

# REAL ALGEBRAIC SURFACES WITH ISOLATED SINGULARITIES

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ABSTRACT. Given a real algebraic surface  $S$  in  $\mathbb{R}P^3$ , we propose a constructive procedure to determine the topology of  $S$  and to compute non-trivial topological invariants for the pair  $(\mathbb{R}P^3, S)$  under the hypothesis that the real singularities of  $S$  are isolated. In particular, starting from an implicit equation of the surface, we compute the number of connected components of  $S$ , their Euler characteristics and the weighted 2-adjacency graph of the surface.

## 1. INTRODUCTION

Given a real algebraic surface  $S$  in  $\mathbb{R}P^3$  by means of an implicit equation, the problem of recognizing the topology of the surface can be addressed at two different levels: either considering  $S$  only as an abstract topological space, or taking into account also its embedding in  $\mathbb{R}P^3$  and looking at the topology of the pair  $(\mathbb{R}P^3, S)$ .

When the surface  $S$  is non-singular, the possible topological models for the connected components of  $S$  are given by the topological classification theorem for surfaces. Thus, if  $S$  is implicitly defined by an equation of even degree, all its connected components are orientable topological 2-manifolds and hence homeomorphic to a torus with  $g$  holes ( $g \geq 0$ ); if the equation that defines  $S$  has an odd degree, then  $S$  contains exactly one non-orientable connected component homeomorphic to the connected sum of a projective plane and a torus with  $g$  holes, while all the other components are orientable.

If we want to consider also how a surface is embedded in  $\mathbb{R}P^3$ , we say that two surfaces  $S, S'$  are ambient-homeomorphic if there exists a homeomorphism  $\varphi : \mathbb{R}P^3 \rightarrow \mathbb{R}P^3$  such that  $\varphi(S) = S'$ ; in this case we also say that the pairs  $(\mathbb{R}P^3, S)$  and  $(\mathbb{R}P^3, S')$  are homeomorphic. At present there is no classification of the pairs  $(\mathbb{R}P^3, S)$  up to homeomorphism even in the non-singular case and deciding whether two pairs  $(\mathbb{R}P^3, S)$  and  $(\mathbb{R}P^3, S')$  are homeomorphic is a very hard problem, also for simple classes of surfaces such as tori (with one hole). Hence a useful contribution in this direction is to find topological invariants of the pair  $(\mathbb{R}P^3, S)$ .

The papers [FGPT] and [FGL] give constructive answers to the problem of recognizing topologically a real algebraic non-singular surface proposing algorithms that compute the number of its connected components and the Euler characteristic of each of them, which determines them up to homeomorphism. In [FGLP] the non-singular surface is considered together with its embedding in  $\mathbb{R}P^3$ ; the authors describe an algorithmical method to compute the “weighted adjacency graph” of the surface, which gives information both on the mutual disposition of the connected components and on their contractibility.

In this paper we address from a constructive point of view the same questions when the surface contains only isolated singularities. Our aim is to find a discrete set of data that are algorithmically computable, sufficient to determine the topology of the surface and that are non-trivial topological invariants for the pair  $(\mathbb{R}P^3, S)$ .

The basic topological information is that in a small 3-dimensional disk  $D$ , centered at an isolated singular point,  $S \cap D$  is homeomorphic to the cone over the curve  $C$  obtained as the intersection of

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$S$  with the boundary of the disk (see [M]). Then, up to homeomorphism, the portion of  $S$  inside the disk can be seen as the space obtained taking the union of as many 2-dimensional disks as the connected components of  $C$ , choosing a point in each of these disks and collapsing these points to a single point. In this way we see the isolated singularity as the effect of two successive operations: first the glueing of a 2-cell (i.e. a subset homeomorphic to a closed 2-dimensional disk) along each connected component of  $C$  and then the collapse of a set containing a point in each attached 2-cell.

Applying this procedure to all the singularities, we obtain a compact topological surface  $T$  without boundary such that  $S$  is homeomorphic to the topological quotient  $T/\mathcal{R}$  where  $\mathcal{R}$  is the equivalence relation that collapses suitable finite families of points of  $T$ . Thus our algorithm will determine topologically  $S$  by computing the Euler characteristics of the connected components of  $T$  and the families of points that, through a collapsing process, produce the isolated singularities of  $S$ .

Furthermore, after defining the *weighted 2-adjacency graph* of  $S$ , we show that it is an invariant of the pair  $(\mathbb{R}P^3, S)$  and we describe an algorithmical method to compute it.

This paper is a natural evolution of the articles [FGPT], [FGL] and [FGLP], which dealt with non-singular surfaces. Here we use the same basic ideas and techniques (use of a Morse projection, connecting paths, reduction to the affine case, etc.) but we insert them in a new procedure able to detect the presence of isolated singularities and to investigate their topological nature. Basically, while at a critical point most of the needed information is given by the index of that point, at an isolated singularity the necessary topological information will be obtained through the investigation of the curve where  $S$  intersects a small disk centered at the singular point. Also to maintain the paper at a reasonable length, we have chosen to describe in detail only the topological results on which the algorithm bases its correctness and the organization of the main algorithm. As for the instrumental algorithmical techniques used as “black boxes”, we only recall their essential features and refer the reader to the papers previously mentioned for a detailed presentation.

The main definitions, the necessary theoretical background and the list  $D(S)$  of data invariant up to homeomorphism of the pair  $(\mathbb{R}P^3, S)$  to be computed are contained in Section 2. In Section 3 we describe a constructive procedure to compute  $D(S)$  when the surface is contained in an affine chart of  $\mathbb{R}P^3$ . This procedure can be applied if some preliminary tests have been positively passed, i.e. if the singularities of  $S$  are isolated and the working system of coordinates is a “good frame”; in Section 4 we present algorithms to perform both these tests and also some preliminary computations concerning the critical and singular points and some related data. When  $S$  is not affine, it is possible to construct a suitable compact algebraic surface  $\widehat{S}$  in  $\mathbb{R}^3$  and to recover  $D(S)$  from  $D(\widehat{S})$ , which can be computed by means of the affine-case algorithm. This reduction procedure and the general-case algorithm are presented in Section 5 which contains also some examples.

## 2. SOME REMARKS ON THE TOPOLOGY OF SURFACES WITH ISOLATED SINGULARITIES

Let  $S$  be the real projective algebraic surface in  $\mathbb{R}P^3$  defined by the equation  $F(x, y, z, t) = 0$ , where  $F$  is a square-free homogeneous polynomial of degree  $d$  with real coefficients. A point  $P \in S$  is called a singular point of the surface if it annihilates all the first partial derivatives of  $F$ ; thus the set  $Sing S$  of the singular points of  $S$  is an algebraic set.

We will consider the case when each singular point is isolated in  $Sing S$ , i.e.  $Sing S$  is a discrete set containing finitely many points. Note that, if  $S$  contains some isolated points, all of them are singular points for  $S$ ; though, a point can be an isolated singular point without being isolated in  $S$ . Note also that we make no assumption on the singular locus  $Sing S_{\mathbb{C}}$  of the complex projective surface  $S_{\mathbb{C}}$  in  $\mathbb{C}P^3$  defined by the equation  $F = 0$ ; since  $F$  is square-free,  $Sing S_{\mathbb{C}}$  cannot have dimension 2, but it can be a complex curve.

If  $Q \in \mathbb{R}^3$  and  $\epsilon \in \mathbb{R}, \epsilon > 0$ , we will use the following notations:

- $B(Q, \epsilon) = \{X \in \mathbb{R}^3 \mid d(X, Q) < \epsilon\}$
- $D(Q, \epsilon) = \{X \in \mathbb{R}^3 \mid d(X, Q) \leq \epsilon\}$
- $S(Q, \epsilon) = \{X \in \mathbb{R}^3 \mid d(X, Q) = \epsilon\}$ ,

where  $d(\cdot, \cdot)$  denotes the Euclidean distance in  $\mathbb{R}^3$ . The previous notations make sense also for points  $Q$  in  $\mathbb{R}\mathbb{P}^3$  working in an affine chart  $U \simeq \mathbb{R}^3$  containing  $Q$ .

The local topological structure of  $S$  at an isolated singularity is described by the following

**Theorem 2.1.** (Milnor, [M] Proposition 2.10) *Let  $Q$  be an isolated singular point of  $S$ . Then there exists a sufficiently small  $r > 0$  such that for all positive  $\epsilon \leq r$*

- i)  $C(Q, \epsilon) = S \cap S(Q, \epsilon)$  is a non-singular curve (possibly empty)
- ii)  $S \cap D(Q, \epsilon)$  is homeomorphic to the cone over  $C(Q, \epsilon)$ .

Any  $r > 0$  such that  $D(Q, r) \setminus \{Q\}$  contains no singular points of  $S$  and no critical points of the restriction to  $S$  of the function  $X \rightarrow d(X, Q)^2$  satisfies the thesis of the previous theorem. For any  $\epsilon \leq r$  we will call  $D(Q, \epsilon)$  a *Milnor disk* at  $Q$ .

Our strategy to study  $S$  will be based on the possibility of modifying  $S$  inside Milnor disks centered at the singular points, getting a topological surface  $T \subset \mathbb{R}\mathbb{P}^3$  (i.e. a 2-dimensional topological manifold) from which we can obtain again  $S$ , except its isolated points if any, by means of a suitable quotient. We will construct  $T$  as an application of the following

**Lemma 2.2.** *Let  $X$  be a non-empty topological 1-dimensional submanifold of a sphere  $S(Q, r)$ . Then there exist finitely many disjoint 2-cells embedded in  $D(Q, r)$  such that, denoting by  $W(X)$  the union of such 2-cells,*

- (1) *the boundary of  $W(X)$  is the curve  $X$ ,*
- (2)  *$W(X) \cap S(Q, r) = X$ .*

*Proof.* Let  $L(X)$  be the adjacency graph of the pair  $(S(Q, r), X)$  whose vertices are the connected components of  $S(Q, r) \setminus X$  and in which two distinct vertices are joined by an edge if and only if they share a common boundary (for the general definition of the adjacency graph of a pair of topological spaces, see also [FGLP]). It is a tree having at least 2 vertices. We will prove the lemma by induction on the number  $n$  of vertices of  $L(X)$ . If  $n = 2$ , then  $X$  is connected and it is sufficient to take as  $W(X)$  the cone over  $X$  with vertex  $Q$ .

Assume now that  $L(X)$  has  $n$  vertices with  $n \geq 3$  and let  $v$  be a vertex of  $L(X)$  of valency 1. Then  $v$  is a connected component of  $S(Q, r) \setminus X$  homeomorphic to an open disk bounded by an oval  $\omega(v)$  which appears in  $L(X)$  as the unique edge having  $v$  in its boundary. For any subset  $Z$  of  $\mathbb{R}^3$  denote by  $C(Z)$  the cone over  $Z$  with vertex  $Q$ . Let  $\tilde{X} = X \setminus \omega(v)$  and consider the curve  $X' = C(\tilde{X}) \cap S(Q, \frac{r}{2})$ . By construction the adjacency graph  $L(X')$  of the pair  $(S(Q, \frac{r}{2}), X')$  has one vertex less than  $L(X)$ , hence by the inductive hypothesis there exists a union of 2-cells  $W(X')$  embedded in  $D(Q, \frac{r}{2})$  whose boundary is  $X'$  and such that  $W(X') \cap S(Q, \frac{r}{2}) = X'$ . If we denote by  $\theta : S(Q, r) \rightarrow S(Q, \frac{3}{4}r)$  the function defined by  $\theta(Y) = \frac{3}{4}(Y - Q) + Q$ , it is sufficient to take

$$W(X) = W(X') \cup \left( C(\tilde{X}) \setminus D(Q, \frac{r}{2}) \right) \cup \left( C(\omega(v)) \setminus D(Q, \frac{3}{4}r) \right) \cup \theta(v).$$

□

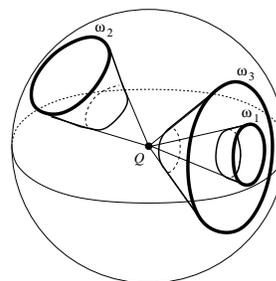


FIGURE 1. *The construction described in the proof of Lemma 2.2*

Assume at first, for simplicity, that  $Q$  is the only singular point of  $S$ . There are two possible situations:

- if  $Q$  is isolated in  $S$ , then  $S$  can be seen as the disjoint union of the point  $Q$  and the compact topological (not algebraic, in general) surface without boundary  $S \setminus \{Q\}$ ,
- if  $Q$  is not isolated in  $S$  and  $D(Q, \epsilon)$  is a Milnor disk at  $Q$ , then  $S \setminus B(Q, \epsilon)$  is a topological surface having as its boundary the non-empty curve  $C(Q, \epsilon)$ . Denote by  $T$  the topological surface without boundary embedded in  $\mathbb{R}P^3$  which is the union of  $S \setminus B(Q, \epsilon)$  and the set  $W(C(Q, \epsilon))$  obtained applying Lemma 2.2 to the curve  $C(Q, \epsilon)$ . Thus  $T$  is obtained from  $S$  removing  $S \cap B(Q, \epsilon)$  and attaching a 2-cell along each connected component of  $C(Q, \epsilon)$ . Choose a point in each of the attached 2-cells and denote by  $Z(Q)$  the set of these points. Then  $S$  is homeomorphic to the topological quotient of  $T$  with respect to the equivalence relation that collapses  $Z(Q)$  to a single point.

Coming back to the general case, henceforth we will denote by

- $Q_1, \dots, Q_m$  the singularities of the surface that are not isolated points in  $S$ ,
- $R_1, \dots, R_s$  the isolated points in  $S$ .

Let  $\epsilon$  be a small positive rational number such that  $D(Q_i, \epsilon)$  is a Milnor disk at  $Q_i$  for  $i = 1, \dots, m$  and such that  $D(Q_i, \epsilon) \cap D(Q_j, \epsilon) = \emptyset$  whenever  $i \neq j$ .

Let  $T$  be the embedded topological surface without boundary obtained from  $S \setminus (\bigcup_{i=1}^m B(Q_i, \epsilon) \cup \{R_1, \dots, R_s\})$  by means of Lemma 2.2, i.e. attaching a 2-cell along each connected component of  $S \cap S(Q_i, \epsilon)$  for all  $i = 1, \dots, m$ . Again denote by  $Z(Q_i)$  the set obtained choosing a point in each 2-cell attached to  $C(Q_i, \epsilon)$ . If  $\mathcal{R}$  is the equivalence relation on  $T$  that collapses to a point each of the sets  $Z(Q_1), \dots, Z(Q_m)$ , then  $S$  is homeomorphic to the disjoint union of  $T/\mathcal{R}$  and of the set  $\{R_1, \dots, R_s\}$ .

The topological type of the space obtained by collapsing finitely many points in a compact connected surface does not depend on the choice of these points, but only on their number. Therefore, if  $T_1, \dots, T_r$  are the connected components of  $T$ , the topological type of  $T/\mathcal{R} = (T_1 \cup \dots \cup T_r)/\mathcal{R}$  is completely determined by the number  $n_{ij}$  of points in  $Z(Q_i) \cap T_j$  for  $i = 1, \dots, m$  and  $j = 1, \dots, r$ .

Recall (see for instance [FGL]) that each connected component  $T_j$ , being embedded in  $\mathbb{R}P^3$ , is topologically determined by its Euler characteristic  $\chi_j$ : if  $T_j$  is orientable, then it is homeomorphic to a torus with  $g_j$  holes (meaning that a torus with 0 holes is a sphere),  $\chi_j$  is even and  $g_j = \frac{2-\chi_j}{2}$ ; if  $T_j$  is non-orientable, then it is homeomorphic to the connected sum of a projective plane and a torus with  $g_j$  holes,  $\chi_j$  is odd and  $g_j = \frac{1-\chi_j}{2}$ .

Thus, in order to determine the topological type of  $S$ , it will be sufficient to compute:

- (1) the list  $\chi(T) = [\chi_1, \dots, \chi_r]$  of the Euler characteristics of the connected components  $T_1, \dots, T_r$  of  $T$ ,
- (2)  $m$  lists of non-negative integers, each having length  $r$ , say

$$l_1 = [n_{11}, \dots, n_{1r}], \dots, l_m = [n_{m1}, \dots, n_{mr}]$$

where  $n_{ij} = \#(Z(Q_i) \cap T_j)$ ,

- (3) the number  $s$  of isolated points in  $S$ .

**Example 2.3.** The surface represented in Figure 2 has three isolated singularities  $Q_1, Q_2, R_1$  and  $R_1$  is an isolated point, so that  $m = 2, s = 1$ . The topological surface  $T$  constructed as explained above has three connected components  $T_1, T_2, T_3$  and all of them are spheres. The singularity  $Q_1$  can be obtained collapsing one point of  $T_1$  and two points of  $T_2$  to a single point; similarly  $Q_2$  can be obtained collapsing three points chosen respectively in  $T_1, T_2$  and  $T_3$ . Thus the topology of  $S$  is determined by the following data:  $\chi(T) = [2, 2, 2]$ ,  $l_1 = [1, 2, 0]$ ,  $l_2 = [1, 1, 1]$ ,  $s = 1$ .  $\square$

So far our topological investigation has not taken into account the way in which the surface is embedded in  $\mathbb{R}P^3$  and in particular it gives no information about the mutual disposition of the

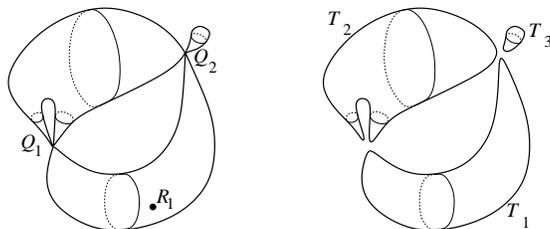


FIGURE 2. A surface with three singular points (left-hand side) and the topological surface  $T$  associated to it (right-hand side).

connected components of  $S$  and the connected components (or regions) of  $\mathbb{RP}^3 \setminus S$ . In [FGLP] it was shown how additional information can be obtained in the case of a non-singular surface by computing the "adjacency graph" of the surface. This is the graph whose vertices are the regions of  $\mathbb{RP}^3 \setminus S$  and two distinct vertices are joined by an edge if and only if their topological closures are not disjoint. When  $S$  is non-singular, two adjacent regions of  $\mathbb{RP}^3 \setminus S$  share in their boundaries a connected component of  $S$ . Hence the edges of the adjacency graph of  $S$  are in 1-1 correspondence with the connected components of the surface, with the only exception that, when  $S$  has an odd degree, the unique non-orientable component of  $S$  is not represented in the graph.

If  $S$  is singular, it may occur that the closures of two regions of the complement of  $S$  share only finitely many points; think for instance of the surface consisting of two spheres tangent at a common point. We will not consider two such regions "really adjacent" and will be interested in the *2-adjacency graph*  $G(S)$  in which two distinct vertices are joined by an edge if and only if the closures of the two regions of  $\mathbb{RP}^3 \setminus S$  meet in a 2-dimensional subset of  $S$ . Observe that

- (1) the 2-adjacency graph  $G(S)$  just defined coincides with the ordinary adjacency graph when  $S$  is non-singular,
- (2) the graph  $G(S)$  is a topological invariant of the pair  $(\mathbb{RP}^3, S)$ ,
- (3) the isolated points of  $S$ , if any, are not represented in  $G(S)$ .

Unlike the non-singular case, there is not a bijective correspondence between the set of the edges of  $G(S)$  and the set of the connected components of the surface even if  $S$  has an even degree: for instance if  $S$  consists of two cones with the same vertex,  $S$  is connected but  $G(S)$  has two edges. Actually the next proposition shows that we recover similar properties if we consider the connected components of  $S \setminus \text{Sing } S$ ; the proof of this result, that we insert for completeness, may be omitted with no influence on the comprehension of the rest of the paper.

**Proposition 2.4.** *Let  $S_1, \dots, S_n$  be the connected components of  $S \setminus \text{Sing } S$ . Then*

- (1) *If the degree of  $S$  is odd, then there exists a unique  $i$  such that, if we set  $\Gamma = S_i$ , the set  $\mathbb{RP}^3 \setminus \overline{\Gamma}$  is connected, while for any  $j \neq i$  the set  $\overline{S_j}$  disconnects  $\mathbb{RP}^3$  into two connected regions.*
- (2) *If the degree of  $S$  is even, for all  $j$   $\overline{S_j}$  disconnects  $\mathbb{RP}^3$  into two connected regions.*

*Proof.* (1) The homology class  $[S]$  in  $H_2(\mathbb{RP}^3, \mathbb{Z}/2\mathbb{Z})$  is given by  $[S] = \sum_{i=1}^n [\overline{S_i}]$ . If  $S$  has an odd degree,  $[S]$  is non-zero and hence there exists a component  $S_i$  of  $S \setminus \text{Sing } S$  such that  $[\overline{S_i}] \neq 0$ .

Moreover for any  $j \neq i$  necessarily  $[\overline{S_j}] = 0$  because otherwise  $[\overline{S_i}] \cdot [\overline{S_j}] \in H_1(\mathbb{RP}^3, \mathbb{Z}/2\mathbb{Z})$  would be non-trivial, which is impossible since  $\overline{S_i} \cap \overline{S_j} \subset \text{Sing } S$  consists of isolated points.

Let  $\Gamma = S_i$ ; we claim that  $\mathbb{RP}^3 \setminus \overline{\Gamma}$  is arcwise connected (and hence connected). Namely for each  $P, Q \in \mathbb{RP}^3 \setminus \overline{\Gamma}$  let  $\alpha : [0, 1] \rightarrow \mathbb{RP}^3$  be a continuous path in  $\mathbb{RP}^3$  joining  $P$  and  $Q$ . If  $\alpha$  does not intersect  $\overline{\Gamma}$ , the claim is proved; otherwise we can assume that  $\alpha$  meets  $\overline{\Gamma}$  in non-singular points of  $S$  (i.e. lying in  $\Gamma$ ) and transversally. Since  $\overline{\Gamma}$  is homologically non-trivial, then  $\Gamma$  is non-orientable; thus, if  $\alpha(t_0) \in \Gamma$ , there exists a loop  $\gamma$  in  $\Gamma$  passing through  $\alpha(t_0)$  and orientation-reversing for  $\Gamma$ .

If  $n(t)$  is a normal vector to  $\Gamma$  along  $\gamma$ , since  $\mathbb{RP}^3$  is orientable, then  $n(1) = -n(0)$ . It is therefore possible to join  $\alpha(t_0 - \epsilon)$  with  $\alpha(t_0 + \epsilon)$  without intersecting  $\Gamma$  following the normal  $n(t)$ . Repeating this construction for each point where  $\alpha$  meets  $\Gamma$ , eventually we get a path lying in  $\mathbb{RP}^3 \setminus \bar{\Gamma}$  and joining  $P$  and  $Q$ .

We have only to prove that, for each  $S_j \neq \Gamma$ ,  $\mathbb{RP}^3 \setminus \bar{S}_j$  is not connected. Otherwise, choosing a segment that meets transversally  $S_j$  in its medium point, and connecting the extremal points of this segment by means of a continuous path disjoint from  $\bar{S}_j$ , we would find a closed curve  $\delta$  that meets  $\bar{S}_j$  only in one point. Then if we consider the homology classes  $[\delta] \in H_1(\mathbb{RP}^3, \mathbb{Z}/2\mathbb{Z})$  and  $[\bar{S}_j] \in H_2(\mathbb{RP}^3, \mathbb{Z}/2\mathbb{Z})$ , we would have  $[\delta] \cdot [\bar{S}_j] = 1$ , in contradiction with the fact that  $[\bar{S}_j] = 0$ .

(2) When  $S$  is even-degree, we get the thesis arguing in a similar way.  $\square$

As a consequence of the previous result, there is a 1-1 correspondence between the closures of the connected components of  $S \setminus \text{Sing } S$  and the edges of  $G(S)$ , except for odd-degree surfaces when the closure of the component  $\Gamma$  given by Proposition 2.4 is not represented in  $G(S)$ .

Moreover, as a consequence of Lemma 2.2, it is not hard to see that

**Proposition 2.5.** *The 2-adjacency graph  $G(S)$  is isomorphic to the adjacency graph  $G(T)$ .*

Apparently, not all information about the mutual position of distinct regions of  $\mathbb{RP}^3 \setminus S$  can be derived from the 2-adjacency graph  $G(S)$ . For instance, if  $S$  consists of two spheres tangent at a point and each external to the other, the graph  $G(S)$  has 3 vertices and the two vertices corresponding to the interior parts of the two spheres are not joined by an edge, just as if the two spheres were disjoint. As a matter of fact, using the lists  $l_1, \dots, l_m$  relative to the singularities which are not isolated points in  $S$ , we can realize whether two regions not joined by an edge in  $G(S) \simeq G(T)$ , and therefore not adjacent with respect to the surface  $T$ , meet at one or more points of their boundaries after the collapsing process that yields  $S$  starting from  $T$ .

Recall that a subset  $A \subset \mathbb{RP}^3$  is called *contractible* if any loop in  $A$  is contractible (i.e. homotopically trivial) as a loop in  $\mathbb{RP}^3$ , *non-contractible* otherwise. Using this notion we can endow the vertices of  $G(T)$  with weights by means of the function  $w_T : \{\text{vertices of } G(T)\} \rightarrow \{c, nc\}$  that marks each vertex of  $G(T)$  (i.e. each region of  $\mathbb{RP}^3 \setminus T$ ) as *contractible* or *non-contractible*.

Observe that an edge of  $G(T)$  (i.e. a connected component of  $T$ ) is non-contractible if and only if its two vertices are both non-contractible; hence the knowledge of the function  $w_T$  is sufficient to know which components of  $T$  are contractible and which components are not. Let us recall that a contractible connected component  $W$  of  $T$  disconnects  $\mathbb{RP}^3$  in two connected regions, only one of which is contractible and called the interior part of  $W$ . For such components it is possible to define a natural partial order relation. The graph  $G(T)$  can be endowed with a set of roots  $r(T)$  from which it is possible to reconstruct such a partial order; for a detailed presentation of these notions and some properties of them, we refer to the paper [FGLP].

The triple  $(G(T), w_T, r(T))$  will be called the *weighted adjacency graph* of  $T$ .

We can fix the same system of weights  $\{c, nc\}$  also on the vertices of the 2-adjacency graph  $G(S)$  by means of  $w_S : \{\text{vertices of } G(S)\} \rightarrow \{c, nc\}$ . We will denote by  $G_{nc}(S)$  the subgraph of  $G(S)$  formed by the non-contractible vertices and by the edges having both vertices marked  $nc$ ; instead we will denote by  $\overline{G_c(S)}$  the subgraph formed by all the contractible vertices, all the edges where at least one vertex is contractible and all the vertices of these edges. By means of arguments similar to those used in the proof of Proposition 2.4, one can see that, if  $S$  has an odd degree, all the regions of  $\mathbb{RP}^3 \setminus S$  are contractible and hence  $G_{nc}(S) = \emptyset$ . Instead, when the degree of  $S$  is even, the closure  $\bar{S}_j$  of each connected component of  $S \setminus \text{Sing } S$  disconnects  $\mathbb{RP}^3$  into two connected regions and at least one of them is non-contractible (possibly both, as in the case of a one-sheeted hyperboloid); in particular  $G_{nc}(S)$  is not empty.

In the singular case, it is no longer true that the weights on the two vertices of an edge are sufficient to determine the contractibility of the edge. For instance, if  $S$  is a cone,  $G(S)$  has two vertices, one marked  $c$ , the other  $nc$  and still the only edge of  $G(S)$  (i.e. the cone itself) is non-contractible. However the knowledge of the weights on the vertices of  $G(S)$  is sufficient to define a partial order relation in the set of the closures of the connected components of  $S \setminus \text{Sing } S$  as precised in the following:

**Definition 2.6.** *Let  $S_i$  and  $S_j$  be distinct connected components of  $S \setminus \text{Sing } S$ . We say that  $\overline{S_i}$  is inside  $\overline{S_j}$  if the following two conditions hold:*

- (1)  $\overline{S_j}$  disconnects  $\mathbb{R}\mathbb{P}^3$  into two regions and one of these is contractible,
- (2)  $S_i$  is contained in the contractible component of  $\mathbb{R}\mathbb{P}^3 \setminus \overline{S_j}$ .

All the information necessary to know the previous partial order among the sets  $\overline{S_i}$  can be obtained by choosing some roots in the 2-adjacency graph  $G(S)$ . The way in which this can be done depends on the degree of  $S$ .

If  $S$  has an even degree, each connected component of  $\overline{G_c(S)}$  is a tree that contains exactly one vertex weighted  $nc$ : we choose these vertices as a *set of roots* of  $G(S)$ . In this way the order induced on each connected component of  $\overline{G_c(S)}$  by the only root contained in it coincides with the partial order described in Definition 2.6.

If  $S$  has an odd degree,  $G_{nc}(S)$  is empty; however we are able to choose a root in  $G(S)$  also in this case: we will call *root* of  $G(S)$  the unique region of  $\mathbb{R}\mathbb{P}^3 \setminus S$  which is adherent to the connected component  $\Gamma$  given by Proposition 2.4.

Note that, while for even-degree surfaces the information about which vertices are the roots of  $G(S)$  is obtained from the weights, for odd-degree surfaces this notion is independent of the weights on  $G(S)$  because in this case each vertex is marked  $c$ . However this information is quite important since sometimes it is the only one that allows to realize that two pairs  $(\mathbb{R}\mathbb{P}^3, S)$  and  $(\mathbb{R}\mathbb{P}^3, S')$  are not homeomorphic: if, for instance,  $S$  consists of a projective plane and two topological spheres, only the knowledge of the root allows to recognize whether the two spheres are mutually external or one of them encircles the other one.

If  $r(S)$  denotes the set of roots of  $S$  defined as above, the triple  $(G(S), w_S, r(S))$  will be called the *weighted 2-adjacency graph* of  $S$ . With the previous definitions it is easy to see that

**Proposition 2.7.** (i) *The weighted 2-adjacency graph of  $S$  is an invariant of the pair  $(\mathbb{R}\mathbb{P}^3, S)$  up to homeomorphism.*

(ii) *The weighted 2-adjacency graph of  $S$  is isomorphic to the weighted adjacency graph of  $T$ .*

The isolated points of  $S$  do not appear at all in  $G(S)$ . While their number is sufficient for the topological characterization of  $S$ , in order to take into account the embedding of  $S$  in  $\mathbb{R}\mathbb{P}^3$  we need to know in which regions of the complement they lie. For that it will be sufficient to compute a list  $q = [q_1, \dots, q_s]$  where  $q_i$  is the region of  $\mathbb{R}\mathbb{P}^3 \setminus T$  containing the  $i$ -th isolated point  $R_i$ .

We will collect all the mentioned data concerning the surface in a single list of data

$$D(S) = [\chi(T), G(T), w_T, r(T), l_1, \dots, l_m, q].$$

By the previous considerations we have that

- (1)  $D(S)$  is an invariant up to homeomorphism of the pair  $(\mathbb{R}\mathbb{P}^3, S)$ ,
- (2)  $D(S)$  completely determines the topological type of  $S$ ,
- (3) though not sufficient to determine the pair  $(\mathbb{R}\mathbb{P}^3, S)$ , the set  $D(S)$  gives useful information on the surface up to ambient-homeomorphism. For instance the surfaces  $S$  and  $S'$  represented in Figure 3 are homeomorphic but the pairs  $(\mathbb{R}\mathbb{P}^3, S)$  and  $(\mathbb{R}\mathbb{P}^3, S')$  are not homeomorphic since the weighted adjacency graphs of  $T$  and  $T'$  are not isomorphic.

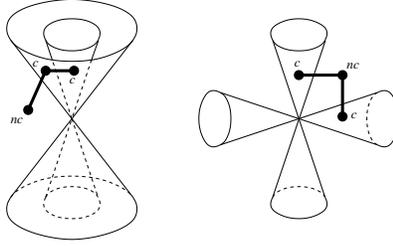


FIGURE 3. Two surfaces that are homeomorphic, but not ambient-homeomorphic.

The next sections will be devoted to show that all the data in  $D(S)$  can be computed starting from an equation of  $S$ , even if  $T$  is not algebraic.

### 3. THE COMPACT AFFINE CASE

In this section we describe a constructive procedure to compute the list of data  $D(S) = [\chi(T), G(T), w_T, r(T), l_1, \dots, l_m, q]$  when the real algebraic surface  $S = \{F(x, y, z, t) = 0\}$  having only isolated singularities does not intersect in real points the plane “at infinity”  $\{t = 0\} \subset \mathbb{RP}^3$ . In this case  $S$  is contained in the affine chart  $\{[x, y, z, t] \in \mathbb{RP}^3 \mid t \neq 0\} \simeq \mathbb{R}^3$  and can be studied working in affine coordinates; namely  $f(x, y, z) = F(x, y, z, 1) = 0$  is an affine equation for  $S$ .

Denote by  $p : S \rightarrow \mathbb{R}$  the projection defined by  $p(x, y, z) = z$ . A point  $P \in S$  is a *critical point* for  $p$  if it is non-singular but it annihilates the first partial derivatives  $f_x$  and  $f_y$ ; in this case  $p(P)$  is called a *critical value*. Recall (see [M] Corollary 2.8) that  $p$  can have at most finitely many critical values, and that under our hypotheses  $S$  can have at most finitely many (real) singular points.

Up to a generic linear change of coordinates we can assume (see for instance [BPR]) that our system of coordinates  $(x, y, z)$  is a “good frame”, that is

- i) the projection  $p$  is a Morse function (i.e. any critical point for  $p$  is *non-degenerate* or equivalently it does not annihilate the determinant of the Hessian matrix  $\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$ )
- ii) if  $P_1$  and  $P_2$  are either singular or critical and  $P_1 \neq P_2$ , then  $p(P_1) \neq p(P_2)$ .

Recall that by index of a non-degenerate critical point  $P$  for  $p$  one means the number of negative eigenvalues of the Hessian matrix of  $p$  at  $P$  with respect to local coordinates; it can be computed from  $f_z$  and the Hessian matrix of  $f$ . In this section we assume to have already checked that  $S$  has only isolated singularities and that the given system of coordinates  $(x, y, z)$  is a good frame; also we assume to have already computed the critical points and their indexes, the singularities and the radius of a Milnor disk at each of them. In Section 4 we will see how these preliminary tests and computations can be performed.

Let  $[-N, N]$  be an interval containing all the critical values of  $p$  and all the images through  $p$  of the singular points of  $S$  (that, for simplicity, we will call *singular values*). We can subdivide it as  $[-N, N] = [-N = a_0, a_1] \cup [a_1, a_2] \cup \dots \cup [a_u, a_{u+1} = N]$  so that each  $a_i$  is neither critical nor singular and each interval  $(a_i, a_{i+1})$  contains only one critical value or one singular value.

For each  $a \in \mathbb{R}$  we will denote by  $C_a$  the level curve  $p^{-1}(a) = S \cap \{z = a\}$  and by  $S_a$  the level surface  $p^{-1}([-N, a]) = S \cap \{z \leq a\}$  having  $C_a$  as its boundary.

Assume that  $\epsilon \in \mathbb{Q}$  is positive and so small that

- $D(Q_i, \epsilon)$  is a Milnor disk at the singular point  $Q_i$  for any  $i = 1, \dots, m$ ,
- $D(R_j, \epsilon)$  is a Milnor disk at the isolated point  $R_j$  for any  $j = 1, \dots, s$ ,
- $D(Q_i, \epsilon) \cap \{z = a_h\} = \emptyset \quad \forall i = 1, \dots, m \text{ and } \forall h = 0, \dots, u + 1$
- $D(R_j, \epsilon) \cap \{z = a_h\} = \emptyset \quad \forall j = 1, \dots, s \text{ and } \forall h = 0, \dots, u + 1$ .

Thus each Milnor disk of radius  $\epsilon$  centered at a singular point is contained in a single open strip  $\mathbb{R}^2 \times (a_h, a_{h+1})$  and does not intersect any level plane  $\{z = a_h\}$ .

Denote by  $T$  the embedded topological surface without boundary obtained from  $S$  removing the points  $R_1, \dots, R_s$  and applying Lemma 2.2 to all the Milnor disks  $D(Q_i, \epsilon)$ ; then, for each  $h = 0, \dots, u + 1$ , we have that

- (1)  $C_{a_h} = S \cap \{z = a_h\} = T \cap \{z = a_h\}$
- (2)  $T_{a_h} = T \cap \{z \leq a_h\}$  is a topological surface with boundary  $C_{a_h}$  obtained from  $S_{a_h}$  removing the points  $R_i$  that lie in  $\{z < a_h\}$  and applying Lemma 2.2 to the Milnor disks  $D(Q_i, \epsilon)$  contained in  $\{z < a_h\}$ .

Thus each  $S_{a_h}$  is homeomorphic to the disjoint union of the isolated points  $R_j$  lying in  $\{z < a_h\}$  and the quotient space  $T_{a_h}/\mathcal{R}_h$ , where  $\mathcal{R}_h$  is the restriction to  $T_{a_h}$  of the equivalence relation  $\mathcal{R}$  introduced in Section 2. Hence at each level we can compute the needed topological data concerning  $S_{a_h}$  studying the topological level surface  $T_{a_h}$ .

More precisely, since  $T_N = T$ , the data  $\chi(T), G(T), l_1, \dots, l_m, q$  contained in the list  $D(S)$  will be obtained after applying to the intervals  $[-N, a_1], \dots, [a_u, N]$  the following

Iterative Step : if  $[a, b]$  is an interval such that  $a$  and  $b$  are regular values and  $\mathbb{R}^2 \times (a, b)$  contains only one point  $Q$  which is either critical for  $p$  or singular for  $S$ , then one computes

$$\text{Output}(S_b) = \{G(C_b), \chi(T_b), G(T_b), M_b, l_1(T_b), \dots, l_m(T_b), q(T_b)\}$$

starting from  $\text{Output}(S_a)$ , where

- i)  $C_b = S \cap \{z = b\} = T \cap \{z = b\}$  and  $G(C_b)$  is the adjacency graph of the pair  $(\{z = b\}, C_b)$  whose vertices are the regions of  $\{z = b\} \setminus C_b$  and where two distinct vertices are joined by an edge if their closures are not disjoint,
- ii)  $T_b = T \cap \{z \leq b\}$  and  $\chi(T_b)$  is the list of the Euler characteristics of the connected components of  $T_b$
- iii)  $G(T_b)$  is the adjacency graph of the pair  $(\{z \leq b\}, T_b)$  whose vertices are the regions of  $\{z \leq b\} \setminus T_b$  and where two distinct vertices are joined by an edge if their boundaries share a 2-dimensional subset,
- iv)  $M_b : G(C_b) \rightarrow G(T_b)$  is the graph morphism that associates to each vertex  $v$  of  $G(C_b)$  the vertex of  $G(T_b)$  representing the region of  $\{z \leq b\} \setminus T_b$  having in its boundary the region of  $\{z = b\} \setminus C_b$  represented by  $v$
- v)  $l_i(T_b) = [n_{i1}, \dots, n_{ir}]$  where  $n_{ij}$  is the number of points of  $Z(Q_i)$  lying in the  $j$ -th component of  $T_b$  (note that  $r$  depends on  $b$ )
- vi)  $q(T_b)$  is a list of length  $s$  where the  $i$ -th element is the region of  $\{z \leq b\} \setminus T_b$  containing the  $i$ -th isolated point  $R_i$  if  $R_i \in \{z \leq b\}$ , it is 0 otherwise.

Note that at the initial step both  $C_{-N}$  and  $T_{-N}$  are empty, so that  $G(C_{-N})$  and  $G(T_{-N})$  consist of a single vertex. The lists  $l_1, \dots, l_m, q$  are initialized as the zero lists; during the iterative procedure only the lists  $l_i(T_b)$  concerning the singular points  $Q_i$  lying in  $\{z < b\}$  are non-zero.

At the end of the iterative procedure, having computed  $\chi(T), G(T), l_1, \dots, l_m, q$ , the only data that we still need to compute to get  $D(S)$  are the function  $w_T$  and the roots  $r(T)$ . In the affine case this is straightforward: since  $T$  is contained in the affine chart  $\{t \neq 0\}$  of  $\mathbb{RP}^3$ , all its components are contractible and all the regions of  $\mathbb{RP}^3 \setminus T$  are contractible except the only one external to all the components of  $T$ . The algorithm easily recognizes this external region as the only vertex in  $G(T_{-N})$ ; we choose it as the only root of  $G(T)$ , mark it as non-contractible and mark as contractible all other vertices in  $G(T)$ .

**Example 3.1.** Consider again the surface of Example 2.3 represented in the left-hand side of Figure 2. Focusing for instance our attention on the reconstruction of  $\chi(T), l_1, l_2, q$ , we want to

see how these data (already announced in Example 2.3) are obtained at the end of the iterative procedure in the strips represented in Figure 4.

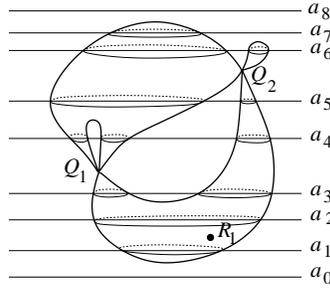


FIGURE 4. Level planes and strips for the iterative reconstruction process.

*Output*( $S_{a_1}$ ):  $S_{a_1} = T_{a_1}$  is a disk, hence  $\chi(T_{a_1}) = [1]$ ,  $l_1 = [0]$ ,  $l_2 = [0]$ ,  $q = [0]$ .

*Output*( $S_{a_2}$ ): we pass through a point which is isolated in  $S$ , hence  $\chi(T_{a_2}) = [1]$ ,  $l_1 = [0]$ ,  $l_2 = [0]$ ,  $q = [1]$  (where we label 1 the region of  $\{z \leq a_2\} \setminus T_{a_2}$  containing the isolated point).

*Output*( $S_{a_3}$ ): passing through a critical point of index 1 influences only the Euler characteristic and we get  $\chi(T_{a_3}) = [0]$ ,  $l_1 = [0]$ ,  $l_2 = [0]$ ,  $q = [1]$ .

*Output*( $S_{a_4}$ ): the strip  $\{a_3 \leq z \leq a_4\}$  contains the singular point  $Q_1$ ;  $T_{a_4}$  is the disjoint union of three disks and, apart from the isolated point  $R_1$ ,  $S_{a_4}$  is homeomorphic to  $T_{a_4}/\mathcal{R}$  where  $\mathcal{R}$  collapses to one point a set of three points lying respectively in the three connected components of  $T_{a_4}$ . Hence  $\chi(T_{a_4}) = [1, 1, 1]$ ,  $l_1 = [1, 1, 1]$ ,  $l_2 = [0, 0, 0]$ ,  $q = [1]$ .

*Output*( $S_{a_5}$ ): two connected components of  $T_{a_4}$  glue together, so that  $T_{a_5}$  is the union of two disks and consequently the length of the lists  $l_1$  and  $l_2$  becomes 2. We get  $\chi(T_{a_5}) = [1, 1]$ ,  $l_1 = [1, 2]$ ,  $l_2 = [0, 0]$ ,  $q = [1]$ .

*Output*( $S_{a_6}$ ): we pass through the singular point  $Q_2$ ;  $T_{a_6}$  is the union of a sphere and two disks;  $Q_2$  is obtained collapsing one point in that sphere with two points chosen respectively in the two disks so that  $\chi(T_{a_6}) = [2, 1, 1]$ ,  $l_1 = [1, 2, 0]$ ,  $l_2 = [1, 1, 1]$ ,  $q = [1]$ .

*Output*( $S_{a_7}$ ): only the Euler characteristics are modified passing through the critical point of index 2 contained in this strip and we get  $\chi(T_{a_7}) = [2, 1, 2]$ ,  $l_1 = [1, 2, 0]$ ,  $l_2 = [1, 1, 1]$ ,  $q = [1]$ .

*Output*( $S_{a_8}$ ): passing through the last critical point of index 2 we get the final expected data

$$\chi(T_{a_8}) = \chi(T) = [2, 2, 2], \quad l_1 = [1, 2, 0], \quad l_2 = [1, 1, 1], \quad q = [1]. \quad \square$$

The remaining part of the section will be devoted to see how it is possible to compute *Output*( $S_b$ ) from *Output*( $S_a$ ) in the Iterative Step described above.

A crucial remark that we want preliminarily to emphasize is that all the needed computations, which have an algebraic nature, can be performed even if we are reconstructing the topology of  $T$  which is not an algebraic surface but a 2-dimensional topological manifold. In particular  $T$  is not defined by a polynomial equation. Nevertheless, as we will see, the only information about  $T$  that we need to compute concerns the behaviour of  $T$  outside the union of the Milnor disks centered at the singular points, where  $T$  coincides with  $S$ ; this will enable us to study  $T$  outside those disks starting from the equation defining  $S$ .

In the iterative step the main algorithm makes use of two special-purpose procedures. At first we need to study the shape of the level curve  $C_b$ . Since  $b$  is a regular value for  $p$ , the affine compact algebraic curve  $C_b$  is non-singular, so that all its connected components are ovals. Recall that an oval is called empty if it contains no other oval in its interior part, and a list  $[\omega_1, \dots, \omega_t]$  of ovals of a curve is called a *nest* of depth  $t$  if  $\omega_1$  is empty,  $\omega_i$  is contained in the interior part of  $\omega_{i+1}$  for

all  $i = 1, \dots, t-1$  (and any other oval containing  $\omega_i$  contains also  $\omega_{i+1}$ ) and  $\omega_t$  is not contained in the interior part of any oval of the curve.

The pair  $(\{z = b\}, C_b)$  is determined up to homeomorphism by the list of its nests or equivalently by the adjacency graph  $G(C_b)$ . The first “black box” algorithmically computes  $G(C_b)$  starting from the equation  $f(x, y, b) = 0$  of  $C_b$ . It is also possible (see for instance [FGPT]), by means of standard techniques, to enrich the curve-algorithm with special functions; namely, for a non-singular curve  $C$ ,

- the function *findRegion*, given a point  $P \in \mathbb{R}^2 \setminus C$ , returns the connected component (or region)  $\text{findRegion}(P)$  of  $\mathbb{R}^2 \setminus C$  containing  $P$ ,
- the function *findOvals*, given a point  $P \in \mathbb{R}^2$ , returns the list of the ovals of  $C$  containing  $P$  ordered by inclusion starting from the innermost oval,
- the function *findPoint*, given an oval  $\omega$  of  $C$ , returns a point lying inside  $\omega$ , more precisely a point  $R$  such that  $\omega$  is the first oval of the sequence  $\text{findOvals}(R)$ .

The second goal in the iterative step is to lift and relate information from the level  $\{z = a\}$  to the level  $\{z = b\}$ . This will be done by means of *connecting paths*: if  $P \in \{a \leq z \leq b\} \setminus S$ , we will denote by *pathUp*( $P, b$ ) (resp. *pathDown*( $P, a$ )) the final point  $\alpha(1)$  of a continuous path  $\alpha : [0, 1] \rightarrow \{a \leq z \leq b\}$  not intersecting  $S$  and such that  $\alpha(0) = P$  and  $\alpha(1) \in \{z = b\}$  (resp.  $\alpha(1) \in \{z = a\}$ ).

Let us now see how, using these techniques, we can reconstruct  $\text{Output}(S_b)$  according to the nature of the “special point”  $Q$  contained in the strip  $\mathbb{R}^2 \times (a, b)$ .

Case 1:  $Q$  is a critical point for  $p$ .

When the strip contains a critical point  $Q$ , using only the index of  $Q$  and computing finitely many connecting paths, it is possible (see [FGPT], [FGL] and [FGLP]) to detect the correspondence among the regions of  $\{z = a\} \setminus C_a$  and those of  $\{z = b\} \setminus C_b$ , and hence to reconstruct  $\chi(T_b)$ ,  $G(T_b)$  and  $M_b$ . As for the lists  $l_i(T_b)$  and  $q(T_b)$ , we observe that

- if  $Q$  has index 0, a new component appears in  $T_b$  so that, for each  $i$ , we have that  $\text{length}(l_i(T_b)) = \text{length}(l_i(T_a)) + 1$ . More precisely,  $l_i(T_b)$  is obtained from  $l_i(T_a)$  inserting a zero in the new additional position, while  $q(T_b) = q(T_a)$ ;
- if  $Q$  has index 1 and passing through it we have the glueing of two distinct connected components of  $T_a$ , say  $\Sigma_1$  and  $\Sigma_2$ , then, for each  $i$ ,  $\text{length}(l_i(T_b)) = \text{length}(l_i(T_a)) - 1$ . Each list  $l_i(T_b)$  is obtained from  $l_i(T_a)$  removing the position corresponding to one of the glued components, say  $\Sigma_2$ , and adding the integer number contained in the cancelled position to the integer appearing in  $l_i(T_a)$  in the position corresponding to the component  $\Sigma_1$ . The glueing of two components goes along with the glueing of two regions of the complement, so it may occur that isolated points lying in different regions of  $\{z \leq a\} \setminus T_a$  lie in the same region of  $\{z \leq b\} \setminus T_b$ : we modify  $q(T_b)$  starting from  $q(T_a)$  accordingly;
- if  $Q$  has index 2 or index 1 but there is no glueing of distinct components of  $T_a$ , then  $l_i(T_b) = l_i(T_a)$  for all  $i$  and  $q(T_b) = q(T_a)$ .

Case 2:  $Q$  is a singular point for  $S$ .

As announced in the introduction, when passing through a singular point we will get the necessary information from the curve  $C(Q, \epsilon) = S \cap S(Q, \epsilon)$  where the surface meets the Milnor disk centered at  $Q = (\alpha, \beta, \gamma)$ . The algebraic curve  $C(Q, \epsilon)$  is not plane, but it can be studied by investigating a plane curve homeomorphic to it. Namely, assuming that the “north pole”  $N = (\alpha, \beta, \gamma + \epsilon)$  of  $S(Q, \epsilon)$  does not lie in  $C(Q, \epsilon)$  and denoting by  $\psi : S(Q, \epsilon) \setminus \{N\} \rightarrow \mathbb{R}^2$  a stereographic projection from  $N$ , the curve  $\tilde{C} = \psi(C(Q, \epsilon))$  is algebraic, compact, non-singular, homeomorphic to  $C(Q, \epsilon)$  and the pairs  $(S(Q, \epsilon) \setminus \{N\}, C(Q, \epsilon))$  and  $(\mathbb{R}^2, \tilde{C})$  are homeomorphic. Up to a linear change of coordinates, we can also assume that  $C(Q, \epsilon)$  is transversal to  $S(Q, \epsilon) \cap \{z = 0\}$ .

A preliminary use of this section curve is to realize whether  $Q$  is an isolated point for the surface, since this happens if and only if  $C(Q, \epsilon)$  (and hence  $\tilde{C}$ ) is empty. In this case the reconstruction of  $Output(S_b)$  is easy, because it is sufficient to lift the data from the level  $a$  to the level  $b$  by means of finitely many connecting paths. In order to update  $q(T_b)$ , we only need to detect the region of  $\{z \leq b\} \setminus T_b$  containing  $Q$ . To do that, we compute the point  $Q' = pathUp(Q, b)$ : if  $findRegion(Q') = \Sigma$  and  $M_b(\Sigma) = R_\Sigma$ , then  $Q$  lies in the region  $R_\Sigma$  of  $\{z \leq b\} \setminus T_b$ .

The case when  $Q$  is singular but not an isolated point in  $S$  is far less trivial. In this situation  $C(Q, \epsilon)$  is non-empty and we know that  $S \cap D(Q, \epsilon)$  is topologically a cone over  $C(Q, \epsilon)$ . In order to compute the topology of  $T_b$ , and hence of  $S_b$ , we will need to determine, for each connected component of the curve  $C(Q, \epsilon)$ , the connected component of  $T_b \setminus D(Q, \epsilon)$  in whose boundary it lies.

Though  $C(Q, \epsilon)$  is not a plane curve, we will call its connected components ovals. The position of the ovals of this curve on the sphere  $S(Q, \epsilon)$  is very important and gives crucial information on the connected components of  $T_b$  that we will use to reconstruct  $Output(S_b)$ . Namely, if  $\omega$  is an oval of  $C(Q, \epsilon)$  and  $S_\omega$  is the connected component of  $(S \setminus D(Q, \epsilon)) \cap \{a \leq z \leq b\}$  that contains  $\omega$  in its boundary, then

- (i) if  $\omega$  is contained in the positive halfsphere  $S(Q, \epsilon) \cap \{z > \gamma\}$ , then  $S_\omega$ , besides  $\omega$ , has in its boundary only another oval  $\omega'$  of  $C_b$ ; attaching a 2-cell along  $\omega$ , there appears in  $T_b$  a new connected component homeomorphic to a disk and with boundary  $\omega'$ ; it is therefore topologically equivalent to passing through a critical point of index 0;
- (ii) if  $\omega$  is contained in the negative halfsphere  $S(Q, \epsilon) \cap \{z < \gamma\}$ , then the boundary of  $S_\omega$  is the union of  $\omega$  and a unique oval  $\omega'$  of  $C_a$ ; the attachment of a 2-cell along  $\omega$  is therefore topologically equivalent to the attachment of a 2-cell along  $\omega'$  and hence equivalent to passing through a critical point of index 2;
- (iii) if  $\omega$  intersects the plane  $\{z = \gamma\}$  (transversally, as precised above), then the boundary of  $S_\omega$  contains both an oval of  $C_a$  and an oval of  $C_b$ ; the component of  $T_b$  containing  $\omega$  intersects the planes  $\{z = a\}$  and  $\{z = b\}$  in those two ovals that it will be necessary to recognize and relate in the process of data lifting.

In other words, passing through a singular point,  $T_b$  can be obtained from  $T_a$  by the simultaneous attachment of a few 0-cells and a few 2-cells with combined effects on the topology of the level surface. This explains why the situation is more complex with respect to passing through a critical point, so that the reconstruction of the data is much more delicate and has to be organized accurately. This is precisely what we are going to do, starting from the analysis of the surface components that meet the Milnor sphere  $S(Q, \epsilon)$ .

**Definition 3.2.** *If  $Q = (\alpha, \beta, \gamma)$ , a subset  $X$  of  $S(Q, \epsilon)$  is called*

- (1) *of type (+) if  $X \subseteq \{z > \gamma\}$*
- (2) *of type (-) if  $X \subseteq \{z < \gamma\}$*
- (3) *of type (+-) if  $X \cap \{z = \gamma\} \neq \emptyset$ .*

If  $\omega$  is an oval of  $C(Q, \epsilon)$  of type (+) (resp. of type (-)),  $\omega$  disconnects the positive (resp. negative) halfsphere containing it into two parts: the one containing the circle  $S(Q, \epsilon) \cap \{z = \gamma\}$  will be called the exterior part of  $\omega$ , while the other one will be called the interior part of  $\omega$ . Using this terminology it is possible to arrange the ovals of  $C(Q, \epsilon)$  of type (+) and those of type (-) in *nests*, extending the usual definition for plane curves recalled above.

- In Section 4 we will see how, via stereographic projection, it is possible
- to compute the list of the nests of type (+) and of type (-) of  $C(Q, \epsilon)$ ,
- for each nest  $[\omega_1, \dots, \omega_n]$  of type (-), to compute a point  $\xi^-$  lying in the interior part of  $\omega_1$ ,
- for each nest  $[\omega_1, \dots, \omega_n]$  of type (+), to compute a point  $\xi^+$  lying in the interior part of  $\omega_1$ ,

– to detect the regions of  $S(Q, \epsilon) \setminus C(Q, \epsilon)$  of type  $(+-)$  and for each such region  $A$  to compute a pair of points  $(\xi^+, \xi^-)$  such that  $\xi^+ \in A \cap \{z > \gamma\}$  and  $\xi^- \in A \cap \{z < \gamma\}$ .

These data will enable us to reconstruct  $Output(S_b)$ .

To some extent the knowledge of the types of all ovals of  $C(Q, \epsilon)$  combined with the observations made above already give some partial information. For instance, as for the effect on the Euler characteristic, each oval of type  $(-)$  (resp. of type  $(+)$ ) is “equivalent” to passing through a critical point of index 2 (resp. of index 0); instead, no effect is caused by ovals of type  $(+-)$ . But the reconstruction of  $G(T_b)$  and  $M_b$ , being related to the embedding of the various components, requires a more careful analysis. For this reason we will proceed in the following fixed order of reconstruction.

First of all we consider the nests of type  $(-)$ . The previous considerations guarantee that, if  $[\omega_1, \dots, \omega_n]$  is a nest of type  $(-)$  of  $C(Q, \epsilon)$ , when using Lemma 2.2 in  $D(Q, \epsilon)$  to construct  $T_b$  in the Milnor disk, we have the attachment of a 2-cell along each of the corresponding  $n$  ovals of  $C_a$ . To determine these ovals, it is sufficient to take a point  $\xi^-$  contained in the interior part of  $\omega_1$  and to compute  $E^- = pathDown(\xi^-, a)$ : the desired  $n$  ovals of  $C_a$  are the first  $n$  ovals of the list  $findOvals(E^-)$ . We can therefore modify the Euler characteristic of the connected components of  $T_a$  having those  $n$  ovals in their boundaries exactly as when we pass through a critical point of index 2. Accordingly we update  $G(T_b)$  and  $M_b$ : in particular the presence of ovals of type  $(-)$  causes no change in  $G(T_b)$ .

As a second step, we consider the regions of  $S(Q, \epsilon) \setminus C(Q, \epsilon)$  of type  $(+-)$ : for each such region  $A$  we use the pair of points  $(\xi^+, \xi^-)$  relative to  $A$  to compute the points  $E^+ = pathUp(\xi^+, b)$  and  $E^- = pathDown(\xi^-, a)$ . If  $findRegion(E^+) = R^+$  and  $findRegion(E^-) = R^-$ , then we realize that  $R^+$  and  $R^-$  bound a connected region of  $\{a \leq z \leq b\} \setminus T_b$  and consequently we set  $M_b(R^+) = M_a(R^-)$ . No change occurs in  $G(T_b)$  and  $\chi(T_b)$ .

The only ovals that we still need to consider are those of type  $(+)$ . We already know that each such oval originates a new connected component in  $T_b$  and that each component is topologically a disk, which is sufficient to update  $\chi(T_b)$ . Also we know that  $G(T_b)$  contains one more vertex and one more edge than  $G(T_a)$  for each oval of type  $(+)$ , but it is necessary to decide how the new edges have to be attached to the graph  $G(T_a)$ . To do that, if  $[\omega_1, \dots, \omega_n]$  is a nest of type  $(+)$  and  $\xi^+$  is a point inside  $\omega_1$ , we compute  $E^+ = pathUp(\xi^+, b)$ . The first  $n$  ovals of  $findOvals(E^+)$  respectively lie in the boundaries of the new components of  $T_b$ . These new components have to be attached to a vertex  $v$  of  $G(T_a)$  as a path formed by  $n$  edges and  $n + 1$  vertices. In order to determine  $v$ , recall that the outermost oval  $\omega_n$  of the nest lies in the closure of a unique region  $A$  of type  $(+-)$  that we have already considered in the previous step of this reconstructive process. If we have found that the pair of connecting paths starting from the points  $(\xi^+, \xi^-)$  in  $A$  relates the regions  $R^+$  of  $\{z = b\} \setminus C_b$  and  $R^-$  of  $\{z = a\} \setminus C_a$ , then the  $n$ -length path corresponding to the nest  $[\omega_1, \dots, \omega_n]$  has to be attached to  $G(T_a)$  in the vertex  $M_a(R^-)$ .

We have so completed the computation of  $\chi(T_b)$ ,  $G(T_b)$  and of the action of  $M_b$  on the vertices and edges of  $G(C_b)$  reached by means of the connecting paths starting from points in  $S(Q, \epsilon)$  computed so far. At this point it is sufficient to complete the reconstruction of  $M_b$  via connecting paths starting from the regions of  $\{z = a\} \setminus C_a$  not reached in the previous steps of this procedure.

A schematic representation of the procedure just explained to reconstruct  $G(T_b)$  and  $M_b$  passing through an isolated singularity is shown in Figure 5.

As for  $q(T_b)$ , since  $G(T_a)$  is a subgraph of  $G(T_b)$  and no regions were glued together, we have that  $q(T_b) = q(T_a)$ .

The only remaining task is the computation of the lists  $l_1(T_b), \dots, l_m(T_b)$ . Since the length of each of them coincides with the number of connected components of  $T_b$ , each list  $l_i(T_b)$  has a length equal to  $length(l_i(T_a))$  increased by the number of ovals of  $C(Q, \epsilon)$  of type  $(+)$ . Moreover, if  $Q$  is

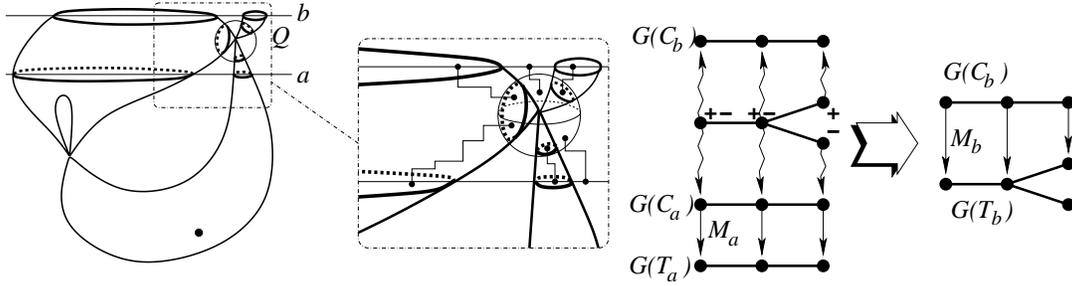


FIGURE 5. Reconstruction of  $G(T_b)$  and  $M_b$  passing through an isolated singular point; the wavy lines represent the connecting paths starting from the six sample points on the Milnor sphere centered at the singularity.

the  $j$ -th element of the list  $[Q_1, \dots, Q_m]$ , then we reconstruct the lists  $l_i(T_b)$  as follows:

1) for all  $i \neq j$ ,  $l_i(T_b)$  is obtained from  $l_i(T_a)$  appending to the list  $l_i(T_a)$  as many zeros as the number of ovals of  $C(Q, \epsilon)$  of type (+),

2)  $l_j(T_b)$  is obtained from  $l_j(T_a)$  appending as many integers 1 as the number of ovals of  $C(Q, \epsilon)$  of type (+) and then iteratively, for each oval of  $C(Q, \epsilon)$  of type (-) and of type (+-), increasing by 1 the integer that appears in the position relative to the component of  $T_b$  containing that oval in its boundary.

#### 4. PRELIMINARY TESTS AND COMPUTATIONS

In the previous section we have seen how it is possible to compute the set of data  $D(S)$  assuming that  $S$  is an affine surface in  $\mathbb{R}^3$  having at most isolated singularities, that the working system of coordinates is a good frame and assuming to have already computed the singular points, the critical points and their indexes and the other needed data concerning the behaviour of the surface locally at the singular points. In this section we describe how these preliminary tests and computations can be performed.

##### 1. Computation of the singular points.

Using the notation of the previous section, assume that  $S$  is given as an affine surface in  $\mathbb{R}^3$  by means of the defining equation  $f(x, y, z) = 0$  with  $f$  a square-free polynomial with real coefficients.

Denote by  $J = (f, f_x, f_y, f_z)$  the ideal generated by  $f$  and its first partial derivatives, and by  $V(J)$  the set of the complex zeros of  $J$ . We need to decide whether the set  $V_{\mathbb{R}}(J)$  of the real zeros of  $J$  contains only finitely many points and, in this case, to compute all of them. This can be done by suitably modifying the procedure described in [FGPT], where the problem was deciding about the emptiness of  $V_{\mathbb{R}}(J)$ .

Since  $f$  is square-free,  $J$  cannot have dimension 2; if it has dimension 0 we can compute the finitely many points in  $V(J)$  by means of any of the methods available in the literature and then select the real ones.

If  $J$  has dimension 1,  $V(J)$  is a complex curve in  $\mathbb{C}^3$ . Up to a generic linear change of coordinates we can assume that the projection  $\pi : \mathbb{C}^3 \rightarrow \mathbb{C}^2$ ,  $\pi(x, y, z) = (y, z)$  is a “good projection” for  $V(J)$ , i.e. the restriction  $\pi|_{V(J)} : V(J) \rightarrow \pi(V(J))$  is finite and 1-1 except for at most a finite number of points. In [FGP] one can find a method, based on the use of the reduced lex Gröbner basis for  $J$ , to test whether  $\pi$  is a good projection. The Zariski closure of  $\pi(V(J))$  is defined by the ideal  $I = J \cap \mathbb{C}[y, z]$  which is the product of a principal ideal ( $g$ ) generated by the  $gcd$  of a set

of generators of  $I$  and a zero-dimensional ideal  $I_0$  obtained from  $I$  by dividing out  $g$  from the generators of  $I$ .

The real points in  $V(J)$  project to real points in  $V(I)$ , that is  $\pi(V_{\mathbb{R}}(J)) \subseteq V_{\mathbb{R}}(I) = V_{\mathbb{R}}(I_0) \cup V_{\mathbb{R}}(g)$ , but the inclusion can be strict because some complex points in  $V(J)$  can project to real points in  $V_{\mathbb{R}}(I)$ . Since the projection  $\pi$  is good, this can happen only for finitely many points. As a consequence,  $V_{\mathbb{R}}(g)$  cannot contain any 1-dimensional connected component intersecting the line at infinity, because  $S$  has no point at infinity and therefore  $V_{\mathbb{R}}(J) \cap \{t = 0\} = \emptyset$ . Moreover, the surface  $S$  has at most isolated real singularities if and only if  $V_{\mathbb{R}}(g)$  contains no 1-dimensional compact components.

Note that, without changing  $V(g)$ , we can assume that  $g$  is square-free, so that  $V(g)$  has at most finitely many singular points; moreover, since  $V_{\mathbb{R}}(g)$  cannot contain any real line, dividing  $g$  by its univariate factors in  $z$  (necessarily without real roots) we can also assume that  $V(g)$  does not contain any 1-dimensional irreducible component of critical points with respect to the projection to the  $z$ -axis  $\sigma : \mathbb{C}^2 \rightarrow \mathbb{C}$ ,  $\sigma(y, z) = z$ . The zero-set defined by the ideal  $K = (g, \frac{\partial g}{\partial y}) \subseteq \mathbb{C}[y, z]$  contains the points of  $V(g)$  which are either singular for  $V(g)$  or critical with respect to the projection  $\sigma$ . Thus we can assume that the dimension of  $K$  is at most 0, so that we can easily compute  $V_{\mathbb{R}}(K)$ .

If  $V_{\mathbb{R}}(g)$  contains some 1-dimensional compact component, then it necessarily contains some singular points or some critical points with respect to  $\sigma$ . Then, in order to check that  $V_{\mathbb{R}}(g)$  contains no 1-dimensional components, it is sufficient to check that  $V_{\mathbb{R}}(g) = V_{\mathbb{R}}(K)$ .

If  $\omega_1, \dots, \omega_h$  are the points in  $\sigma(V_{\mathbb{R}}(K))$  and we choose  $\eta_0, \dots, \eta_h \in \mathbb{Q}$  such that  $\eta_0 < \omega_1 < \eta_1 < \omega_2 < \dots < \eta_{h-1} < \omega_h < \eta_h$ , then  $V_{\mathbb{R}}(g) = V_{\mathbb{R}}(K)$  if and only if  $V_{\mathbb{R}}(g) \cap \{z = \eta_i\} = \emptyset$ , i.e. if and only if, for all  $i$ , the equation  $g(y, \eta_i) = 0$  has no real roots, which is easy to check. If that is true, we are sure that  $S$  has only finitely many real singular points that lie in the fibers over the points in  $V_{\mathbb{R}}(I_0 \cdot K)$ ; in order to compute them it is sufficient to compute the points in  $V_{\mathbb{R}}(I_0 \cdot K, J)$ , where the ideal  $(I_0 \cdot K, J) \subseteq \mathbb{C}[x, y, z]$  is at most zero-dimensional since  $\pi$  has finite fibers.

## 2. Good frame test and computation of the critical points.

We want now to test that all (real) critical points for  $p$  are non-degenerate and, if so, to compute them.

The zero-set defined in  $\mathbb{C}^3$  by the ideal  $K = (f, f_x, f_y)$  contains all the singular points of  $S_{\mathbb{C}} = \{f = 0\} \subset \mathbb{C}^3$  and all the critical points of the projection  $S_{\mathbb{C}} \subset \mathbb{C}^3 \rightarrow \mathbb{C}$ ,  $p(x, y, z) = z$ .

Let  $u$  denote the product of all the univariate factors of  $f$  in the variable  $z$ . Then  $u$  cannot have any real root  $z_0$ , because otherwise the plane  $\{z = z_0\}$  would be contained in  $S$  which has no points at infinity. Thus, dividing  $f$  by  $u$  (which does not modify the real zero-set), we can assume that  $f$  is not divisible by any univariate polynomial in  $z$ .

As a consequence, the ideal  $K$  cannot have dimension 2: otherwise, if  $h = 0$  is the equation of a 2-dimensional irreducible component of  $V(K)$ , since  $h$  divides  $f, f_x$  and  $f_y$ , then  $h$  would be a univariate polynomial in  $z$  dividing  $f$  which cannot exist after our previous reduction.

A critical point  $P \in S$  is degenerate for  $p$  if it annihilates the function  $D(x, y, z) = \det H$ , where  $\det H$  denotes the determinant of the matrix  $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$ . Thus  $p$  is a Morse function if and only if  $V_{\mathbb{R}}(K, D) \subseteq V_{\mathbb{R}}(J)$ . In order to test whether this condition holds, it is helpful to use the following result (for a proof see [FGPT], Proposition 6.5)

**Lemma 4.1.** *Any point  $P$  lying in a 1-dimensional component of  $V(K)$  is necessarily either singular for  $S_{\mathbb{C}}$  or degenerate.*

Hence, if we remove from  $V(K)$  the points lying in  $V(J) \cup V(D)$ , we remove from  $V(K)$  all the 1-dimensional components; in other words the ideal  $L$  defining the set  $V(K) \setminus (V(J) \cup V(D)) = V(K) \setminus V(f_z \cdot D)$  is zero-dimensional and  $V_{\mathbb{R}}(L)$  contains exactly the real non-degenerate critical points. Recall that  $L$  can be easily computed by saturating  $K$  with respect to  $(f_z \cdot D)$ .

If we denote by  $\#A$  the number of elements of a finite set  $A$ , we get

**Proposition 4.2.** *The projection  $p$  is a Morse function if and only if  $V_{\mathbb{R}}(K)$  is a finite set and  $\#V_{\mathbb{R}}(K) = \#V_{\mathbb{R}}(L) + \#V_{\mathbb{R}}(J)$ .*

*Proof.* We can split  $V(K)$  as

$$V(K) = V(L) \cup V(J) \cup V(K, D)$$

where  $V(L) \cap V(J) = \emptyset$  by construction. Since both  $V(L)$  and  $V_{\mathbb{R}}(J)$  are finite sets, from the previous splitting we get that  $V_{\mathbb{R}}(K, D)$  is finite if and only if  $V_{\mathbb{R}}(K)$  is finite.

If  $V_{\mathbb{R}}(K)$  is not finite, then  $V_{\mathbb{R}}(K, D)$  cannot be contained in  $V_{\mathbb{R}}(J)$  and hence  $p$  is not a Morse function. If  $V_{\mathbb{R}}(K)$  is finite, then  $V_{\mathbb{R}}(K, D) \subseteq V_{\mathbb{R}}(J)$  if and only if  $\#V_{\mathbb{R}}(K) = \#V_{\mathbb{R}}(L) + \#V_{\mathbb{R}}(J)$ .  $\square$

Since we can check whether  $V_{\mathbb{R}}(K)$  is a finite set by means of the procedure used to investigate  $V_{\mathbb{R}}(J)$  in the previous Step 1, Proposition 4.2 gives a method to test whether  $p$  is a Morse function. If this is true, we need only to compute  $V_{\mathbb{R}}(L)$ , which contains exactly the (real) critical points of  $p$ , and the indexes of these points. Additional remarks concerning the possibility of computing only the real critical values and their indexes via eigenvalue computations, avoiding the whole computation of the real critical points, can be found in Section 5 of [FGLP].

### 3. Computation of the radius of a Milnor disk at a singular point.

Let  $Q = (\alpha, \beta, \gamma)$  be a (real) singular point of  $S$ ; by Theorem 2.1 there exists a real  $r_0 > 0$  such that, for all positive  $\epsilon \leq r_0$ ,  $S \cap D(Q, \epsilon)$  is homeomorphic to the cone over  $S \cap S(Q, \epsilon)$ . In order to compute such a radius it suffices to compute an  $r_0$  such that the disk  $D(Q, r_0)$  contains neither singular points of  $S$  nor critical points for the function  $\rho : S \rightarrow \mathbb{R}$ ,  $\rho(x, y, z) = (x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2$  except  $Q$  itself.

The points that are either singular for  $S$  or critical for  $\rho$  are the points  $P = (x, y, z) \in S$  such that the rank of the matrix  $M = \begin{pmatrix} f_x & f_y & f_z \\ x - \alpha & y - \beta & z - \gamma \end{pmatrix}$  is lower or equal to 1. If we denote by  $M_1, M_2, M_3$  the determinants of the three square submatrices of order 2 of  $M$ , then the mentioned points are the real solutions of the system of four equations  $f = 0, M_1 = 0, M_2 = 0, M_3 = 0$ .

Let  $d(x, y, z, r) = r - (x - \alpha)^2 - (y - \beta)^2 - (z - \gamma)^2$  and consider the ideal  $G = (f, M_1, M_2, M_3, d) \subset \mathbb{C}[x, y, z, r]$ . Then  $V(G) = \Sigma \cup \Gamma \subset \mathbb{C}^4$ , where

$$\Sigma = \{(P, \rho(P)) \mid P \in \text{Sing } S_{\mathbb{C}}\} \quad \text{and}$$

$$\Gamma = \{(P, \rho(P)) \mid f(P) = 0, P \notin \text{Sing } S_{\mathbb{C}} \text{ and } P \text{ is a critical point for } \rho\}.$$

If  $\sigma : \mathbb{C}^4 \rightarrow \mathbb{C}$  is the projection defined by  $\sigma(x, y, z, r) = r$ , then  $0 \in \sigma(V(G))$  since  $(Q, 0) \in V(G)$ ; we look for a real positive  $r_0$  such that  $\{r \in \mathbb{R} \mid 0 < r < r_0\}$  does not contain any point in  $\sigma(V_{\mathbb{R}}(G))$ .

At first we can compute  $\delta \in \mathbb{R}^+$  such that  $\{z \in \mathbb{C} \mid 0 < |z| < \delta\} \cap \sigma(\Gamma) = \emptyset$ . Namely, by Milnor's result on the critical values of polynomial maps already recalled, the set  $\sigma(\Gamma)$  is finite in  $\mathbb{C}$ ; in particular it coincides with its Zariski closure  $\overline{\sigma(\Gamma)}^Z$ . Then  $\sigma(\Gamma) \subseteq \sigma(\overline{\Gamma}^Z) \subseteq \overline{\sigma(\Gamma)}^Z = \sigma(\Gamma)$  and hence  $\sigma(\Gamma) = \sigma(\overline{\Gamma}^Z)$ .

Since  $\overline{\Gamma}^Z = \overline{V(G) \setminus \Sigma}^Z$  and  $\Sigma = V(J, d)$ , the algebraic set  $\overline{\Gamma}^Z$  is the zero-set of the ideal  $I_{\Gamma}$  obtained by saturating  $G$  with respect to the ideal  $(J, d)$ . Hence  $\sigma(\Gamma) = \sigma(\overline{\Gamma}^Z)$  is the zero-set of the elimination ideal  $I_{\Gamma} \cap \mathbb{C}[r]$  that we can easily compute. It is then sufficient to take  $\delta = \min\{|\theta| : \theta \in V(I_{\Gamma} \cap \mathbb{C}[r]), \theta \neq 0\}$ .

Thus the interval  $(0, \delta) \subset \mathbb{R}$  does not contain any point in  $\sigma(\Gamma_{\mathbb{R}})$ , but it can still possibly contain some point in  $\sigma(\Sigma_{\mathbb{R}})$ . We can easily avoid this: since  $\Sigma_{\mathbb{R}}$  is finite (by our hypothesis that the real singular locus  $\text{Sing } S$  is finite), it is sufficient to compute  $\eta = \min\{\|Q - P_i\| \mid P_i \in \text{Sing } S, P_i \neq Q\}$  and to take  $r_0 = \min(\delta, \eta)$ .

#### 4. Computation of the “sample points” on a Milnor sphere.

The reconstruction of  $Output(S_b)$  when passing through a singular point  $Q$  used the ability to decide whether an oval on the Milnor sphere  $S(Q, \epsilon)$  and a region of  $S(Q, \epsilon) \setminus C(Q, \epsilon)$  is of type  $(+), (-)$  or  $(+-)$  and the ability to choose on the sphere some “sample points” to be used as starting points for connecting paths. We want now to see how all this can be done effectively.

Using the notation introduced in Section 3, if  $\psi : S(Q, \epsilon) \setminus \{N\} \rightarrow \mathbb{R}^2$  is the stereographic projection that transforms the circle  $S(Q, \epsilon) \cap \{z = \gamma\}$  onto the unit sphere  $S^1 \subset \mathbb{R}^2$ , the ovals of  $C(Q, \epsilon)$  of type  $(-)$  (resp. of type  $(+)$ ) are mapped onto ovals of  $\tilde{C} = \psi(C(Q, \epsilon))$  internal to  $S^1$  (resp. external to  $S^1$ ), while the images of the ovals of type  $(+-)$  intersect transversally  $S^1$ . A similar correspondence holds for the regions of  $S(Q, \epsilon) \setminus C(Q, \epsilon)$ .

This allows us to translate our problem on  $C(Q, \epsilon)$  into the analogous one on  $\tilde{C}$  and to solve it working with a plane curve.

Here we can compute the needed data through the following steps:

- 1) by means of the curve-algorithm compute the nests of  $\tilde{C}$ ,
- 2) compute the points in  $\tilde{C} \cap S^1$  using for instance a rational parametrization of  $S^1$ ,
- 3) choose a point in each of the arcs of  $S^1 \setminus \tilde{C}$  and collect them into a set  $\mathcal{M} = \{M_1, \dots, M_k\}$ ,
- 4) for each  $M_i \in \mathcal{M}$  compute the region  $findRegion(M_i)$  of  $\mathbb{R}^2 \setminus \tilde{C}$  containing it. Since distinct points  $M_i, M_j$  may belong to the same region, remove duplicates from  $\mathcal{M}$  so that distinct points lie in different regions. Denote by  $R^{+-} = \{R_1, \dots, R_t\}$  the set of regions so found: they are precisely the regions of type  $(+-)$ . Since an oval is of type  $(+-)$  if and only if the two adjacent regions are both  $(+-)$ , looking at the graph  $G(\tilde{C})$  we determine also the ovals of type  $(+-)$ ,
- 5) moving a little each point  $M \in \mathcal{M}$  outside and inside  $S^1$ , following for instance a radius, we compute a pair  $(\xi^+, \xi^-)$  in each region of type  $(+-)$ ,
- 6) for each nest  $n = [\sigma_1, \dots, \sigma_h]$  of  $\tilde{C}$ , by means of  $findPoint$ , choose a point  $P(n)$  in the center of the nest (i.e. inside the innermost oval of  $n$ ). If we denote by  $R(\sigma_i)$  the region comprised between  $\sigma_{i-1}$  and  $\sigma_i$ , let  $j$  be the least integer in  $\{1, \dots, h\}$  such that  $R(\sigma_j)$  is a region  $(+-)$ . Then the list of ovals  $[\sigma_1, \dots, \sigma_{j-1}]$  is a nest of type either  $(+)$  or  $(-)$ : precisely it is of type  $(+)$  if  $\|P(n)\| > 1$ , of type  $(-)$  otherwise. Respectively we choose  $P(n)$  as  $\xi^+$  or  $\xi^-$ ,
- 7) all the ovals of  $\tilde{C}$  not found so far are ovals containing  $S^1$  in their interior part, thus they are of type  $(+)$ . In particular they are necessarily ordered by inclusion into a unique list  $[\eta_1, \dots, \eta_p]$  that can be computed removing for instance from  $findOvals(0)$  the ovals already found in the previous steps. Then the list  $[\psi^{-1}(\eta_p), \dots, \psi^{-1}(\eta_1)]$  is a nest of type  $(+)$  on  $S(Q, \epsilon)$  for which we can choose the point  $N$  as  $\xi^+$ .

#### 5. THE GENERAL CASE AND EXAMPLES

The Affine-Case-Algorithm described in Section 3 can be used to compute the set of data  $D(S)$  only when  $S$  is contained in an affine chart of  $\mathbb{R}P^3$ . In this section we show that in the general case we can achieve the same goal constructing an affine real algebraic surface  $\hat{S} \subset \mathbb{R}^3$ , computing  $D(\hat{S})$  by means of the Affine-Case-Algorithm and then recovering  $D(S)$  from  $D(\hat{S})$ .

The strategy is the same already used in [FGL] and [FGLP] respectively to compute the topological type and the weighted adjacency graph in the case of a non-singular algebraic surface. Here we see that the previous construction can be helpful even when  $S$  has isolated singularities and can be used to compute also the lists  $l_1, \dots, l_m$  and  $q$  containing the needed information about the singularities of  $S$ . In the same spirit of the previous sections we only briefly recall the essential features of the construction of  $\hat{S}$  and some of its properties that can be found in detail in the two papers mentioned above; here we focus our attention on the new aspects due to the presence of the singularities and on the way to “descend” the data in  $D(\hat{S})$  concerning the singular points to recover the lists  $l_1, \dots, l_m, q$ .

Denote by  $\pi : S^3 \rightarrow \mathbb{RP}^3$  the map that associates to any point  $(x, y, z, t)$  of the 3-sphere  $S^3$  the point of homogeneous coordinates  $[x, y, z, t]$  in  $\mathbb{RP}^3$ ; then each fiber contains two antipodal points on the sphere and  $S^3$  turns out to be a 2-sheeted covering space of  $\mathbb{RP}^3$ . If  $S$  is defined by the homogeneous equation  $F(x, y, z, t) = 0$  and we lift  $S$  through  $\pi$ , the surface

$$\tilde{S} = \pi^{-1}(S) = \{(x, y, z, t) \in \mathbb{R}^4 \mid F(x, y, z, t) = 0\} \cap S^3$$

is invariant with respect to the antipodal map  $ap : S^3 \rightarrow S^3$  defined by  $ap(v) = -v$ . If  $(0, 0, 0, 1) \notin \tilde{S}$  (what we can assume up to an affine translation of  $S$ ) and if  $\varphi : S^3 \setminus \{(0, 0, 0, 1)\} \rightarrow \mathbb{R}^3$  denotes the stereographic projection given by  $\varphi(x, y, z, t) = (\frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t})$ , then the image  $\hat{S} = \varphi(\tilde{S})$  is a compact algebraic surface in  $\mathbb{R}^3$ , homeomorphic to  $\tilde{S}$  and defined implicitly by the polynomial equation  $F(2X, \|X\|^2 - 1) = 0$  where  $X = (x, y, z)$ . Furthermore, if  $inv = \varphi \circ ap \circ \varphi^{-1} : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{R}^3 \setminus \{0\}$  denotes the involution  $inv(X) = -\frac{X}{\|X\|^2}$  corresponding to  $ap$  via the stereographic projection, then  $\hat{S}$  is invariant with respect to  $inv$ .

In [FGL] and [FGLP] it was shown that, when  $S$  is non-singular, the ability to recognize the action of  $inv$  on the set of the connected components of  $\hat{S}$  and on the set of the regions of  $\mathbb{R}^3 \setminus \hat{S}$ , together with the topology of  $\hat{S}$  and the adjacency graph  $G(\hat{S})$ , is sufficient to compute  $\chi(S)$  and the weighted adjacency graph of  $S$ .

In the case we are examining, when  $S$  has at most isolated singularities, in order to compute  $D(S)$  we make use of the topological surface  $T$  obtained from  $S$  applying the modifications of Lemma 2.2 inside the Milnor disks at the singularities of  $S$  that are not isolated points. Also the topological surface  $\varphi(\pi^{-1}(T))$  is invariant with respect to  $inv$ ; more precisely  $inv$  induces an involution on the set  $\mathcal{F}$  of the connected components of  $\varphi(\pi^{-1}(T))$  and on the set  $\mathcal{R}$  of the regions of  $\mathbb{R}^3 \setminus \varphi(\pi^{-1}(T))$ .

Hence we can split  $\mathcal{F}$  as the union of  $\mathcal{F}_1 \cup \mathcal{F}_2$ , where

$$\mathcal{F}_1 = \{\hat{Y} \in \mathcal{F} \mid inv(\hat{Y}) = \hat{Y}\} \quad \text{and} \quad \mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$$

and split  $\mathcal{R}$  as the union of  $\mathcal{R}_1 \cup \mathcal{R}_2$ , where

$$\mathcal{R}_1 = \{\hat{\Sigma} \in \mathcal{R} \mid inv(\hat{\Sigma}) = \hat{\Sigma}\} \quad \text{and} \quad \mathcal{R}_2 = \mathcal{R} \setminus \mathcal{R}_1.$$

Our *descending procedure* to derive the needed data relative to  $T$  (and hence to  $S$ ) from the data on  $\varphi(\pi^{-1}(T))$  is based on the following characterization:

**Proposition 5.1.** *Let  $Y$  be a connected component of  $T$  and  $\Sigma$  a region of  $\mathbb{RP}^3 \setminus T$ . Then*

- (1)  $\varphi(\pi^{-1}(Y))$  is either a connected component of  $\varphi(\pi^{-1}(T))$  (so that it belongs to  $\mathcal{F}_1$ ) or it is the union of two distinct connected components of  $\varphi(\pi^{-1}(T))$  transformed each into the other by  $inv$ ,
- (2)  $Y$  is non-contractible if and only if  $\varphi(\pi^{-1}(Y)) \in \mathcal{F}_1$ ,
- (3)  $\Sigma$  is non-contractible if and only if  $\varphi(\pi^{-1}(\Sigma)) \in \mathcal{R}_1$ .

The previous results were proved in the mentioned papers [FGL] and [FGLP] for the components and regions of a non-singular algebraic surface; since the proof uses only the fact that  $(S^3, \pi, \mathbb{RP}^3)$  is a double covering, it holds also for singular surfaces and even for topological 2-manifolds contained in  $\mathbb{RP}^3$ .

Since  $S$  has only isolated singularities, also the singularities of  $\hat{S}$  are isolated, so that we can compute  $D(\hat{S})$  by means of the Affine-Case-Algorithm; we obtain these data from the study of the topological 2-manifold  $\hat{T}$  associated to  $\hat{S}$  after applying the modification of Lemma 2.2 inside the Milnor disks at the singularities of  $\hat{S}$ . Note that  $\hat{T}$  is homeomorphic to  $\varphi(\pi^{-1}(T))$  and also the pairs  $(\mathbb{R}^3, \hat{T})$  and  $(\mathbb{R}^3, \varphi(\pi^{-1}(T)))$  are homeomorphic.

The fact that the pairs  $(\mathbb{R}^3, \hat{T})$  and  $(\mathbb{R}^3, \varphi(\pi^{-1}(T)))$  are homeomorphic is very important because it allows us to recover  $\chi(T)$ ,  $G(T)$ ,  $w_T$  and  $r(T)$  by means of a “descending procedure” based on the

properties of  $\varphi(\pi^{-1}(T))$  with respect to  $inv$  but using the data relative to the surface  $\widehat{T}$  computed by the Affine-Case-Algorithm. In particular the procedure to split the set  $\mathcal{F}$  of the connected components of  $\varphi(\pi^{-1}(T))$  and the set  $\mathcal{R}$  of the regions of its complement can be performed working with the connected components of  $\widehat{T}$ . The procedure, described in detail in the previous papers, is based on the investigation of the plane level curve  $\widehat{T} \cap \{z = 0\}$ . Since we can assume that 0 is neither a critical value for the projection  $p$  nor a singular value for  $\widehat{S}$ , we can choose 0 as one of the levels  $a_i$  to be studied in the iterative procedure applied to  $\widehat{S}$ . In particular  $\widehat{T} \cap \{z = 0\}$  coincides with  $\widehat{S} \cap \{z = 0\}$  and it is a non-singular algebraic curve, invariant w.r.t.  $inv$ .

The only data of  $D(S)$  that we still need to compute are the lists  $l_1, \dots, l_m, q$  that we want to derive from the analogous lists computed for  $\widehat{S}$  by means of  $\widehat{T}$ . The reconstruction of  $q(T)$  is straightforward: if  $R$  is an isolated point for  $S$  contained in a region  $\Sigma$  of  $\mathbb{R}\mathbb{P}^3 \setminus T$ , then  $\varphi(\pi^{-1}(R))$  consists of two points that lie in the same region of  $\mathbb{R}^3 \setminus \widehat{T}$  if  $\varphi(\pi^{-1}(\Sigma)) \in \mathcal{R}_1$  (i.e. it is connected) or that lie in different components if  $\Sigma$  splits into two regions of  $\mathcal{R}_2$ . Since we know the sets  $\mathcal{R}_1$  and  $\mathcal{R}_2$ , accordingly we recover the list  $q(T)$  of length  $s$  from the list  $q(\widehat{T})$  of length  $2s$ .

Also the number of singularities that are not isolated points doubles when passing from  $S$  to  $\widehat{S}$ , hence for  $\widehat{S}$  we get  $2m$  lists  $\widehat{l}_1, \dots, \widehat{l}_{2m}$ . Up to reordering we can assume that  $\widehat{l}_1, \dots, \widehat{l}_m$  are the lists relative to the points  $\widehat{Q}_1, \dots, \widehat{Q}_m$  and  $\widehat{l}_{m+1}, \dots, \widehat{l}_{2m}$  are relative to the points  $inv(\widehat{Q}_1), \dots, inv(\widehat{Q}_m)$ .

Let us see how, for instance, from the lists  $\widehat{l}_1$  and  $\widehat{l}_{m+1}$  we recover the list  $l_1$ .

If  $\mathcal{F}_1$  contains  $h$  components, say  $\widehat{Y}_1, \dots, \widehat{Y}_h$ , and  $\mathcal{F}_2$  contains  $2k$  components  $\widehat{Y}_{h+1}, \dots, \widehat{Y}_{h+k}$ ,  $inv(\widehat{Y}_{h+1}), \dots, inv(\widehat{Y}_{h+k})$ , then the length of both  $\widehat{l}_1$  and  $\widehat{l}_{m+1}$  is  $h + 2k$ , while the length of  $l_1$  will be  $h + k$ . For simplicity assume that  $\widehat{l}_1$  contains in the first  $h$  positions the data relative to the components in  $\mathcal{F}_1$ , in the successive  $k$  positions the data relative to  $\widehat{Y}_{h+1}, \dots, \widehat{Y}_{h+k}$  and in the last  $k$  positions the data relative to  $inv(\widehat{Y}_{h+1}), \dots, inv(\widehat{Y}_{h+k})$ . In particular (if we denote by  $l(i)$  the  $i$ -th element in a list  $l$ ) we have that  $\widehat{l}_1(j) = \widehat{l}_{m+1}(j)$  for each  $j = 1, \dots, h$  and, because of the pairing induced in  $\mathcal{F}_2$  by  $inv$ , that  $\widehat{l}_1(h + j) = \widehat{l}_{m+1}(h + k + j)$  and  $\widehat{l}_1(h + k + j) = \widehat{l}_{m+1}(h + j)$  for all  $j = 1, \dots, k$ .

Then it is easy to see that the list  $l_1$  has to be filled according to the following rule

$$\begin{aligned} l_1(j) &= \widehat{l}_1(j) = \widehat{l}_{m+1}(j) \quad \forall j = 1, \dots, h \quad \text{and} \\ l_1(h + j) &= \widehat{l}_1(h + j) + \widehat{l}_1(h + k + j) \quad \forall j = 1, \dots, k. \end{aligned}$$

Before exemplifying the descending procedure on some simple non-affine surfaces, let us conclude the paper with a brief schematic summary of how our algorithm and its two main functions work.

#### AFFINE-CASE-ALG

Input:  $f(x, y, z) = 0$  with  $f$  a square-free polynomial in  $\mathbb{Q}[x, y, z]$

Output:  $D(S) = [\chi(T), G(T), w_T, r(T), l_1, \dots, l_m, q]$  if  $S$  has at most isolated singularities, error otherwise.

- Compute the singular locus: if  $S$  has non-isolated singularities, error
- Compute the real critical points, check that  $(x, y, z)$  is a good frame (otherwise perform a linear change of coordinates and start again) and compute the indexes of the critical points
- Split  $[-N, N] = [-N = a_0, a_1] \cup [a_1, a_2] \cup \dots \cup [a_u, a_{u+1} = N]$
- Initialize  $Output(S_{-N})$
- for  $i = 1, \dots, u$  repeat  $Output(S_{a_i}) = lift(Output(S_{a_{i-1}}))$   
(the lifting and reconstruction process is explained in Section 3)
- Compute  $r(T)$  choosing as root of  $G(T_N)$  the only vertex in  $G(T_{-N})$
- Compute  $w_T$ : mark the root as non-contractible vertex and mark all other vertices as contractible

- Assemble the list  $D(S)$  extracting  $\chi(T), G(T), l_1, \dots, l_m, q$  from  $Output(S_N)$  and completing with  $r(T)$  and  $w_T$ .

#### GENERAL-CASE-ALG

Input:  $F(x, y, z, t) = 0$  with  $F$  a homogeneous square-free polynomial in  $\mathbb{Q}[x, y, z, t]$

Output:  $D(S) = [\chi(T), G(T), w_T, r(T), l_1, \dots, l_m, q]$  if  $S$  has at most isolated singularities, error otherwise.

- If the curve  $C_\infty = \{F(x, y, z, 0) = 0\}$  is empty, then  
 $D(S) = \text{AFFINE-CASE-ALG}(F(x, y, z, 1))$
- else  
 $f(x, y, z) = F(2X, \|X\|^2 - 1)$  with  $X = (x, y, z)$   
 $D(\hat{S}) = \text{AFFINE-CASE-ALG}(f(x, y, z))$   
 $D(S) = \text{descend } D(\hat{S})$  (the descending process is explained in Section 5)

**Example 5.2.** Consider the surface represented in the left-hand side of Figure 6 consisting of a cone and two isolated points.

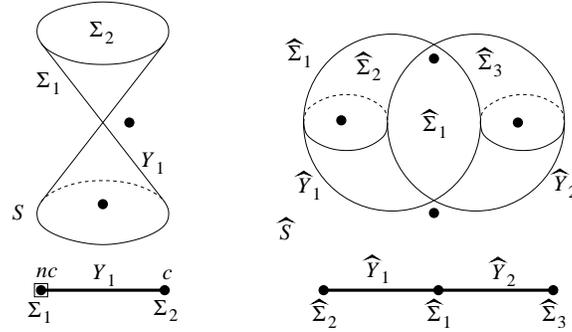


FIGURE 6. An even degree non-affine surface.

The right-hand side represents the doubled surface  $\hat{S}$  and the adjacency graph  $G(\hat{T})$ , where  $\hat{T}$  is the union of two spheres. Note that the symbols used in the figure, according to the notation used in this section, indicate the components and regions of  $T$  and  $\hat{T}$ , even if these surfaces are not represented in the figure.

Using the method described above we get the following data concerning  $\hat{T}$ :

Regions:  $inv(\hat{\Sigma}_1) = \hat{\Sigma}_1$  and  $inv(\hat{\Sigma}_2) = \hat{\Sigma}_3$ ; hence, labelling by means of the index  $i$  each region  $\hat{\Sigma}_i$ , we have  $\mathcal{R}_1 = \{1\}$  and  $\mathcal{R}_2 = \{2, 3\}$

Components:  $inv(\hat{Y}_1) = \hat{Y}_2$ ; hence, again labelling by means of the index  $i$  each connected component  $\hat{Y}_i$ , we get  $\mathcal{F}_1 = \emptyset$  and  $\mathcal{F}_2 = \{1, 2\}$ . Moreover  $\chi(\hat{T}) = [2, 2]$

Singularities:  $\hat{l}_1 = [1, 1]$ ,  $\hat{l}_2 = [1, 1]$ ,  $q(\hat{T}) = [1, 1, 2, 3]$ .

The descending procedure yields  $\chi(T) = [2]$ ,  $l_1 = [2]$ ,  $q(T) = [1, 2]$ , so we recognize that  $T$  is a sphere and that  $S$  is the union of two isolated points and the space obtained collapsing two points in the sphere  $T$ . The weighted 2-adjacency graph of  $S$  is represented in Figure 6 below  $S$ .

**Example 5.3.** The surface  $S$  represented in the left-hand side of Figure 7 contains a cone and a plane passing through the vertex of the cone, thus there is only one singular point which is not isolated in  $S$ .

Proceeding as in the previous example, using the notations appearing in the figure and the same way of labelling, we compute:

Regions:  $inv(3) = 4$  and  $inv(1) = 2$ ; hence  $\mathcal{R}_1 = \emptyset$  and  $\mathcal{R}_2 = \{1, 2, 3, 4\}$

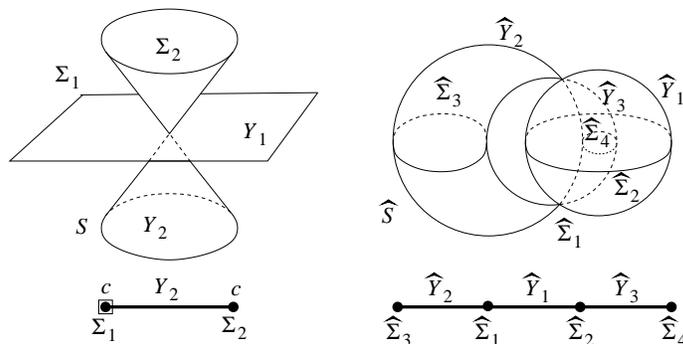


FIGURE 7. An odd degree non-affine surface.

Components:  $inv(1) = 1, inv(2) = 3$ ; hence  $\mathcal{F}_1 = \{1\}$  and  $\mathcal{F}_2 = \{2, 3\}$ . Moreover  $\chi(\widehat{T}) = [2, 2, 2]$   
 Singularities:  $\widehat{l}_1 = [1, 1, 1], \widehat{l}_2 = [1, 1, 1], q(\widehat{T}) = []$ .

By means of the descending procedure we get  $\chi(T) = [1, 2], l_1 = [1, 2], q(T) = []$ , i.e.  $T$  is the disjoint union of a projective plane and a sphere and  $S$  is obtained from it collapsing a point in the plane with two points in the sphere.

**Example 5.4.** Also the surface in Figure 8 contains a projective plane and only one singular point.

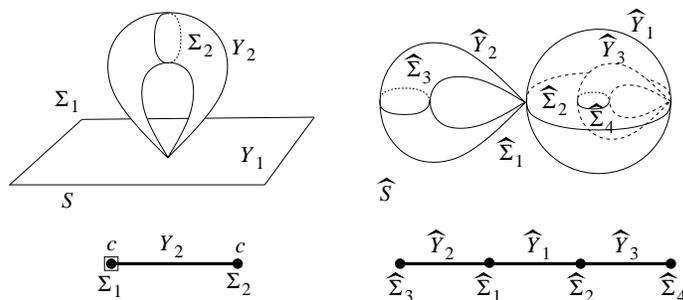


FIGURE 8. Another odd degree non-affine surface.

The usual procedure yields:

Regions:  $inv(3) = 4$  and  $inv(1) = 2$ ; hence  $\mathcal{R}_1 = \emptyset$  and  $\mathcal{R}_2 = \{1, 2, 3, 4\}$

Components:  $inv(1) = 1, inv(2) = 3$ ; hence  $\mathcal{F}_1 = \{1\}$  and  $\mathcal{F}_2 = \{2, 3\}$ . Moreover  $\chi(\widehat{T}) = [2, 2, 2]$

Singularities:  $\widehat{l}_1 = [1, 2, 0], \widehat{l}_2 = [1, 0, 2], q(\widehat{T}) = []$ .

By means of the descending procedure we get  $\chi(T) = [1, 2], l_1 = [1, 2], q(T) = []$ , i.e.  $T$  is the disjoint union of a projective plane and a sphere and  $S$  is obtained from it collapsing a point in the plane with two points in the sphere.

Observe that the output  $D(S)$  obtained coincides with the one of the surface  $S$  of Example 7, in spite of the fact that the two surfaces cannot be mapped each into the other by means of a homeomorphism of  $\mathbb{R}P^3$ . However also in this case there is an invariant that distinguishes the two surfaces: the lists  $\widehat{l}_1, \widehat{l}_2$ . This shows that the method of “doubling”  $S$  into  $\widehat{S}$  is not only a useful technical device to compute  $D(S)$  but also provides new additional invariants by homeomorphism.

REFERENCES

[BPR] BASU, S., POLLACK, R., AND ROY, M.-F.: Algorithms in Real Algebraic Geometry. *Algorithms and Computat. in Mathematics, Vol. 10*, Springer-Verlag, 2003

- [FGL] FORTUNA, E., GIANNI, P. AND LUMINATI, D.: Algorithmical determination of the topology of real algebraic surfaces. *J. Symbolic Comput.*, vol. 38, n. 6 (2004), pp. 1551-1567
- [FGLP] FORTUNA, E., GIANNI, P., LUMINATI, D. AND PARENTI, P.: The adjacency graph of a real algebraic surface. Preprint Università di Trento, 671, (2004)
- [FGPT] FORTUNA, E., GIANNI, P., PARENTI, P. AND TRAVERSO, C.: Algorithms to compute the topology of orientable real algebraic surfaces. *J. Symbolic Comput.*, vol. 36, n. 3-4 (2003), pp. 343-364
- [FGP] FORTUNA, E., GIANNI, P. AND PARENTI, P.: Some constructions for real algebraic curves *J. Symbolic Comput.*, to appear
- [M] MILNOR, J.: Singular points of complex hypersurfaces. *Annals of Mathematics Studies, No. 61*, Princeton University Press, Princeton, N.J. 1968

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