t C(\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x)ds} e^{\frac{1}{2\hbar}\int_0^t(\sqrt{\hbar}\omega(s)+x)\Omega^2(\sqrt{\hbar}\omega(s)+x)ds}\bar{\phi}(e^{i\pi/4}x)\psi_0(e^{i\pi/4}\sqrt{\hbar}\omega(t)+e^{i\pi/4}x)W(d\omega)dx. \quad (5.41)$$

Proof. By the assumptions on ϕ, one can easily verify that the function $(2\pi i\hbar)^{d/2}e^{-\frac{i}{2\hbar}|x|^2}\bar{\phi}(x)$ is the Fourier transform of the bounded variation measure on $\mathbb{R}^d\times\mathcal{H}$, which is the product measure $\mu_\phi(dx)\times\delta_0(d\gamma)$, where $\delta_0(d\gamma)$ is the measure on H_t concentrated on the vector $0\in H_t$. Analogously the function $(x,\gamma)\mapsto\psi_0(\gamma(t)+x)$ is the Fourier transform of the bounded variation measure μ_ψ on $\mathbb{R}^d\times\mathcal{H}$ given by :

$$\int_{\mathbb{R}^d\times H_t} f(x,\gamma)\mu_\psi(dxd\gamma) = \int_{\mathbb{R}^d\times H_t} f(x,x\gamma)\delta_{G_t}(d\gamma)\mu_0(dx),$$

where G_t is the vector in $\mathcal{H}$ given by $G_t(s)=s$. As $\mathcal{F}(\mathbb{R}^d\times\mathcal{H})$ is a Banach algebra, the product $f(x,\gamma) := (2\pi i\hbar)^{d/2}e^{-\frac{i}{2\hbar}|x|^2}\bar{\phi}(x)\psi(\gamma(t)+x)$ still belongs to $\mathcal{F}(\mathbb{R}^d\times\mathcal{H})$, in fact it is the Fourier transform of the convolution $\mu_f \equiv (\mu_\phi\times\delta_0)*\mu_\psi$. A direct computation shows that μ_f satisfies assumptions

(5.20) of theorem 22, that is (5.39), if and only if μ_0 and μ_ϕ satisfy (5.38).
By simple calculations one can verify that the operator B given by (5.33) is bounded symmetric and trace class. Moreover if assumptions (5.37) are satisfied, $I - B$ is positive definite (see appendix B for more details).
A direct computation shows that the function $V : \mathcal{H} \to \mathbb{R}$,

$$V(x,\gamma) = A((x,\gamma),(x,\gamma),(x,\gamma),(x,\gamma))$$

is continuous in the norm of the Banach space B and extends to a function $\bar{V}$ on it.
By applying theorem 22 and theorem 23 the conclusion follows. $\square$

Remark 19. *The class of states $\phi, \psi_0 \in L^2(\mathbb{R}^d) \cap \mathcal{F}(\mathbb{R}^d)$ satisfying assumption* (5.38) *is sufficiently rich. Indeed both ϕ and ψ_0 can be chosen in two dense subsets of the Hilbert space $L^2(\mathbb{R}^d)$. More precisely one can take for instance $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$ of the form $\psi_0(x) = P(x)e^{-\alpha\frac{x\Omega^{-1}x}{2\hbar}}$, and $\phi \in \mathcal{S}(\mathbb{R}^d)$ of the form $\phi(x) = Q(x)e^{-x(\frac{\beta\Omega}{2\hbar}+i\gamma)x}$, with $\alpha, \beta, \gamma > 0$ and with P, Q arbitrary polynomials. Moreover α and β have to satisfy the following conditions, for all $i = 1, \dots, d$:*

$$\begin{cases} \frac{1}{\beta\Omega_i} - \frac{1}{2(1-\Omega_i\tan(\Omega_i t)} > 0 \\ \frac{1}{\alpha} - \frac{1}{2}\left(\frac{\tan(\Omega_i t)+\Omega_i}{1-\Omega_i\tan(\Omega_i t)}\right) > 0) \\ \left(\frac{1-\Omega_i\tan(\Omega_i t)}{\alpha\Omega_i} - \frac{(\tan(\Omega_i t)+\Omega_i)}{2\Omega_i}\right)\left(\frac{1-\Omega_i\tan(\Omega_i t)}{\beta\omega_i} - \frac{1}{2}\right) > \left(\frac{1}{2\cos(\Omega_i t)}\right)^2 \end{cases} \tag{5.42}$$

Let us denote respectively by D_1 and D_2 the set of vectors ϕ and ψ_0 of the above form. It is easy to see that both D_1 and D_2 are dense in $L^2(\mathbb{R}^d)$.

The oscillatory integral (5.36) can heuristically be written in the following form:

$$(\phi, \psi(t)) = \text{“}\int_{\mathbb{R}^d} \bar{\phi}(x) \quad \int_{\{\gamma|\gamma(t)=x\}} e^{\frac{i}{\hbar}S_t(\gamma)}\psi_0(\gamma(0))D\gamma dx\text{”}$$

and interpreted as a rigorous realization of the Feynman path integral representing the inner product between the vector $\phi \in L^2(\mathbb{R}^d)$ and the solution of the Schrödinger equation (5.30) with initial datum ψ_0. However the infinite dimensional oscillatory integral (5.36) is well defined only if $\lambda \leq 0$. By the considerations in remark 18 the absolutely convergent integrals (5.40) and (5.41) are analytic functions of the complex variable λ if $Im(\lambda) > 0$, continuous in $Im(\lambda) = 0$ and coinciding with (5.36) if $\lambda \leq 0$. We shall prove that when $\lambda \geq 0$ the Gaussian integrals (5.40) and (5.41) represent the inner product $\langle \phi, \psi(t) \rangle$, where $\psi(t)$ is the solution of the Schrödinger equation. We will prove moreover the Borel summability of the formal Dyson expansion for $\langle \phi, \psi(t) \rangle$.

Lemma 8. *Let $\lambda = 0$, $\psi_0, \phi \in \mathcal{S}(\mathbb{R}^d)$. Let μ_0, resp μ_ϕ, be such that $\hat{\mu}_0 = \psi_0$, resp. $\hat{\mu}_\phi(x) = (2\pi i\hbar)^{d/2} e^{-\frac{i}{2\hbar}|x|^2} \bar{\phi}(x)$. Assume moreover that μ_0, μ_ϕ satisfy condition (5.38). Then the scalar product between ϕ and the solution ψ_t of the Schrödinger equation with initial datum ψ_0 is given by:*

$$\langle \phi, \psi_t \rangle = \int_{\mathbb{R}^d \times H_t} \Big(\int_{\mathbb{R}^d \times C_t} e^{ie^{i\pi/4}\sqrt{\hbar}(x\cdot y + n(\gamma)(\omega))} e^{\frac{1}{2}\langle (x,\omega), B(x,\omega)\rangle} W(d\omega) \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{d/2}} dx \Big) \mu_f(dy d\gamma) \quad (5.43)$$

where μ_f is the complex bounded variation measure on $\mathbb{R}^d \times H_t$ whose Fourier transform is the function $f : \mathcal{H} \to \mathbb{C}$, given by $f(x,\gamma) := (2\pi i\hbar)^{d/2} e^{-\frac{i}{2\hbar}|x|^2} \bar{\phi}(x) \psi(\gamma(t) + x)$ and B is the continuous extension on $\mathbb{R}^d \times C_t$ of the operator (5.33).

Proof. In order to avoid the use of a complicated notation we assume $d = 1$. The proof holds in a completely similar way in the case $d > 1$. As $\psi_0 \in S(\mathbb{R}^d)$, the solution of the Schrödinger equation with $\lambda = 0$, i.e. with the free Hamiltonian, and initial datum ψ_0 is given by

$$\psi(t,x) = (2\pi i\hbar)^{-1/2} \sqrt{\frac{\Omega}{\sin \Omega t}} \int_{\mathbb{R}} e^{\frac{i\Omega}{2\hbar \sin \Omega t}(\cos \Omega t (x^2+y^2) - 2xy)} \psi_0(y) dy, \quad (5.44)$$

$t > 0, x \in \mathbb{R}$, so that

$$\langle \phi, \psi_t \rangle = (2\pi i\hbar)^{-1/2} \sqrt{\frac{\Omega}{\sin \Omega t}} \int_{\mathbb{R}} \bar{\phi}(x) \int_{\mathbb{R}} e^{\frac{i\Omega}{2\hbar \sin \Omega t}(\cos \Omega t (x^2+y^2) - 2xy)} \psi_0(y) dy dx \quad (5.45)$$

Let $(2\pi i\hbar)^{1/2} e^{\frac{-i|x|^2}{2\hbar}} \bar{\phi}(x) = \int_{\mathbb{R}} e^{ik\cdot x} \mu_\phi(dk)$ and $\psi_0(y) = \int_{\mathbb{R}} e^{il\cdot y} \mu_0(dl)$, so that (5.45) becomes:

$$\frac{1}{\sqrt{\cos \Omega t}} \int_{\mathbb{R}} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ie^{i\pi/4}\sqrt{\hbar} xk} e^{-\frac{i\hbar \tan \Omega t l^2}{2\Omega}} e^{\frac{\Omega \tan \Omega t x^2}{2}} e^{\frac{i\sqrt{\hbar}e^{i\pi/4} xl}{\cos \Omega t}} \mu_\phi(dk) \mu_0(dl) dx.$$

A direct computation (see appendix B) shows that the latter expression is exactly equal to the integral (5.43), that is to

$$\int_{\mathbb{R} \times H_t} \Big(\int_{\mathbb{R}^d \times C_t} e^{ie^{i\pi/4}\sqrt{\hbar}(x\cdot k + x\cdot l + n(G_t^l)(\omega))} e^{\frac{1}{2}\langle (x,\omega), B(x,\omega)\rangle} W(d\omega) \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{d/2}} dx \Big) \mu_f(dy d\gamma) \quad (5.46)$$

(where $G_t^l(s) = ls$ and n has been defined in section 5.2) and the conclusion follows. □

Remark 20. *By Fubini's theorem expression* (5.43) *is also equal to*

$$(i)^{d/2}\int_{\mathbb{R}^d\times C_t} e^{\frac{1}{2\hbar}\int_0^t(\sqrt{\hbar}\omega(s)+x)\Omega^2(\sqrt{\hbar}\omega(s)+x)ds}$$

$$\bar{\phi}(e^{i\pi/4}x)\psi_0(e^{i\pi/4}\sqrt{\hbar}\omega(t)+e^{i\pi/4}x)W(d\omega)dx \quad (5.47)$$

Lemma 9. *Let* $\lambda = 0$ *and* $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$*, such that for each* $x \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} e^{kx}e^{\frac{\hbar}{4}\langle k,\Omega^{-1}\tan\Omega t k\rangle}|\tilde{\psi}_0(k)|dk < \infty. \quad (5.48)$$

Then the solution ψ_t *of the Schrödinger equation* (5.30) *is an analytic function in the variable* $z \in \mathbb{C}^d$ *and its value in* $z = e^{i\pi/4}x$*,* $x \in \mathbb{R}^d$ *is given by:*

$$\psi_t(e^{i\pi/4}x) = \int_{C_t}\psi_0(e^{i\pi/4}x + e^{i\pi/4}\sqrt{\hbar}\omega(t))e^{\frac{1}{2\hbar}\langle(x,\sqrt{\hbar}\omega),B(x,\sqrt{\hbar}\omega)\rangle}W(d\omega)$$

Proof. In order to avoid the use of a complicated notation we assume $d = 1$. The proof holds in a completely similar way in the case $d > 1$.
Since $\psi_0 \in \mathcal{S}(\mathbb{R})$, $\lambda = 0$, one has (5.44). By Parseval's equality this is also equal to

$$\psi_t(x) = \sqrt{\frac{1}{\cos\Omega t}}e^{-\frac{i\Omega\tan(\Omega t)x^2}{2\hbar}}\int_{\mathbb{R}} e^{-\frac{i\hbar\tan(\Omega t)k^2}{2\Omega}}e^{\frac{ikx}{\cos\Omega t}}\tilde{\psi}_0(k)dk$$

The analyticity of $\psi_t(z)$, $z \in \mathbb{C}$, follows by Morera and Fubini theorems. Moreover $\psi_t(e^{i\pi/4}x)$ is given by

$$\psi_t(e^{i\pi/4}x) = \sqrt{\frac{1}{\cos\Omega t}}e^{\frac{\Omega\tan(\Omega t)x^2}{2\hbar}}\int_{\mathbb{R}} e^{-\frac{i\hbar\tan(\Omega t)k^2}{2\Omega}}e^{\frac{ie^{i\pi/4}kx}{\cos\Omega t}}\tilde{\psi}_0(k)dk \quad (5.49)$$

On the other hand, by Fubini's theorem (which holds thanks to the assumption (5.48)), one has:

$$\int_{C_t}\psi_0(e^{i\pi/4}x + e^{i\pi/4}\sqrt{\hbar}\omega(t))e^{\frac{\Omega^2}{2\hbar}\int_0^t(\sqrt{\hbar}\omega(s)+x)^2ds}W(d\omega) =$$

$$= \int_{\mathbb{R}}\tilde{\psi}_0(k)e^{ikxe^{i\pi/4}}e^{\frac{\Omega^2tx^2}{2\hbar}}\int_{C_t} e^{\frac{\Omega^2}{2}\int_0^t\omega^2(s)ds}e^{\frac{\Omega^2x}{\sqrt{\hbar}}\int_0^t\omega(s)ds}e^{ik\sqrt{\hbar}e^{i\pi/4}\omega(t)}W(d\omega)dk \quad (5.50)$$

By a direct computation (see appendix B) the latter expression is equal to (5.49) and the conclusion follows. □

Theorem 25. *Let $\phi, \psi_0 \in \mathcal{S}(\mathbb{R}^d)$ satisfy assumption* (5.38)*. Then the power series expansions (in powers of λ) of the expression* (5.41) *coincides with the Dyson expansion for the scalar product between ϕ and the solution of the Schrödinger equation* (5.30)*.*

Proof. In order to avoid a complicated notation we assume $d = 1$, but the proof is valid also in the case $d > 1$.
First of all one can easily verify that expression (5.41) is an analytic function of the variable $\lambda \in \mathbb{C}$ in the upper halfplane $Im(\lambda) > 0$ and continuous in $\lambda \in \mathbb{R}$. By expanding it in power series of λ around $\lambda = 0$ we have for any $N \in \mathbb{N}$, that (5.41) is equal to:

$$(i)^{d/2} \sum_{n=0}^{N-1} \frac{1}{n!}\Big(\frac{i\lambda}{\hbar}\Big)^n \int_0^t ds_1 \cdots \int_0^t ds_n \int_{\mathbb{R}\times C_t} \prod_{i=1}^n (\sqrt{\hbar}\omega(s_i) + x)^4$$
$$e^{\frac{\Omega^2}{2\hbar}\int_0^t (\sqrt{\hbar}\omega(s)+x)^2 ds} \bar{\phi}(e^{i\pi/4}x)\psi_0(e^{i\pi/4}\sqrt{\hbar}\omega(t) + e^{i\pi/4}x) W(d\omega)dx + R_N, \quad (5.51)$$

with R_N a remainder term. Because of the symmetry of the integrand, (5.51) is equal to

$$(i)^{d/2} \sum_{n=0}^{N-1} \Big(\frac{i\lambda}{\hbar}\Big)^n \int \cdots \int_{\Delta_n} ds_1 \cdots ds_n \int_{\mathbb{R}\times C_t} \prod_{i=1}^n (\sqrt{\hbar}\omega(s_i) + x)^4$$
$$e^{\frac{\Omega^2}{2\hbar}\int_0^t (\sqrt{\hbar}\omega(s)+x)^2 ds} \bar{\phi}(e^{i\pi/4}x)\psi_0(e^{i\pi/4}\sqrt{\hbar}\omega(t) + e^{i\pi/4}x) W(d\omega)dx + R_N \quad (5.52)$$

where $\Delta_n = \{(s_1, \dots, s_n) \in [0,t]^n : 0 \le s_1 \le \cdots \le s_n \le t\}$. The integral over $\mathbb{R} \times C_t$ can be evaluated by partitioning the interval $[0,t]$ into $n+1$ subintervals $[s_0 \equiv 0, s_1]$, $[s_1, s_2]$, ..., $[s_{n-1}, s_n]$, $[s_n, s_{n+1} \equiv t]$. Let us denote by $\omega_i : [s_i, s_{i+1}] \to \mathbb{R}$ the Wiener process on the interval $[s_i, s_{i+1}]$, $\omega_i(s_i) = 0$, by C_i the space of continuous paths on $[s_i, s_{i+1}]$ and by $\mathbb{E}_{[s_i,s_{i+1}]}$ the expectation with respect to the Wiener measure on it. With these notations expression (5.52) becomes

$$(i)^{d/2} \sum_{n=0}^{N-1} \Big(\frac{-i\lambda}{\hbar}\Big)^n \int \cdots \int_{\Delta_n} ds_1 \cdots ds_n \int_{\mathbb{R}} dx \bar{\phi}(e^{i\pi/4}x) \mathbb{E}_{[0,s_1]}[(\sqrt{\hbar}e^{i\pi/4}\omega_0(s_1)+$$
$$+ xe^{i\pi/4})^4 e^{\frac{\Omega^2}{2\hbar}\int_0^{s_1}(\sqrt{\hbar}\omega_0(s)+x)^2 ds} \mathbb{E}_{[s_1,s_2]}[(\sqrt{\hbar}e^{i\pi/4}\omega_1(s_2) + \sqrt{\hbar}e^{i\pi/4}\omega_0(s_1) + xe^{i\pi/4})^4$$
$$e^{\frac{\Omega^2}{2\hbar}\int_{s_1}^{s_2}(\sqrt{\hbar}\omega_1(s)++\sqrt{\hbar}\omega_0(s_1)+x)^2 ds} \cdots \mathbb{E}_{[s_n,t]}[e^{\frac{\Omega^2}{2\hbar}\int_{s_n}^t (\sqrt{\hbar}\omega_n(s)+\sqrt{\hbar}\sum_{i=0}^{n-1}\omega_i(s_{i+1})+x)^2 ds}$$
$$\psi_0(e^{i\pi/4}\sqrt{\hbar}\sum_{i=0}^n \omega_i(s_{i+1}) + e^{i\pi/4}x)] \dots]] + R_N$$

By lemma 8 and lemma 9 the latter expression is equal to

$$\sum_{n=0}^{N-1}\left(\frac{-i\lambda}{\hbar}\right)^n\int\cdots\int_{\Delta_n}ds_1\cdots ds_n\langle\phi, e^{-i\frac{s_1}{\hbar}H_0}Ve^{-i\frac{(s_2-s_1)}{\hbar}H_0}V\cdots$$
$$\cdots e^{-i\frac{(s_n-s_{n-1})}{\hbar}H_0}Ve^{-i\frac{(t-s_n)}{\hbar}H_0}\psi_0\rangle + R_N$$

and, by the change of variables $s_i \to t - s_{n+1-i}$, to

$$\sum_{n=0}^{N-1}\left(\frac{-i\lambda}{\hbar}\right)^n\int\cdots\int_{\Delta_n}ds_1\cdots ds_n\langle\phi, e^{-i\frac{(t-s_n)}{\hbar}H_0}Ve^{-i\frac{(s_n-s_{n-1})}{\hbar}H_0}V\cdots$$
$$\cdots e^{-i\frac{(s_2-s_1)}{\hbar}H_0}Ve^{-i\frac{s_1}{\hbar}H_0}\psi_0\rangle + R_N$$

where $H_0 \equiv -\frac{\hbar^2}{2}\Delta + \frac{x\Omega^2 x}{2}$ is the harmonic oscillator Hamiltonian and $V(x) \equiv x^4$. The latter expression is Dyson's expansion for the scalar product between ϕ and the solution ψ_t of the Schrödinger equation (5.30) with Hamiltonian $H = H_0 + \lambda V$ and the conclusion follows. □

Theorem 26. *Let $\lambda \geq 0$, and let $\phi, \psi_0 \in \mathcal{S}(\mathbb{R}^d)$ satisfy assumption* (5.38). *Then the scalar product between ϕ and the solution of the Schrödinger equation* (5.30) *with initial datum ψ_0 is given by the absolutely convergent integrals* (5.40) *and* (5.41).

Proof. Let us consider the anharmonic oscillator Hamiltonian H given by (5.31). H is a positive selfadjoint operator and generates an analytic semigroup $T^z(t) = e^{-\frac{ztH}{\hbar}}$, $t \geq 0$, $z \in \mathbb{C}$, $Re(z) \geq 0$ (see for instance [91]). Given $t \geq 0$ and $\phi, \psi_0 \in L^2(\mathbb{R}^d)$, the function $F : \bar{D} \to \mathbb{C}$, where $D = \{z \in \mathbb{C}, Re(z) > 0\}$ and $\bar{D}$ is the closure of D,

$$F(z) \equiv \langle\phi, T^z(t)\psi_0\rangle \tag{5.53}$$

is analytic in D and continuous in $\bar{D}$. If $z = i$, $F(z)$ is the scalar product between ϕ and the solution $\psi(t)$ of the Schrödinger equation (5.30) with initial datum ψ_0, while if $z \in \mathbb{R}^+$, $F(z)$ is the scalar product between ϕ and the solution of the heat equation

$$\frac{\partial}{\partial t}\psi = -\frac{z}{\hbar}H\psi \tag{5.54}$$

In this case $F(z)$ can be computed by means of the Feynman-Kac formula (see for instance [92]):

$$F(z) = \int_{\mathbb{R}^d} \bar{\phi}(x) \int_{C_t} e^{-\frac{z}{2\hbar}\int_0^t (\sqrt{\hbar z}\omega(s)+x)\Omega^2(\sqrt{\hbar z}\omega(s)+x)ds}$$

$$e^{-\frac{z\lambda}{\hbar}\int_0^t C(\sqrt{\hbar z}\omega(s)+x,\sqrt{\hbar z}\omega(s)+x,\sqrt{\hbar z}\omega(s)+x,\sqrt{\hbar z}\omega(s)+x)ds}\psi_0(\sqrt{\hbar z}\omega(t)+x)W(d\omega)dx$$

$$= z^{d/2}\int_{\mathbb{R}^d} \bar{\phi}(\sqrt{z}x) \int_{C_t} e^{-\frac{z^2}{2\hbar}\int_0^t (\sqrt{\hbar}\omega(s)+x)\Omega^2(\sqrt{\hbar}\omega(s)+x)ds}$$

$$e^{-\frac{z^3\lambda}{\hbar}\int_0^t C(\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x)ds}\psi_0(\sqrt{\hbar z}\omega(t)+\sqrt{z}x)W(d\omega)dx \tag{5.55}$$

By the assumptions on the vectors ϕ, ψ_0, the r.h.s. of (5.55) makes sense for $z \in \bar{D}$. Moreover, by the analyticity of the semigroup $T^z(t)$, it represents for $z = i$ the scalar product $\langle \phi, e^{-\frac{it}{\hbar}H}\psi_0\rangle$, that is:

$$i^{d/2}\int_{\mathbb{R}^d} \bar{\phi}(e^{i\pi/4}x) \int_{C_t} e^{\frac{1}{2\hbar}\int_0^t (\sqrt{\hbar}\omega(s)+x)\Omega^2(\sqrt{\hbar}\omega(s)+x)ds}$$

$$e^{\frac{i\lambda}{\hbar}\int_0^t C(\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x)ds}\psi_0(\sqrt{\hbar}e^{i\pi/4}\omega(t)+e^{i\pi/4}x)W(d\omega)dx \tag{5.56}$$

This coincides with expression (5.41) and the conclusion follows. □

Theorem 27. *Let $\lambda \geq 0$, and let $\phi, \psi_0 \in \mathcal{S}(\mathbb{R}^d)$ satisfy assumption*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{(y+\cos[\Omega(t+t\delta)]^{-1}x)(1-(1+\delta)\Omega\tan[\Omega(t+t\delta)])^{-1}(y+\cos[\Omega(t+t\delta)]^{-1}x)}$$

$$e^{\frac{\hbar}{4(1+\delta)}x\Omega^{-1}\tan[\Omega(t+t\delta)]x}|\mu_0|(dx)|\mu_\phi|(dy) < \infty \tag{5.57}$$

for some $\delta > 0$. Then the Dyson expansion for the scalar product between ϕ and the solution of the Schrödinger equation (5.30) *with initial datum ψ_0 is Borel summable.*

Proof. By theorems 25 and 26 it is sufficient to show the Borel summability of the power series expansions (in powers of λ) of expression (5.41).
In order to avoid a complicated notation we assume $d = 1$, but the proof is valid also in the case $d \geq 1$.
As already remarked before lemma 3, the expression (5.41) is an analytic function of the variable $\lambda \in \mathbb{C}$ in the upper halfplane $Im(\lambda) > 0$ and continuous in $\lambda \in \mathbb{R}$. Moreover the rest R_N of its asymptotic expansion (5.51) is equal to:

$$R_N = \int_{\mathbb{R}^d\times H_t}\Big(\int_{\mathbb{R}\times C_t}\sum_{n=N}^{\infty}\frac{1}{n!}\Big(\frac{i\lambda}{\hbar}\Big)^n\int_0^t ds_1\cdots\int_0^t ds_n\prod_{i=1}^n(\sqrt{\hbar}\omega(s_i)+x)^4$$

$$e^{\frac{\Omega^2}{2\hbar}\int_0^t(\sqrt{\hbar}\omega(s)+x)^2ds}e^{ie^{i\pi/4}(x\cdot y+\sqrt{\hbar}n(\gamma)(\omega))}W(d\omega)\frac{e^{-\frac{|x|^2}{2\hbar}}}{(2\pi\hbar)^{d/2}}dx\Big)\mu_f(dyd\gamma)$$

R_N satisfies the following uniform estimate in $Im(\lambda) \geq 0$:

$$|R_N| = |\int_{\mathbb{R}^d \times H_t} \Big(\int_{\mathbb{R}\times C_t} \frac{1}{N-1!} \Big(\frac{i\lambda}{\hbar}\Big)^N \int_0^t ds_1 \cdots \int_0^t ds_N \prod_{i=1}^N (\sqrt{\hbar}\omega(s_i)+x)^4$$

$$\int_0^1 du(1-u)^{N-1} e^{i\frac{u\lambda}{\hbar}\int_0^t(\sqrt{\hbar}\omega(s)+x)^4 ds} e^{\frac{\Omega^2}{2\hbar}\int_0^t(\sqrt{\hbar}\omega(s)+x)^2 ds}$$

$$e^{ie^{i\pi/4}(x\cdot y+\sqrt{\hbar}n(\gamma)(\omega))} W(d\omega) \frac{e^{-\frac{|x|^2}{2\hbar}}}{(2\pi\hbar)^{d/2}} dx \Big) \mu_f(dyd\gamma)|$$

$$\leq |\lambda|^N \hbar^N \frac{1}{N!} \int_0^t ds_1 \cdots \int_0^t ds_N \int_{\mathbb{R}^d \times H_t} \int_{\mathbb{R}\times C_t} e^{\frac{\Omega^2}{2}\int_0^t(\omega(s)+x)^2 ds}$$

$$\prod_{i=1}^N (\omega(s_i)+x)^4 e^{-\frac{\sqrt{2}}{2}\sqrt{\hbar}(x\cdot y+n(\gamma)(\omega))} W(d\omega) \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{d/2}} dx |\mu_f|(dyd\gamma) \quad (5.58)$$

By denoting G_i the vector in $C_t^* \subset H_t$ equal to $G_i(s) = 1_{[0,s_i]}s$, $(|G_i|_{H_t} = s_i)$, the Gaussian integral

$$\int_{\mathbb{R}\times C_t} e^{\frac{\Omega^2}{2}\int_0^t(\omega(s)+x)^2 ds} \prod_{i=1}^N (\omega(s_i)+x)^4 e^{-\frac{\sqrt{2}}{2}\sqrt{\hbar}(x\cdot y+n(\gamma)(\omega))} W(d\omega) \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{d/2}} dx =$$

$$= \int_{\mathbb{R}\times C_t} e^{\frac{1}{2}\langle (x,\omega), B(x,\omega)\rangle} \prod_{i=1}^N (n(G_i)(\omega)+x)^4 e^{-\frac{\sqrt{2}}{2}\sqrt{\hbar}(x\cdot y+n(\gamma)(\omega))} W(d\omega) \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{d/2}} dx$$

is equal to

$$H_{4N}\Big(i\frac{\sqrt{\hbar}}{2}(I-B)^{-1/2}(G_1,1), \ldots, i\frac{\sqrt{\hbar}}{2}(I-B)^{-1/2}(G_N,1) \Big| (\sqrt{\frac{\hbar}{2}}\gamma, \sqrt{\frac{\hbar}{2}}y)\Big) \frac{e^{\frac{\hbar}{4}\langle(\gamma,y),(I-B)^{-1}(\gamma,y)\rangle}}{\sqrt{\det(I-B)}} \quad (5.59)$$

where

$$D_{x_1} \ldots D_{x_n} e^{-x^2} = (-1)^n H_n(x_1, \ldots, x_n|x) e^{-x^2}$$

By the assumption (5.57) on ψ_0, ϕ involving a $\delta > 0$, we have

$$\int_{H_t \times R} e^{(1+\delta)\frac{\hbar}{4}\langle(\gamma,y),(I-B)^{-1}(\gamma,y)\rangle} |\mu_f|(d\gamma dy) < \infty$$

By using this and the estimate on Hermite polynomials H_n derived in [88] (formula (2.9)) we see that expression (5.59) is bounded by

$$ac^N \prod_{i=1}^N (s_i+1)^4 \Big(\frac{1+\delta}{\delta}\Big)^{2N} 2N!,$$

where $a, c > 0$ are suitable constants. By inserting such an estimate into (5.58) and by using the identity $2N! = 2^{2N}N!(N-1/2)!/\sqrt{\pi}$, we have:

$$|R_N| \leq AC^N|\lambda|^N N!$$

This and the analyticity of (5.41) in $Im(\lambda) > 0$, by Nevanlinna theorem [84], assure the Borel summability of asymptotic expansion (5.51). □

5.4 Concluding remarks

There are relations between our approach in the definition of the Feynman integral and those in [40, 72, 55, 83]. Indeed formula (5.41) often appears in the literature for a more restricted class of potentials and initial conditions. We would like however to underline that here we achieved to prove (5.41) and related formulas for potentials of polynomial growth. This involves our extension of the definition of infinite dimensional oscillatory integrals (in the spirit of [12, 57, 4]) to a class of phase functions much larger than the usual "quadratic + Fourier transform of measure". In [40, 72, 55, 83] the authors define the Feynman functional by means of a Gaussian integral depending on a parameter (which in some cases can be identified with the mass m), prove the analyticity of such a functional in a suitable region of the complex plane and show that when it approaches the imaginary axis the corresponding functional gives a representation of the solution of the Schrödinger equation for a restricted class of potentials. In work of the euclidean approach to quantum field theory, the representation of solution of the perturbed heat equation via a Feynman-Kac formula and integrals with respect to Gaussian (Wiener resp. Orstein-Uhlenbeck) measures are used to provide via an "analytic continuation in time" solutions of the Schrödinger equation. In [32] this approach provides a semiclassical expansion for the Schrödinger equation. In our case, under suitable assumptions on the initial datum ψ_0, we prove that the infinite dimensional oscillatory integral we define *coincides* with a Gaussian integral. In the case of the quartic potential $V = \lambda x^4$ we prove that the Gaussian integral representing the solution of the Schrödinger equation is an analytic function of the complex variable λ in the upper halfplane which coincides for $\lambda \leq 0$ with a well defined infinite dimensional oscillatory integral. We plan to use our representation for discussing rigorously asymptotic expansions in fractional powers of $\hbar$ (semiclassical expansions).

Appendix A

The Fourier transform of $e^{\frac{i}{\hbar}P(x)}$

A.1 Proof of lemma 5

Let us denote D the region of the complex plane:

$$D \subset \mathbb{C}, \qquad D \equiv \{z \in \mathbb{C} \mid Im(z) < 0\}$$

Let us assume $\hbar$ is a complex variable belonging to the region $\bar{D} \setminus \{0\}$. We are going to compute the Fourier transform of $e^{\frac{i}{\hbar}P(x)}$.
Let us introduce the polar coordinates in $\mathbb{R}^N$:

$$\int_{\mathbb{R}^N} e^{ik\cdot x} e^{\frac{i}{\hbar}P(x)} dx = \\ = \int_{S_{N-1}} \Big(\int_0^{+\infty} e^{i|k|rf(\phi_1,\ldots,\phi_{N-1})} e^{\frac{i}{\hbar}\mathcal{P}_{(\phi_1,\ldots,\phi_{N-1})}(r)} r^{N-1} dr \Big) d\Omega_{N-1} \quad \text{(A.1)}$$

where instead of N Cartesian coordinates we use $N-1$ angular coordinates $(\phi_1, \ldots, \phi_{N-1})$ and the variable $r = |x|$. S_{N-1} denotes the $(N-1)$-dimensional spherical surface, $d\Omega_{N-1}$ is the measure on it, $\mathcal{P}_{(\phi_1,\ldots,\phi_{N-1})}(r)$ is a $2M_{th}$ order polynomial in the variable r with coefficients depending on the $N-1$ angular variables $(\phi_1, \ldots, \phi_{N-1})$, namely:

$$P(x) = r^{2M} A_{2M}\Big(\frac{x}{|x|}, \ldots, \frac{x}{|x|}\Big) + r^{2M-1} A_{2M-1}\Big(\frac{x}{|x|}, \ldots, \frac{x}{|x|}\Big) + \ldots + \\ + \ldots + rA_1\Big(\frac{x}{|x|}\Big) + A_0 = \\ = a_{2M}(\phi_1, \ldots, \phi_{N-1})r^{2M} + a_{2M-1}(\phi_1, \ldots, \phi_{N-1})r^{2M-1} + \\ + \ldots + a_1(\phi_1, \ldots, \phi_{N-1})r + a_0 \\ = \mathcal{P}_{(\phi_1,\ldots,\phi_{N-1})}(r) \quad \text{(A.2)}$$

where $a_{2M}(\phi_1,\dots,\phi_{N-1})>0$ for all $(\phi_1,\dots,\phi_{N-1})\in S_{N-1}$.
Let us focus on the integral

$$\int_0^{+\infty} e^{i|k|rf(\phi_1,\dots,\phi_{N-1})}e^{\frac{i}{\hbar}\mathcal{P}_{(\phi_1,\dots,\phi_{N-1})}(r)}r^{N-1}dr, \tag{A.3}$$

which can be interpreted as the Fourier transform of the distribution on the real line

$$F(r)=\Theta(r)r^{N-1}e^{\frac{i}{\hbar}\mathcal{P}_{(\phi_1,\dots,\phi_{N-1})}(r)},$$

with $\Theta(r)=1$ for $r\geq 0$ and $\Theta(r)=0$ for $r<0$. Let us introduce the notation $k'\equiv kf(\phi_1,\dots,\phi_{N-1})$, $a_k\equiv a_k(\phi_1,\dots,\phi_{N-1})$, $k=0,\dots,2M$, $P'(r)=\sum_{k=0}^{2M}a_kr^k$ and $\hbar\in\mathbb{C}$, $\hbar=|\hbar|e^{i\phi}$, with $-\pi\leq\phi\leq 0$.
Let us consider the complex plane and set $z=\rho e^{i\theta}$. If $Im(\hbar)<0$ the integral (A.3) is absolutely convergent, while if $\hbar\in\mathbb{R}\setminus\{0\}$ it needs a regularization. If $\hbar\in\mathbb{R}$, $\hbar>0$ we have

$$\int_0^{+\infty} e^{ik'r}e^{\frac{i}{\hbar}P'(r)}r^{N-1}dr=\lim_{\epsilon\downarrow 0}\int_{z=\rho e^{i\epsilon}} e^{ik'z}e^{\frac{i}{\hbar}P'(z)}z^{N-1}dz \tag{A.4}$$

while if $\hbar<0$

$$\int_0^{+\infty} e^{ik'r}e^{\frac{i}{\hbar}P'(r)}r^{N-1}dr=\lim_{\epsilon\downarrow 0}\int_{z=\rho e^{-i\epsilon}} e^{ik'z}e^{\frac{i}{\hbar}P'(z)}z^{N-1}dz \tag{A.5}$$

We deal first of all with the case $\hbar\in\mathbb{R}$, $\hbar>0$ (the case $\hbar<0$ can be handled in a completely similar way). Let

$$\gamma_1(R)=\{z\in\mathbb{C}\quad|\quad 0\leq\rho\leq R,\quad\theta=\epsilon\}$$

$$\gamma_2(R)=\{z\in\mathbb{C}\quad|\quad \rho=R,\quad\epsilon\leq\theta\leq\pi/4M\}$$

$$\gamma_3(R)=\{z\in\mathbb{C}\quad|\quad 0\leq\rho\leq R,\quad\theta=\pi/4M\}$$

From the analyticity of the integrand and the Cauchy theorem we have

$$\int_{\gamma_1(R)\cup\gamma_2(R)\cup\gamma_3(R)} e^{ik'z}e^{\frac{i}{\hbar}P'(z)}z^{N-1}dz=0.$$

In particular:

$$\begin{aligned}\Big|\int_{\gamma_2(R)} e^{ik'z}e^{\frac{i}{\hbar}P'(z)}z^{N-1}dz\Big| &= R^N\Big|\int_\epsilon^{\pi/4M} e^{ik'Re^{i\theta}}e^{\frac{i}{\hbar}P'(Re^{i\theta})}e^{iN\theta}d\theta\Big| \\ &\leq R^N\int_\epsilon^{\pi/4M} e^{-k'R\sin(\theta)}e^{-\frac{1}{\hbar}\sum_{k=1}^{2M}a_kR^k\sin(k\theta)}d\theta \\ &\leq R^N\int_\epsilon^{\pi/4M} e^{-k''R\theta}e^{-a_{2M}\frac{4M}{\hbar\pi}R^{2M}\theta}e^{-\sum_{k=1}^{2M-1}a'_kR^k\theta}d\theta \end{aligned}\tag{A.6}$$

where k'', a'_k $k = 1, \ldots, 2M-1$ are suitable constants. We have used the fact that if $\alpha \in [0, \pi/2]$ then $\frac{2}{\pi}\alpha \leq \sin(\alpha) \leq \alpha$. The latter integral can be explicitly computed and gives:

$$R^N \Big(\frac{e^{-\epsilon(a_{2M}\frac{4M}{\hbar\pi}R^{2M} + k''R + \sum_{k=1}^{2M-1} a'_k R^k)} - e^{-\frac{\pi}{4M}(a_{2M}\frac{4M}{\hbar\pi}R^{2M} + k''R + \sum_{k=1}^{2M-1} a'_k R^k)}}{a_{2M}\frac{4M}{\hbar\pi}R^{2M} + k''R + \sum_{k=1}^{2M-1} a'_k R^k} \Big),$$

which converges to 0 as $R \to \infty$. We get

$$\int_{z=\rho e^{i\epsilon}} e^{ik'z} e^{\frac{i}{\hbar}P'(z)} z^{N-1} dz = \int_{z=\rho e^{i(\pi/4M)}} e^{ik'z} e^{\frac{i}{\hbar}P'(z)} z^{N-1} dz$$

By taking the limit as $\epsilon \downarrow 0$ of both sides one gets:

$$\int_0^{+\infty} e^{ik'r} e^{\frac{i}{\hbar}P'(r)} r^{N-1} dr = e^{iN\pi/4M} \int_0^{+\infty} e^{ik\rho e^{i\pi/4M}} e^{\frac{i}{\hbar}P'(re^{i\pi/4M})} \rho^{N-1} d\rho$$

By substituting into (A.12) we get the final result:

$$\tilde{F}(k) = \int_{\mathbb{R}^N} e^{ik\cdot x} e^{\frac{i}{\hbar}P(x)} dx = e^{iN\pi/4M} \int_{\mathbb{R}^N} e^{ie^{i\pi/4M}k\cdot x} e^{\frac{i}{\hbar}P(e^{i\pi/4M}x)} dx. \quad \text{(A.7)}$$

In the case $\hbar < 0$ an analogous reasoning gives:

$$\tilde{F}(k) = \int_{\mathbb{R}^N} e^{ik\cdot x} e^{\frac{i}{\hbar}P(x)} dx = e^{-iN\pi/4M} \int_{\mathbb{R}^N} e^{ie^{-i\pi/4M}k\cdot x} e^{\frac{i}{\hbar}P(e^{-i\pi/4M}x)} dx. \quad \text{(A.8)}$$

The analyticity of $\tilde{F}(k)$ is trivial in the case $Im(\hbar) < 0$, and follows from equations (A.7) and (A.8) when $\hbar \in \mathbb{R} \setminus \{0\}$
If $Im(\hbar) < 0$ a representation of type (A.7) still holds. By setting $\hbar = |\hbar|e^{i\phi}$, with $-\pi \leq \phi \leq 0$ and by deforming the integration contour in the complex z plane, one gets

$$\tilde{F}(k) = \int_{\mathbb{R}^N} e^{ik\cdot x} e^{\frac{i}{\hbar}P(x)} dx =$$
$$= e^{iN(\pi/4M+\phi/2M)} \int_{\mathbb{R}^N} e^{ie^{i(\pi/4M+\phi/2M)}k\cdot x} e^{\frac{i}{\hbar}P(e^{i(\pi/4M+\phi/2M)}x)} dx \quad \text{(A.9)}$$

A.2 The boundedness of $\tilde{F}(k)$ as $|k| \to \infty$.

Let us consider the distribution $e^{\frac{i}{\hbar}P(x)}$ and its Fourier transform $\tilde{F}(k) = \int_{\mathbb{R}^N} e^{ikx} e^{\frac{i}{\hbar}P(x)} dx$. Let us focus on the case $\hbar \in \mathbb{R} \setminus \{0\}$ (in the case $Im(\hbar) < 0$ $|\tilde{F}|$ is trivially bounded by $\int_{\mathbb{R}^N} |e^{\frac{i}{\hbar}P(x)}| dx = \int_{\mathbb{R}^N} e^{\frac{Im(\hbar)}{|\hbar|^2}P(x)} dx < +\infty$). Let us assume for notation simplicity that $\hbar = 1$, the general case can be handled in a completely similar way. In order to study $\int_{\mathbb{R}^N} e^{ikx}e^{iP(x)}dx$ one has to introduce a suitable regularization. Chosen $\psi \in \mathcal{S}(\mathbb{R}^N)$, such that $\psi(0) = 1$ we have

$$e^{iP(x)}\psi(\epsilon x) \to e^{iP(x)}, \qquad \text{in } \mathcal{S}'(\mathbb{R}^N) \text{ as } \epsilon \to 0,$$

$$\tilde{F}(k) = \lim_{\epsilon\to 0}\int_{\mathbb{R}^N} e^{ikx}e^{iP(x)}\psi(\epsilon x)dx.$$

Let us consider first of all the case $N = 1$ and $P(x) = x^{2M}/2m$. The unique real stationary point of the phase function $\Phi(x) = kx+x^{2M}$ is $c_k = -k^{\frac{1}{2M-1}}$ Let χ_1 be a positive C^∞ function such that $\chi_1(x) = 1$ if $|x - c_k| \leq 1/2$, $\chi_1(x) = 0$ if $|x - c_k| \geq 1$ and $0 \leq \chi_1(x) \leq 1$ if $1/2 \leq |x - c_k| \leq 1$. Let $\chi_0 \equiv 1 - \chi_1$. Then $\tilde{F}(k) = I_1(k) + I_0(k)$, where $I_0(k) = \lim_{\epsilon\to 0}\int e^{ikx}e^{ix^{2M}/2m}\chi_0(x)\psi(\epsilon x)dx$ and $I_1(k) = \int e^{ikx}e^{ix^{2M}/2m}\chi_1(x)dx$. For the study of the boundedness of $|\tilde{F}(k)|$ as $|k| \to \infty$ it is enough to look at I_0, since one has, by the choice of χ_1, that $|I_1| \leq 2$. By repeating the same reasoning used in the proof of theorem 16 I_0 can be computed by means of Stokes formula:

$$\lim_{\epsilon\to 0}\int e^{ikx}e^{ix^{2M}/2m}\chi_0(x)\psi(\epsilon x)dx = i\lim_{\epsilon\to 0}\epsilon\int e^{ikx}e^{i\frac{x^{2M}}{2m}}\frac{\chi_0(x)\psi'(\epsilon x)}{k+x^{2M-1}}dx+$$
$$+\, i\lim_{\epsilon\to 0}\int e^{ikx}e^{ix^{2M}/2m}\frac{d}{dx}\Big(\frac{\chi_0(x)}{k+x^{2M-1}}\Big)\psi(\epsilon x)dx \quad \text{(A.10)}$$

Both integrals are absolutely convergent and, by dominated convergence, we can take the limit $\epsilon \to 0$, so that

$$I_0(k) = i\int e^{ikx}e^{ix^{2M}/2m}\frac{d}{dx}\Big(\frac{\chi_0(x)}{k+x^{2M-1}}\Big)dx =$$
$$= i\int e^{ikx}e^{ix^{2M}/2m}\Big(\frac{\chi_0'(x)}{k+x^{2M-1}}\Big)dx - i\int e^{ikx}e^{ix^{2M}/2m}\Big(\frac{(2M-1)\chi_0(x)x^{2M-2}}{(k+x^{2M-1})^2}\Big)dx$$

Thus:

$$|I_0(k)| \leq 2\int_{c_k-1}^{c_k-1/2}\Big|\frac{1}{k+x^{2M-1}}\Big|dx + 2\int_{c_k+1/2}^{c_k+1}\Big|\frac{1}{k+x^{2M-1}}\Big|dx+$$
$$+\,(2M-1)\int_{-\infty}^{c_k-1/2}\Big|\frac{x^{2M-2}}{(k+x^{2M-1})^2}\Big|dx + (2M-1)\int_{c_k+1/2}^{+\infty}\Big|\frac{x^{2M-2}}{(k+x^{2M-1})^2}\Big|dx$$

By a change of variables it is possible to see that both integrals remain bounded as $|k| \to \infty$. Let us consider for instance the first one:

$$\int_{c_k-1}^{c_k-1/2}\Big|\frac{1}{k+x^{2M-1}}\Big|dx = \frac{k^{\frac{1}{2M-1}}}{|k|}\int_{-1-1/k^{\frac{1}{2M-1}}}^{-1-1/2k^{\frac{1}{2M-1}}}\Big|\frac{1}{1+y^{2M-1}}\Big|dy$$

The latter integral diverges logarithmically as $|k| \to \infty$, so that the r.h.s. goes to 0 as $|k| \to \infty$. Let us consider the integral $\int_{-\infty}^{c_k-1/2} \left|\frac{x^{2M-2}}{(k+x^{2M-1})^2}\right| dx$. By a change of variables it is equal to

$$\int_{-\infty}^{c_k-1/2} \left|\frac{x^{2M-2}}{(k+x^{2M-1})^2}\right| dx =$$

$$\frac{1}{|k|} \int_{-\infty}^{-1-1/2k^{\frac{1}{2M-1}}} \left|\frac{y^{2M-2}}{(1+y^{2M-1})^2}\right| dy$$

The latter integral diverges as $O(k)$ as $|k| \to \infty$, so that the r.h.s. remains bounded as $|k| \to \infty$. By such considerations we can deduce that $|\tilde{F}(k)|$ is bounded as $|k| \to \infty$.
A similar reasoning holds also in the case $N = 1$ and $P(x) = \sum_{i=1}^{2M} a_i x$ is a generic polynomial. Indeed for $|k|$ sufficiently large the derivative of the phase function $\Phi'(x) = k + P'(x)$ has only one simple real root, denoted by c_k. One can repeat the same reasoning valid for the case $P(x) = x^{2M}/2M$ and prove that for $|k| \to \infty$ one has $|\int e^{ikx+iP(x)} dx| \leq C$ (where C is a function of the coefficients a_i of P at most with polynomial growth).
The general case $\mathbb{R}^N$ can also be essentially reduced to the one-dimensional case. Indeed let use consider a generic vector $k \in \mathbb{R}^N$, $k = |k|u_1$, and study the behavior of $\tilde{F}(k)$ as $|k| \to \infty$. By choosing as orthonormal base $u_1, \ldots, u_N$ of $\mathbb{R}^N$, where $u_1 = k/|k|$, we have

$$\tilde{F}(k) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^{N-1}} e^{iQ(x_2,\ldots,x_N)} \psi(\epsilon x_2) \cdots \psi(\epsilon x_N) \\ \Big(\int_{\mathbb{R}} e^{i|k|x_1} e^{iP_{x_2,\ldots,x_N}(x_1)} \psi(\epsilon x_1) dx_1 \Big) dx_2 \ldots dx_N \quad \text{(A.11)}$$

where $\psi \in \mathcal{S}(\mathbb{R})$, $\psi(0) = 1$; $x_i = x \cdot u_i$, $P_{x_2,\ldots,x_N}(x_1)$ is the polynomial in the variable x_1 with coefficients depending on powers of the remaining $N-1$ variables $x_2, \ldots, x_N$, obtained by considering in the initial polynomial $P(x_1, x_2, \ldots, x_N)$ all the terms containing some power of x_1. The polynomial Q in the $N-1$ variables $x_2, \ldots, x_N$ is given by $P(x_1, x_2, \ldots, x_N) - P_{x_2,\ldots,x_N}(x_1)$. Let us set $I^\epsilon(k, x_2, \ldots, x_N) \equiv \int_{\mathbb{R}} e^{i|k|x_1} e^{iP_{x_2,\ldots,x_N}(x_1)} \psi(\epsilon x_1) dx_1$. By the previous considerations we know that, for each $\epsilon \geq 0$, $|I^\epsilon(k, x_2, \ldots, x_N)|$ is bounded by a function of $G(x_2, \ldots, x_N)$ of polynomial growth. By the same reasonings as in the proof of theorem 16 we can deduce that the oscillatory integral (A.11) is a well defined bounded function of k.

A.3 Proof of lemma 6

Let us study the Fourier transform of the complex-valued distribution

$$\frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(x)}, \qquad x\in\mathbb{R}^N,$$

where $(I-B)$ is symmetric and strictly positive, $\lambda \leq 0$ and P is given by (5.3):

$$\tilde{F}(k)=\int_{\mathbb{R}^N} e^{ik\cdot x}\frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(x)}dx$$

Without loss of generality we can assume that the quadratic form $x\cdot(I-B)x$ is equal to $x\cdot x$, as it can always be reduced to this form by a change of coordinates.
Let us compute the $N-$dimensional integral defining $\tilde{F}(k)$ by introducing the polar coordinates in $\mathbb{R}^N$:

$$\int_{\mathbb{R}^N} e^{ik\cdot x}\frac{e^{\frac{i}{2\hbar}x\cdot x}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(x)}dx=$$
$$=\int_{S_{N-1}}\Big(\int_0^{+\infty} e^{i|k|rf(\phi_1,\dots,\phi_{N-1})}\frac{e^{\frac{i}{2\hbar}r^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(r)}r^{N-1}dr\Big)d\Omega_{N-1} \quad (A.12)$$

where instead of N Cartesian coordinates we use $N-1$ angular coordinates $(\phi_1,\dots,\phi_{N-1})$ and the variable $r=|x|$. S_{N-1} denotes the $(N-1)$-dimensional spherical surface, $d\Omega_{N-1}$ is the Haar measure on it, $f(\phi_1,\dots,\phi_{N-1})=(k\cdot x)/|k|r$, $P(r)$ is a fourth order polynomial in the variable r with coefficients depending on the $N-1$ angular variables $(\phi_1,\dots,\phi_{N-1})$, namely:

$$P(r)=r^4A\Big(\frac{x}{|x|},\frac{x}{|x|},\frac{x}{|x|},\frac{x}{|x|}\Big)=r^4a(\phi_1,\dots,\phi_{N-1}) \quad (A.13)$$

where $a(\phi_1,\dots,\phi_{N-1})>0$ for all $(\phi_1,\dots,\phi_{N-1})\in S_{N-1}$. Let us focus on the integral

$$\int_0^{+\infty} e^{i|k|rf(\phi_1,\dots,\phi_{N-1})}\frac{e^{\frac{i}{2\hbar}r^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(r)}r^{N-1}dr.$$

This can be interpreted as the Fourier transform of the distribution on the real line

$$F(r)=\theta(r)r^{N-1}\frac{e^{\frac{i}{2\hbar}r^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(r)},$$

with $\theta(r)=1$ if $r\geq 0$ $\theta(r)=0$ otherwise, $\lambda<0$ and $P(r)=ar^4$, $a>0$:

$$\int_0^{+\infty} e^{ikr}\frac{e^{\frac{i}{2\hbar}r^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(r)}r^{N-1}dr. \quad (A.14)$$

Let us consider the complex plane and set $z = re^{i\theta}$. We have

$$\int_0^{+\infty} e^{ikr}\frac{e^{\frac{i}{2\hbar}r^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(r)}r^{N-1}dr =$$

$$= \lim_{\epsilon\downarrow 0}\int_{z=\rho e^{i\epsilon}} e^{ikz}\frac{e^{\frac{i}{2\hbar}z^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(z)}z^{N-1}dz$$

$$= \lim_{\epsilon\downarrow 0}\lim_{R\to+\infty}\int_0^R e^{ik\rho e^{i\epsilon}}\frac{e^{\frac{i}{2\hbar}\rho^2e^{2i\epsilon}}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(\rho e^{i\epsilon})}\rho^{N-1}e^{Ni\epsilon}d\rho \quad (A.15)$$

Given:

$$\begin{array}{lcl} \gamma_1(R) = \{z\in\mathbb{C} & | & 0\le\rho\le R, \quad \theta=\epsilon\} \\ \gamma_2(R) = \{z\in\mathbb{C} & | & \rho=R, \quad \epsilon\le\theta\le\pi/4-\epsilon\} \\ \gamma_3(R) = \{z\in\mathbb{C} & | & 0\le\rho\le R, \quad \theta=\pi/4-\epsilon\} \end{array}$$

with $\epsilon > 0$ small, from the analyticity of the integrand and the Cauchy theorem we have

$$\int_{\gamma_1(R)\cup\gamma_2(R)\cup\gamma_3(R)} e^{ikz}\frac{e^{\frac{i}{2\hbar}z^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(z)}z^{N-1}dz = 0.$$

In particular:

$$\Big|\int_{\gamma_2(R)} e^{ikz}\frac{e^{\frac{i}{2\hbar}z^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(z)}z^{N-1}dz\Big| =$$

$$= R^N\Big|\int_\epsilon^{\pi/4-\epsilon} e^{ikRe^{i\theta}}\frac{e^{\frac{ie^{i2\theta}}{2\hbar}R^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(Re^{i\theta})}e^{iN\theta}d\theta\Big|$$

$$\le R^N\int_\epsilon^{\pi/4-\epsilon} e^{-kR\sin(\theta)}\frac{e^{\frac{-\sin(2\theta)}{2\hbar}R^2}}{(2\pi\hbar)^{N/2}}e^{\frac{\lambda}{\hbar}(aR^4\sin(4\theta))}d\theta$$

$$\le R^N\int_\epsilon^{\pi/8} e^{-k'R\theta}\frac{e^{\frac{-2\theta}{\pi\hbar}R^2}}{(2\pi\hbar)^{N/2}}e^{\frac{\lambda}{\hbar}(aR^4\frac{8}{\pi})\theta}d\theta +$$

$$+ R^Ne^{\frac{\lambda}{\hbar}2aR^4}\int_{\pi/8}^{\pi/4-\epsilon} e^{-k'R\theta}\frac{e^{\frac{-2\theta}{\pi\hbar}R^2}}{(2\pi\hbar)^{N/2}}e^{\frac{\lambda}{\hbar}(-aR^4\frac{8}{\pi})\theta}d\theta$$

$$= \frac{R^N}{(2\pi\hbar)^{N/2}}\Big\{\Big(\frac{e^{(\frac{8a\lambda}{\pi\hbar}R^4-\frac{2}{\pi\hbar}R^2-k'R)\pi/8} - e^{(\frac{8a\lambda}{\pi\hbar}R^4-\frac{2}{\pi\hbar}R^2-k'R)\epsilon}}{(\frac{8a\lambda}{\pi\hbar}R^4-\frac{2}{\pi\hbar}R^2-k'R)}\Big)+$$

$$+\Big(\frac{e^{\frac{8\epsilon a\lambda}{\pi\hbar}R^4}e^{(-\frac{2}{\pi\hbar}R^2-k'R)(\pi/4-\epsilon)} - e^{\frac{a\lambda}{\hbar}R^4}e^{(-\frac{2}{\pi\hbar})R^2-k'R)\pi/8}}{(\frac{-8a\lambda}{\pi\hbar}R^4-\frac{2}{\pi\hbar}R^2+-k'R)}\Big)\Big\} \quad (A.16)$$

where $k'\in\mathbb{R}$ is a suitable constant. We have used the fact that if $\alpha\in[0,\pi/2]$ then $\frac{2}{\pi}\alpha\le\sin(\alpha)\le\alpha$, while if $\alpha\in[\pi/2,\pi]$ then $\sin(\alpha)\ge 2-\frac{2}{\pi}\alpha$. From the last line one can deduce that

$$\Big|\int_{\gamma_2(R)} e^{ikz}\frac{e^{\frac{i}{2\hbar}z^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(z)}z^{N-1}dz\Big| \to 0, \qquad R\to\infty,$$

so that

$$\int_{z=\rho e^{i\epsilon}} e^{ikz} \frac{e^{\frac{i}{2\hbar}z^2}}{(2\pi i\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}P(z)} z^{N-1} dz = \int_{z=\rho e^{i(\pi/4-\epsilon)}} e^{ikz} \frac{e^{\frac{i}{2\hbar}z^2}}{(2\pi i\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}P(z)} z^{N-1} dz$$

By taking the limit as $\epsilon \downarrow 0$ of both sides one gets:

$$\int_0^{+\infty} e^{ikr} \frac{e^{\frac{i}{2\hbar}r^2}}{(2\pi i\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}P(r)} r^{N-1} dr = \int_0^{+\infty} e^{ik\rho e^{i\pi/4}} \frac{e^{\frac{-\rho^2}{2\hbar}\rho^2}}{(2\pi\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}P(\rho e^{i\pi/4})} \rho^{N-1} d\rho \tag{A.17}$$

By substituting into (A.12) we get the final result:

$$\begin{aligned} &\int_{\mathbb{R}^N} e^{ik\cdot x} \frac{e^{\frac{i}{2\hbar}x\cdot x}}{(2\pi i\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}P(x)} dx \\ &\qquad = \int_{\mathbb{R}^N} e^{ie^{i\pi/4}k\cdot x} \frac{e^{\frac{-x\cdot x}{2\hbar}}}{(2\pi\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}P(xe^{i\pi/4})} dx \\ &\qquad\qquad = \mathbb{E}[e^{ie^{i\pi/4}k\cdot x} e^{\frac{-i\lambda}{\hbar}P(xe^{i\pi/4})}] \end{aligned} \tag{A.18}$$

Appendix B

Some explicit calculations

B.1 The positivity of the operator $I - B$

Let us study the spectrum of the self-adjoint operator B on $\mathcal{H}$ given by (5.33). In order to avoid the use of too many indexes we will assume $d = 1$, but our reasonings remain valid also in the case $d > 1$. A positive real number c_l and a vector $(x_l, \gamma_l) \in \mathcal{H}$ are respectively an eigenvalue and an eigenvector of B if and only if:

$$\begin{cases} t\Omega^2 x_l + \Omega^2 \int_0^t \gamma_l(s)ds = c_l x_l \\ \Omega^2 x_l(ts - \frac{s^2}{2}) - \int_0^s \int_t \Omega^2 \gamma_l(r) dr du = c_l \gamma_l(s) \end{cases}$$

More precisely the vector $(x_l, \gamma_l) \in \mathcal{H}$ solves the following system:

$$\begin{cases} t\Omega^2 x_l + \Omega^2 \int_0^t \gamma_l(s)ds = c_l x_l \\ c_l \ddot{\gamma}_l(s) + \Omega^2 \gamma_l(s) = -\Omega^2 x_l \\ \gamma_l(0) = 0 \\ \dot{\gamma}_l(t) = 0 \end{cases}$$

By a direct calculation one can verify that the latter system indeed admits a (unique) solution if and only if C_l satisfies the following equation

$$\frac{\Omega}{\sqrt{c_l}} \tan \frac{\Omega t}{\sqrt{c_l}} = 1$$

A graphical representation of the position of the solutions shows that the operator B is trace class. Moreover if the conditions (5.37) are fulfilled the maximum eigenvalue of B is strictly less than 1, so that $(I - B)$ is positive definite.

B.2 Estimate of a Gaussian integral.

Let us consider the following function $F:\mathcal{H}\to\mathbb{C}$ given by

$$F(y,\eta)=\int_{\mathbb{R}^d\times C_t} e^{\sqrt{\hbar}xy+\sqrt{\hbar}n(\eta)(\omega)}e^{\frac{1}{2}\langle(x,\omega),B(x,\omega)\rangle}N(dx)W(d\omega).$$

Let us assume Ω, t satisfy assumption (5.37). By a direct computation and by Fubini theorem, F is equal to

$$\begin{aligned}F(y,\eta)=(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{\sqrt{\hbar}xy}e^{-x\frac{(I-t\Omega^2)}{2}x}\int_{C_t} e^{\sqrt{\hbar}n(\eta)(\omega)}e^{x\int_0^t\Omega^2\omega(s)ds}\\ e^{\frac{1}{2}\int_0^t\omega(s)\Omega^2\omega(s)ds}W(d\omega)dx\\ =(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{\sqrt{\hbar}xy}e^{-x\frac{(I-t\Omega^2)}{2}x}\int_{C_t} e^{\sqrt{\hbar}n(\eta)(\omega)}e^{n(v_x)(\omega)}e^{\frac{1}{2}\langle\omega,L\omega}W(d\omega)dx,\end{aligned}\quad\text{(B.1)}$$

where $L:H_t\to H_t$ is the operator given by

$$L\gamma(s)=-\int_0^s\int_t^{s'}\Omega^2\gamma(s'')ds''ds'$$

and $v_x\in H_t$ is the vector given by $v_x(s)=\Omega^2x(ts-\frac{s^2}{2})$. One can easily verify that L is symmetric and trace class. Indeed by denoting by α^2,γ respectively the eigenvalues and the eigenvectors of the operator L, we have

$$-\Omega^2\ddot{\gamma}(s)=\alpha^2\gamma(s),\qquad \gamma(0)=0,\dot{\gamma}(t)=0$$

Without loss of generality we can assume Ω^2 is diagonal with eigenvalues Ω_i^2, $i=1,\dots d$. The components γ_i, $i=1,\dots d$, of the eigenvector γ corresponding to the eigenvalue α^2 are equal to

$$\gamma_i(s)=A_i\sin\frac{\Omega_i s}{\alpha}.$$

By imposing the condition $\dot{\gamma}(t)=0$, we have $\Omega_i t/\alpha=\pi/2+n_i\pi$, $n_i\in\mathbb{Z}$. The possible α^2 are of the form $\alpha^2=\Omega_i^2t^2/(n_i+\frac{1}{2})^2\pi^2$. It follows that the operator $I-L$ is positive definite if and only if $\Omega_i t<\pi/2$ for all $i=1,\dots d$. Moreover the Fredholm determinant of L can easily be computed by means of the equality $\cos x=\prod(1-\frac{x^2}{\pi^2(n+1/2)^2})$ and it is equal to $\det\cos\Omega t$.
By the considerations in section 4 the function $G:\mathbb{R}^d\to\mathbb{R}$ given by

$$G(x)=\int_{C_t} e^{\sqrt{\hbar}n(\eta)(\omega)+n(v_x)(\omega)}e^{\frac{1}{2}\langle\omega,L\omega\rangle}W(d\omega)\quad\text{(B.2)}$$

is equal to

$$\frac{1}{\sqrt{\det\cos\Omega t}}e^{\frac{1}{2}\langle\sqrt{\hbar}\eta+v_x,(I-L)^{-1}(\sqrt{\hbar}\eta+v_x)\rangle}$$

where $(I-L)^{-1}$ is given by

$$(I-L)^{-1}\gamma(s) = \Omega^{-1}[\int_0^s \sin[\Omega(s-s')]\ddot{\gamma}(s')ds' + \sin(\Omega s)\dot{\gamma}(0)] + \\ + \sin(\Omega s)[\cos(\Omega t)]^{-1}\int_0^t \sin[\Omega(t-s')]\dot{\gamma}(s')ds' \quad \text{(B.3)}$$

Moreover by direct computation we see that

$$G(x) \;=\; \frac{1}{\sqrt{\det\cos\Omega t}} e^{\frac{1}{2}\langle\sqrt{\hbar}\eta,(I-L)^{-1}\sqrt{\hbar}\eta\rangle} e^{\frac{1}{2}x(-t\Omega^2+\Omega\tan\Omega t)x} e^{\langle v_x,(I-L)^{-1}\sqrt{\hbar}\eta\rangle} \quad \text{(B.4)}$$

By inserting this into (B.1), we have

$$F(y,\eta) = \frac{(2\pi)^{-d/2}}{\sqrt{\det\cos\Omega t}} e^{\frac{1}{2}\langle\sqrt{\hbar}\eta,(I-L)^{-1}\sqrt{\hbar}\eta\rangle} \int_{\mathbb{R}^d} e^{\sqrt{\hbar}xy} e^{-\frac{1}{2}x(I-\Omega\tan\Omega t)x} e^{\langle v_x,(I-L)^{-1}\sqrt{\hbar}\eta\rangle} dx$$

In particular by taking $\eta = \beta G_t$, $\beta \in \mathbb{C}$, $G_t \in H_t$, $G_t(s) = zs$, $z \in \mathbb{R}^d$ we get

$$F(y,\eta) = \frac{e^{\frac{\hbar\beta^2}{2}z(\Omega^{-1}tan\Omega t)z}}{\sqrt{\det(\cos\Omega t - \Omega\sin\Omega t)}} \\ e^{\frac{\hbar}{2}(y+\beta\cos\Omega t^{-1}(1-\cos\Omega t)z)(1-\Omega\tan\Omega t)^{-1}(y+\beta\cos\Omega t^{-1}(1-\cos\Omega t)z)}. \quad \text{(B.5)}$$

Appendix C

Borel-summable asymptotic expansions

Let $V \subset \mathbb{C}$ a domain in the complex plane, such that $0 \in \partial V$ and

$$z \in V \Rightarrow \forall t \in (0,1], \quad tz \in V$$

Let us denote $\hat{V} := V \cup \{0\}$. Both V and $\hat{V}$ will be called *angular neighborhoods* of zero. A domain $U \subset \mathbb{C}$, such that U is the closure of an angular neighborhood of zero, will be called *closed angular neighborhood.*

Definition 11. *Let V an angular neighborhood of zero. An asymptotic sequence of functions $(\phi_i)_{i\in\mathbb{N}}$ for $z \to 0$ in V is a sequence of functions $\phi_i : \hat{V} \to \mathbb{C}$, which do not vanish in V and such that for every $i \in \mathbb{N}$:*

$$\lim_{z\to 0} \frac{\phi_{i+1}}{\phi_i}(z) = 0$$

In the following we shall focus on the asymptotic sequence (in any angular neighborhood of zero) $\phi_n(z) = z^{n/k}$, $n \in \mathbb{N}$, for fixed $k > 0$ and denote by $\mathbb{C}[z^{1/k}]$ the space of formal power series with complex coefficients

$$\hat{f}(z) = \sum_{n=0}^{\infty} a_n z^{n/k}, \quad \{a_n\} \subset \mathbb{C},\ k > 0. \tag{C.1}$$

Definition 12. *A formal power series $\hat{f}$ is called a ($z^{1/k}-$) asymptotic expansion for a function $f : V \to \mathbb{C}$ as $z \to 0$ in an angular neighborhood V if for each closed angular neighborhood U with $U \subsetneq V$ and any $N \in \mathbb{N}$, there exists a number $C(N) > 0$ such that*

$$\forall z \in U \ : \quad |f(z) - \sum_{n=0}^{N} a_n z^{n/k}| \leq C(N)|z^{n/k}|. \tag{C.2}$$

In this case we write $f \sim \hat{f}$, $z \to 0$ in V.

For more details see [62]

Remark 21. *It is important to recall that the domain V in definition 12 plays a crucial role, indeed the existence of an expansion depends strongly on V.*

Remark 22. *An asymptotic expansions is not necessarily convergent (and usually this is the case!). Indeed condition* (C.3) *means that for fixed N the function f in approximated by the sum $\sum_{n=0}^{N} a_n z^{n/k}$ for z sufficiently small, while if the formal power series* (C.1) *is convergent in some domain V to an analytic function f then the following holds:*

$$\forall z \in V \ : \quad \lim_{N\to\infty} |f(z) - \sum_{n=0}^{N} a_n z^{n/k}| = 0 \tag{C.3}$$

which means that for fixed $z \in V$ the function f in approximated by the sum $\sum_{n=0}^{N} a_n z^{n/k}$ for N sufficiently large.

It is easy to see that if a function f admits an $(z^{1/k}-)$ asymptotic expansion in a given domain, then it is unique. On the other hand different functions can have the same asymptotic expansion, for instance the function $f(z) = 0$ and $g(z) = e^{1/\sqrt{z}}$ have both a zero asymptotic expansion in the domain $\{z \in \mathbb{C},\ |z| < r\} \setminus [0, r]$. In other words, if an asymptotic expansion is not convergent (and this is often the case) it does not characterize uniquely a function f asymptotically equivalent to it. In order to construct an 1 to 1 correspondence between formal power series and functions one can apply a very powerful summation tool: Borel summability. It works as follows:

1. transform the given power series $\hat{f}$ into another convergent power series $\hat{B}$;
2. compute the analytic function B obtained in this way;
3. apply an integral transform mapping the analytic function B to analytic function f
4. the function f (the so called sum of $\hat{f}$) obtained in this way has the power series $\hat{f}$ we started

In order to apply Borel summability method it is necessary to impose stronger conditions on the coefficients.

Definition 13. *Given $s > 0$, a formal power series $\hat{f}(z) = \sum_{n=0}^{\infty} a_n z^{n/k} \in \mathbb{C}[z^{1/k}]$ belongs to the s-Gevrey classes $\mathbb{C}[z^{1/k}]_s$ if exist two constants $C, M > 0$, such that*

$$\forall n \in \mathbb{N} : \quad |a_n| \leq CM^n (\Gamma(1 + n/k))^s,$$

where Γ is the Euler Gamma function.

The Gevrey classes are connected via the following transform acting on formal series:

Definition 14. *The map* $\mathbb{B}_{p,k} : \mathbb{C}[z^{1/k}]_s \to \mathbb{C}[z^{1/k}]_{s-p}$ *defined by*

$$\mathbb{B}_{p,k}[\sum_{n=0}^{\infty} a_n z^{n/k}](t) := \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(1+np/k)} t^{np/k} \tag{C.4}$$

is called the (formal) $(p,k)-$*Borel transform.*

It is important to note that the $(s,k)-$Borel transform maps $\mathbb{C}[z^{1/k}]_s$ to convergent series.
We can now define the concept of *Borel summability*:

Definition 15. *A formal power series* $\hat{f}(z) = \sum_{n=0}^{\infty} a_n z^{n/k}$ *is called* $\mu-$*Borel summable to the sum* f *if* f *is an holomorphic function on* V *for some angular neighborhood of zero* V, $f \sim \hat{f}$ *as* $z \to 0$ *in* V *and the following procedure is possible:*

1. *The* $(1,k)-$*Borel transform* $\mathbb{B}_{1,k}[\hat{f}](t)$ *has nonzero radius of convergence and thus converges in a neighborhood of zero to some function* $B(\cdot)$.
2. *This holomorphic function admits an analytic continuation (denoted again by the symbol* $B(\cdot)$*) onto some open neighborhood of* $\mathbb{R}^+$
3. *the Laplace transform of* B *gives a representation of* f *on a subset of* V*:*

$$f(z) = \frac{1}{z}\int_0^{\infty} B(t)e^{-t/z}dt \tag{C.5}$$

In other words if an asymptotic series is Borel summable to a function f, it characterizes uniquely f, even if it is not convergent.
The following criterion for Borel summability is due to F. Nevanlinna [84], see also [94] :

Theorem 28. *Let* $k > 0$, $R \in (0, +\infty]$ *and define* $D_R := \{z \in \mathbb{C} : Re(1/z) > 1/R\}$ *if* $R \neq \infty$ *and* $D_R := \{z \in \mathbb{C} : Re(z) > 0\}$ *else.*
Let f *be an holomorphic function admitting an asymptotic expansion with respect to the asymptotic sequence* $(z^{n/k}$ *in the domain* D_R*, i.e. such that* $f(z) \sim \sum_{n=0}^{\infty} a_n z^{n/k} =: \hat{f}$ *and* $\exists A > 0, \rho > 0\ \forall \epsilon > 0, z \in \{Re(1/z) \geq \epsilon + 1/R\}$, $\hat{\rho} > \rho, n \in \mathbb{N}$*:*

$$|f(z) - \sum_{i=0}^{n-1} a_i z^{i/k}| \leq A\Gamma(1+n/k)\hat{\rho}^n |z|^{n/k}$$

Then the asymptotic power series $\hat{f}$ *is Borel summable to the function* f.

Bibliography

[1] S. Albeverio. Wiener and Feynman Path Integrals and Their Applications. Proceedings of the Norbert Wiener Centenary Congress, 163-194, *Proceedings of Symposia in Applied Mathematics* **52**, Amer. Math. Soc., Providence, RI, 1997.

[2] S. Albeverio, Ph. Blanchard, R. Høegh-Krohn. Feynman path integrals and the trace formula for the Schrödinger operators. *Comm. Math. Phys.*, **83** n.1, 49-76, 1982.

[3] S. Albeverio, A.M. Boutet de Monvel-Berthier, Z. Brzeźniak. The trace formula for Schrödinger operators from infinite dimensional oscillatory integrals. *Math. Nachr.* **182**, 21-65, 1996.

[4] S. Albeverio and Z. Brzeźniak. Finite-dimensional approximation approach to oscillatory integrals and stationary phase in infinite dimensions. *J. Funct. Anal.*, **113** n.1, 177-244, 1993.

[5] S. Albeverio and Z. Brzeźniak. Oscillatory integrals on Hilbert spaces and Schrödinger equation with magnetic fields. *J. Math. Phys.*, **36** n.5, 2135–2156, 1995.

[6] S. Albeverio, Z. Brzeźniak, Z. Haba. On the Schrödinger equation with potentials which are Laplace transform of measures. *Potential Anal.*, **9** n.1, 65-82, 1998.

[7] S.Albeverio, J.E. Fenstad, R. Høegh-Krohn, T. Lindstrøm. Non Standard Methods in Stochastic Analysis and Mathematical Physics. *Pure and Applied Mathematics* **122**, Academic Press, Inc., Orlando, FL, 1986.

[8] S.Albeverio, G. Guatteri, S. Mazzucchi. Phase space Feynman path integrals *J. Math. Phys.* **43**, 2847-2857, 2002.

[9] S. Albeverio, G. Guatteri, S. Mazzucchi. Representation of the Belavkin equation via Feynman path integrals. *Probab. Theory Relat. Fields* **125**, 365-380, (2003).

[10] S.Albeverio, G. Guatteri, S. Mazzucchi. Representation of the Belavkin equation via phase space Feynman path integrals *to appear in* IDAQP.

[11] S. Albeverio, A. Hahn, A. Sengupta. Rigorous Feynman path integrals, with applications to quantum theory, gauge fields, and topological invariants, SFB 611, Preprint, Bonn, **58** (2003) *to appear in* Proc. Conf. Mathematical Legacy of Feynman's Path Integral, Lisbone 2003

[12] S. Albeverio and R. Høegh-Krohn. *Mathematical theory of Feynman path integrals.* Lecture Notes in Mathematics, Vol. 523. Springer-Verlag, Berlin, 1976.

[13] S. Albeverio and R. Høegh-Krohn. Oscillatory integrals and the method of stationary phase in infinitely many dimensions, with applications to the classical limit of quantum mechanics. *Invent. Math.*, **40** n.1, 59-106, 1977.

[14] S. Albeverio, A. Khrennikov, O. G. Smolyanov. The probabilistic Feynman-Kac formula for an infinite-dimensional Schr"odinger equation with exponential and singular potentials. *Potential Anal.*, **11** n.2, 157-181, 1997.

[15] S. Albeverio, V. N. Kolokol′tsov, O. G. Smolyanov. Représentation des solutions de l'équation de Belavkin pour la mesure quantique par une version rigoureuse de la formule d'intégration fonctionnelle de Menski. *C. R. Acad. Sci. Paris Sér. I Math.*, 323(6), 661–664, 1996.

[16] S. Albeverio, V. N. Kolokol′tsov, O. G. Smolyanov. Continuous quantum measurement: local and global approaches. *Rev. Math. Phys.*, **9** n.8, 907–920, 1997.

[17] S. Albeverio, S. Liang. Asymptotic Expansions for the Laplace Approximations of Sums of Banach Space-valued Random Variables. SFB 611 Preprint No.15, Bonn 2002

[18] S. Albeverio and S. Mazzucchi. Generalized Fresnel Integrals. SFB 611 Preprint No. 59, Bonn 2003.

[19] S. Albeverio and S. Mazzucchi. Feynman path integrals for polynomially growing potentials. UTM 638 Preprint of the University of Trento, 2003.

[20] S. Albeverio, H. Röckle, V. Steblovskaya. Asymptotic expansions for Ornstein-Uhlenbeck semigroups perturbed by potentials over Banach spaces. *Stochastics Stochastics Rep.* **69** n.3-4, 195-238, 2000.

[21] S. Albeverio and J. Schäfer. Abelian Chern-Simons theory and linking numbers via oscillatory integrals. *J. Math. Phys.* **36**, 2157-2169, 1995.

[22] S. Albeverio and A. Sengupta. A mathematichal construction of the non-Abelian Chern-Simons functional integral. *Commun. Math. Phys.* **186**, 563-579, 1997.

[23] S. Albeverio, V. Steblovskaya. Asymptotics of infinite-dimensional integrals with respect to smooth measures I. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **2**, n.4, 529-556, 1999.

[24] V.I. Arnold. Integrals of quickly oscillating functions and singularities of projections of Lagrange manifolds. *Funct. Analys. and its Appl.* **6**, 222-224, 1972.

[25] J. P. Aubin. *Applied Functional Analysis. Second Edition* Wiley-Interscience, NY, 2000.

[26] R. Azencott, H. Doss, L'équation de Schrödinger quand h tend vers zéro: une approche probabiliste. (French) [The Schrödinger equation as h tends to zero: a probabilistic approach], *Stochastic aspects of classical and quantum systems* (Marseille, 1983), 1–17, Lecture Notes in Math., 1109, Springer, Berlin, 1985.

[27] A. Barchielli. On the quantum theory of measurements continuous in time. *Rep. Math. Phys.* **33**, n 1-2, 21-34, 1993.

[28] A. Barchielli, V.P. Belavkin. Measurements continuous in time and a posteriori states in quantum mechanics. *J. Phys. A* **24**, n 7, 1495-1514, 1991.

[29] V.P. Belavkin. A new wave equation for a continuous nondemolition measurement. *Phys. Lett.*, vol. A140, no.7/8, 355-358, 1989.

[30] V.V. Belokurov, Yu. P. Solov'ev, E.T. Shavgulidze. Method of approximate evaluation of path integrals using perturbation theory with convergent series. I. *Theor. Math. Phys.* **109**, 1287-1293, 1996.

[31] V.V. Belokurov, Yu. P. Solov'ev, E.T. Shavgulidze. Method of approximate evaluation of path integrals using perturbation theory with convergent series. II. *Theor. Math. Phys.* **109**, 1294-1301, 1996.

[32] G. Ben Arous, F. Castell. A probabilistic approach to semi-classical approximations. *J. Funct. Anal.* **137**, no. 1, 243–280, 1996.

[33] F. Benatti , G.C. Ghirardi, A. Rimini, T. Weber. Quantum mechanichs with spontaneous localization and the quantum theory of measurement. *Nuovo Cimento B (11)* **100**, 27-41, 1987.

[34] M. Berry. Making light of mathematics. *Bull. Amer. Math. Soc. (N.S.)* **40**, n.2, 229-237, 2003.

[35] Ph. Blanchard, A. Jadczyk, Relativistic quantum events. *Found. Phys.*, **26** no.12, 1669-1681, 1996.

[36] Ph. Blanchard, L. JaKobczyk, R. Olkiewicz, Measures of entanglement based on decoherence. *J. Phys. A*, **34** no.41, 8501-8516, 2001.

[37] S. Bonaccorsi and G. Tessitore, Asymptotic behavior of infinite dimensional stochastic differential equations by anticipative variation of constants formula. *Appl. Math. Optim.*, **44** n.3, 203–225, 2001.

[38] Z. Brzeźniak, M. Capiński, F. Flandoli. Approximation for diffusion in random fields. *Stochastic Anal. Appl.* **8** n.3, 293–313, 1990.

[39] A.O. Caldeira, A.J. Leggett. Path integral approach to quantum Brownian motion. *Physica* **121A**, 587-616, 1983.

[40] R.H. Cameron. A family of integrals serving to connect the Wiener and Feynman integrals. *J. Math. and Phys.* **39**, 126-140, 1960.

[41] P. Cartier and C. DeWitt-Morette. Functional integration. *J. Math. Phys.* **41**,no. 6, 4154-4187, 2000.

[42] C. B. Chiu, B. Misra, E. C. G. Sudarshan. Time evolution of unstable quantum states and a resolution of Zeno's paradox. *Phys. Rev. D* **16**, No. 2, 520-529, 1977.

[43] P.R. Chernoff. Product Formulas, Nonlinear Semigroups, and Addition of Unbounded Operators. *Mem. Amer. Math. Soc.* **140** 1974.

[44] E. Combet. *Intégrales exponentielles, développements asymptotiques, propriétés lagrangiennes.* Lecture Notes in Mathematics, Vol. 937. Springer-Verlag, Berlin-New York, 1982.

[45] D. M. Chung. Conditional analytic feynman integrals on Wiener spaces. *Proc. AMS* **112**, 479-488, 1991.

[46] L.K. Chung, J.C. Zambrini. *Introduction to random time and quantum randomness.* Monographs of the Portuguese Mathematical Society. McGraw-Hill, Lisbon, 2001.

[47] G. Da Prato, *Stochastic evolutions equations by semigroups methods.* Quadernes Centre de Recerca Matematica, Barcelona, n.11, 1998.

[48] G. Da Prato and Zabczyk J, *Stochastic Equations in Infinite Dimensions.* Cambridge University Press, Cambridge, 1992.

[49] I. Daubechies, J. R. Klauder. Quantum-mechanical path integrals with Wiener measure for all polynomial Hamiltonians II. *J. Math. Phys.* **26**, 2239-2256, 1985.

[50] E. B. Davies. *Quantum Theory of Open Systems.* Academic, London, 1976.

[51] I. Davies, A. Truman. On the Laplace asymptotic expansion of conditional Wiener integrals and the Bender-Wu formula for x^{2N}-anharmonic oscillator. *J. Math. Phys.* **24**, no.2, 255-266, 1983.

[52] M. De Faria, H.H. Kuo, L. Streit. The Feynman integrand as a Hida distribution. *J. Math. Phys.* **32**, 2123-2127, 1991.

[53] L. Diosi. Continuous quantum measurements and Ito formalism. *Phys. Lett. A* **129** (8), 419-223, 1988.

[54] P. A. M. Dirac. *Quantum Mechanics.* Oxford Univ. Press, London, 1958.

[55] H. Doss. Sur une Résolution Stochastique de l'Equation de Schrödinger à Coefficients Analytiques. *Commun. Math. Phys.*, 73, 247-264, 1980.

[56] J.J. Duistermaat. Oscillatory integrals, Lagrange immersions and unfoldings of singularities. *Comm. Pure Appl. Math.* **27**, 207-281, 1974.

[57] D. Elworthy and A. Truman. Feynman maps, Cameron-Martin formulae and anharmonic oscillators. *Ann. Inst. H. Poincaré Phys. Théor.*, 41(2), 115–142, 1984.

[58] M.V. Fedoriuk, V.P. Maslov. *Semi-Classical Approximation in Quantum Mechanics,* D. Reidel, Dortrecht, 1981.

[59] R.P. Feynman, A.R. Hibbs. *Quantum mechanics and path integrals.* Macgraw Hill, New York, 1965.

[60] D. Gatarek and N.Gisin. Continuous quantum jumps and infinite-dimensional stochastic equation. *J. Math. Physics* **32**, n.8, 2152–2157, 1991.

[61] L. Gross. Abstract Wiener Spaces. Proc. 5^{th} Berkeley Symp. Math. Stat. Prob. **2**, 31-42, 1965.

[62] G. H. Hardy. *Divergent series.* Oxford University Press, London, 1963.

[63] T. Hida, H.H. Kuo, J. Potthoff, L. Streit. *White Noise.* Kluwer, Dordrecht, 1995.

[64] L. Hörmander. Fourier integral operators I. *Acta Math.*, **127** n. 1, 79-183, 1971.

[65] L. Hörmander. *The Analysis of Linear Partial Differential Operators. I. Distribution Theory and Fourier Analysis.* Springer-Verlag, Berlin/Heidelberg/New York/Tokyo, 1983.

[66] R. L. Hudson, K. R. Parthasarathy. Quantum Ito's Formula and Stochastic Evolutions. *Commun. Math. Phys.* **93**, 301-323, (1984).

[67] K. Ito. Wiener integral and Feynman integral. *Proc. Fourth Berkeley Symposium on Mathematical Statistics and Probability.* Vol 2, pp. 227-238, California Univ. Press, Berkeley, 1961.

[68] K. Ito. Generalized uniform complex measures in the hilbertian metric space with their applications to the Feynman path integral. *Proc. Fifth Berkeley Symposium on Mathematical Statistics and Probability.* Vol 2, part 1, pp. 145-161, California Univ. Press, Berkeley, 1967.

[69] G.W. Johnson, M.L. Lapidus, *The Feynman integral and Feynman's operational calculus.* Oxford University Press, New York, 2000.

[70] M. Kac. *Integration in function spaces and some of its applications*, SNS, Pisa, 1980.

[71] M. Kac, Wiener and integration in function space. *Bull. AMS* 72, n.1, 52-68, 1966.

[72] G. Kallianpur, D. Kannan, R.L. Karandikar. Analytic and sequential Feynman integrals on abstract Wiener and Hilbert spaces, and a Cameron Martin Formula. *Ann. Inst. H. Poincaré, Prob. Th.* **21**, 323-361, 1985.

[73] V. N. Kolokol′tsov. *Semiclassical analysis for diffusions and stochastic processes.* Lecture Notes in Mathematics, 1724. Springer-Verlag, Berlin, 2000.

[74] T. Kuna, L. Streit, W. Westerkamp. Feynman integrals for a class of exponentially growing potentials. *J. Math. Phys.* **39**, no. 9, 4476-4491, 1998.

[75] H.H. Kuo. *Gaussian Measures in Banach Spaces,* Lecture Notes in Math., Springer-Verlag Berlin-Heidelberg-New York, 1975.

[76] B. Malgrange. Sommations des séries divergentes. *Expo. Math.* **13**, 163-222, 1995.

[77] V. Mandrekar. Some remarks on various definitions of Feynman integrals. in Lectures Notes Math., Eds K. Jacob Beck, pp.170-177, 1983.

[78] V.P. Maslov. *Méthodes Opérationelles,* Mir. Moscou, 1987.

[79] M.B. Mensky. *Quantum measurements and decoherence. Models and phenomenology.* Fundamental Theories of Physics, 110. Kluwer Academic Publishers, Dordrecht, 2000.

[80] R. Milnos, Generalized random processes and their extension to a measure. *Trudy Mos. Mat. Ob* **8**, 497-518, 1959.

[81] B. Misra, E. C. G. Sudarshan. The Zeno's paradox in quantum theory. *J. Math. Phys.* **18**, No. 4, 756-763, 1977.

[82] P. Mullowney, *A general theory of integration in function spaces,* Pitman R.N., 1987.

[83] E. Nelson. Feynman integrals and the Schrödinger equation. *J. Math. Phys.* **5**, 332-343, 1964.

[84] F. Nevanlinna. Zur Theorie der asymptotischen Potenzreihen. *Ann. Acad. Sci. Fenn. (A)*, 12 (3), 1-81, 1919.

[85] A. Pazy. *Semigroups of linear operators and applications to partial differential equations.* Springer-Verlag, New York, 1983.

[86] A. Peres. Zeno paradox in quantum theory. *Am. J. Phys.* **48**, No. 11, 931-932, 1980.

[87] V. I. Piterbarg, V. R. Fatalov. The Laplace method for probability measures in Banach spaces. *Russian Math. Surveys*, **50** 1151-1239, 1995.

[88] J. Rezende. The method of stationary phase for oscillatory integrals on Hilbert spaces. *Comm. Math. Phys.*, **101**, 187-206, 1985.

[89] L.S. Schulman, *Lectures on path integration* World Sci. Publishing, River Edge NJ, 1993.

[90] M. Reed, B. Simon. *Methods of Modern Mathematical Physics. Functional analysis* Academic Press, New York, 1975.

[91] M. Reed, B. Simon. *Methods of Modern Mathematical Physics. Fourier Analysis, Self-Adjointness* Academic Press, New York, 1975.

[92] B. Simon. *Functional integration and quantum physics.* Academic Press, New York-London, 1979.

[93] O.G. Smolyanov, E.T. Shavgulidze. *Path integrals.* Moskov. Gos. Univ., Moscow, 1990.

[94] A. Sokal. An improvement of Watson's theorem on Borel summability. *J. Math. Phys.* **21**, 261-263, 1980.

[95] H. Thaler. Solution of Schrödinger equations on compact Lie groups via probabilistic methods. to appear in *Potential Anal.*

[96] A. Truman. The polygonal path formulation of the Feynman path integral. in *Feynman path integrals.* edited by S. Albeverio et al. Springer Lecture Notes in Physics **106**, 1979.

[97] A Truman. The Feynman maps and the Wiener integral. *J. Math. Phys.* **19**, 1742-1750, 1978.

[98] A. Truman and T. Zastawniak. Stochastic PDE's of Schrödinger type and stochastic Mehler kernels—a path integral approach. In *Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, 1996)*, pages 275–282. Birkhäuser, Basel, 1999.

[99] A. Truman and T. Zastawniak. Stochastic Mehler kernels via oscillatory path integrals. *J. Korean Math. Soc.*, 38(2), 469–483, 2001.

[100] T. J. Zastawniak. Equivalence of Albeverio-Høegh-Krohn-Feynman integral for anharmonic oscillators and the analytic Feynman integral. *Univ. Iagel. Acta Math.* **28**, 187-199, 1991.

[101] T. J. Zastawniak. Fresnel type path integral for the stochastic Schrödinger equation. *Lett. Math. Phys.*, 41(1), 93–99, 1997.