Limit cycle's uniqueness for a class of plane systems

M. Sabatini

February 18, 2004

Abstract

A uniqueness theorem for limit cycles of a class of plane differential systems is proved. The main result is applicable to secord order O.D.E.'s with dissipative term depending both on the position and the velocity. [0]

1 Introduction

In this paper we are concerned with limit cycle's uniqueness for plane systems of the type

(1) $$x' = \beta(x)\big[\varphi(y) - F(x,y)\big], \qquad y' = -\alpha(y)g(x).$$

The system (1) contains, as special cases, systems equivalent to Liénard or Rayleigh equations. In fact, if we choose $\beta(x) = \alpha(x,y) = 1$, $F(x,y) \equiv F(x)$, $\varphi(y) = y$, we obtain the usual Liénard system, equivalent to

$$x'' + f(x)x' + g(x) = 0,$$

where $f(x) = F'(x)$. On the other hand, taking $\beta(x) = \alpha(x,y) = 1$, $F(x,y) \equiv F(x)$, $g(x) = x$ we obtain the system

$$x' = \varphi(y) - F(x), \qquad y' = -x,$$

equivalent to the Rayleigh equation

$$y'' - F(-y') + \varphi(y) = 0.$$

There exist more general second order equations that can be studied by means of systems reducible to form (1). Also, Lotka-Volterra systems are particular cases of system (1).

Systems of the form (1), with F not depending on y, were considered by several authors in relation to uniqueness of limit cycles, see e. g. [2], [4], [3], [5] and references therein. The main motivation of this paper is the extension of

[0]The author would like to thank prof. G. Villlari of the Univ. di Firenze for some conversations on the subject of this paper.
Key words and phrases: limit cycle, uniqueness, Liénard equation, Rayleigh equation, Lotka-Volterra.
2000 Mathematics Subject Classification: Primary 34C05, Secondary 34C25.

previous results to systems containing the term $F(x,y)$, depending both on x and y. Our results are in the line of [4], [3], concerned with Liénard systems,

$$x' = y - F(x), \qquad y' = -g(x).$$

Such papers are based on the observation that the integral $\int_0^T F(x)g(x)dt$ vanishes on every cycle. Comparing the value of such an integral on different cycles allows to prove, under suitable hypotheses, that at most one limit cycle exists. The argument presented in [4] and [3] can be adapted to the case of F depending on both variables, even replacing y with an increasing function $\varphi(y)$.

2 Results

If h is a function defined on a (possibly generalized) interval, we say that $h \in S$ if $th(t) > 0$ for $t \neq 0$ and $h(0) = 0$. If k is a function defined on a domain D, we say that $k \in L(D)$ if k is lipschitzian on D.

Throughout all of this section we assume that there exist $\bar{a} < 0 < \bar{b}$ such that

o) $\alpha, \varphi \in L(\mathbb{R})$, $\beta, g \in L\left((\bar{a}, \bar{b})\right)$, $F \in L\left((\bar{a}, \bar{b}) \times \mathbb{R}\right)$,

oo) $g, \varphi \in S$, $\alpha > 0$, $\beta > 0$ on their domains.

The assumption on the sign of α and β is not a significant restriction in dealing with limit cycles. In fact, if $\beta(x_0) = 0$ for some $x_0 \in \mathbb{R}$, then the line $x = x_0$ is an invariant line for (1) and no cycle may cross it. Similarly for $\alpha(y)$. Hence we can reparametrize the orbits of (1) multiplying the vector field by $\frac{1}{\alpha(y)\beta(x)}$. The new system is

$$x' = \frac{\varphi(y) - F(x,y)}{\alpha(y)}, \qquad y' = -\frac{g(x)}{\beta(x)},$$

whose orbits coincide with those ones of (1). In particular, uniqueness of limit cycles for such a system is equivalent to uniqueness of limit cycles for (1). The new system can be written as

$$x' = \tilde{\varphi}(y) - \tilde{F}(x,y), \qquad y' = -\tilde{g}(x),$$

with $\tilde{\varphi}(y) = \frac{\varphi(y)}{\alpha(y)}$, $\tilde{F}(x,y) = \frac{F(x,y)}{\alpha(y)}$, $\tilde{g}(x) = \frac{g(x)}{\beta(x)}$. Hence we can restrict, without loss of generality, to the following class of systems

$$x' = \varphi(y) - F(x,y), \qquad y' = -g(x). \tag{2}$$

Let us set

$$G(x) = \int_0^x g(s)\ ds, \qquad \Phi(y) = \int_0^y \varphi(s)\ ds.$$

Theorem 1 *Assume there exist $a, b \in (\bar{a}, \bar{b})$, $a < 0 < b$, such that:*

i) for all $x \in (a,b)$, the function $y \mapsto \varphi(y) - F(x,y)$ is strictly increasing;

ii) for all $x \in (a,b)$: $F(x,y) < 0$;

iii) for all $x \notin (a,b)$: $F(x,y) > 0$; for all $y \in \mathbb{R}$ the function $x \mapsto F(x,y)$ is increasing out of (a,b);

iv) the set $\varphi(y) - F(x,y) = 0$ is the graph of a function $\mu(x)$ defined on $(\bar{a}, \bar{b})$;

Then the system (2) has at most one limit cycle meeting both the lines $x = a$, $x = b$.

Proof. Let us set $V(x,y) = G(x) + \Phi(y)$. By the sign assumptions on g and φ, the function V is positive definite at the origin.

Let $\gamma(t) = (x_\gamma(t), y_\gamma(t))$ be a T-periodic cycle of (2). Denoting by $\dot{V}$ the derivative of V along the solutions of (2), we have

$$0 = V(\gamma(T)) - V(\gamma(0)) = \int_0^T \dot{V}(\gamma(t))\, dt = -\int_0^T F(x_\gamma(t), y_\gamma(t)) g(x_\gamma(t))\, dt.$$

We denote concisely by $-\int_0^T F(x)g(x)dt$ the last integral in the above formula. As in .... we prove the cycle's uniqueness assuming, by absurd, the existence of a second cycle, and showing that $\int_0^T F(x)g(x)dt$ cannot vanish on both cycles. Let γ_1, γ_2 be distinct cycles of (2) both meeting the lines $x = a$, $x = b$. Let T_j, $j = 1,2$, be the period of, respectively, γ_j, $j = 1,2$. The system (2) has a unique critical point, because $g(x)$ vanishes only at the origin and $\varphi(y) - F(x,y) = 0$ intersects the line $x = 0$ at a single point. Hence γ_1 and γ_2 are concentric. Let γ_1 be contained in the interior of γ_2. Let I_j, $j = 1,2$ be the value of $\int_0^{T_j} F(x)g(x)dt$ computed along γ_j. We claim that $I_1 < I_2$.

It is sufficient to show that the proof of theorem 1 in [4] can be performed replacing y with $\varphi(y)$ and $F(x)$ with $F(x,y)$.

The graph of $y = \mu(x)$ divides the strip $\bar{a} < x < \bar{b}$ into two parts, the upper one H^* and the lower one H_*. One has $x' > 0$ on H^*, $x' < 0$ on H_*. The decomposition of the integral $\int_0^T F(x)g(x)dt$ can be done just as in [4], [3]. First one integrates with respect to x on the interval (a,b), separately along $\gamma_1 \cap H^*$ and along $\gamma_1 \cap H_*$. The comparison between the integrals I_1 and I_2 along those arcs is similar to that one in [4], since as y increases, also $\varphi(y) - F(x,y)$ increases.

Similarly for the other portions of arc of γ_1 and γ_2, since, when integrating with respect to y, one uses the sign of $F(x,y)$ and its mononicity with respect to x outside the strip $a < x < b$. ♣

Next corollary is concerned with the special case of systems

$$(3) \qquad x' = \varphi(y) - F(x), \qquad y' = -g(x).$$

It extends theorem 1 in [4].

Corollary 1 *Assume F not to depend on y, φ to be strictly increasing. Let there exist $a < 0 < b$ such that $F(x) > 0$ out of (a,b), $F(x) < 0$ on (a,b), F increasing out of (a,b). Then the system (3) has at most one limit cycle meeting both the lines $x = a$, $x = b$.*

Proof. The hypotheses $i)$, $ii)$, $iii)$ of theorem (1) are trivially satisfied. As for hypothesis $iv)$, it is sufficient to observe that φ is invertible, so that one can take $\mu(x) = \varphi^{-1}(F(x))$. ♣

The limit cycles of (3) do not necessarily meet both lines $x = a$, $x = b$. In [2], [3], additional conditions were considered in order to ensure such a property. Here we prove two corollaries related to such a problem.

Corollary 2 *In the hypotheses of theorem 1, assume additionally that* $\varphi(-y) = -\varphi(y)$, $F(-x,-y) = -F(x,y)$, $g(-x) = -g(x)$. *Then (2) has at most one limit cycle.*

Proof. Using $V(x,y)$ as a Liapunov function, one has

$$\dot{V}(x,y) = -F(x,y)g(x) > 0$$

on the strip $a < x < b$. Hence the origin is a repellor, and no limit cycles are contained in the strip $a < x < b$. If there exist limit cycles, they have to meet at least one of the lines $x = a$ or $x = b = -a$. Now it is sufficient to observe that the orbits of (4) are symmetric with respect to the origin, hence every cycle meeting the line $x = a$ has to meet also the line $x = b = -a$. This gives the uniqueness of the limit cycles. ♣

Next corollary is concerned with the case of F independent of y, that is with system (3).

Corollary 3 *Under the hypotheses of theorem 1, assume additionally* $G(a) = G(b)$. *Then (3) has at most one limit cycle.*

Proof. The level set $V(x,y) = G(a) = G(b)$ contains the points $(a,0)$ and $(b,0)$. The monotonicity of φ and g implies that the sublevel set $V_{G(a)} = \{(x,y) : V(x,y) \leq G(a)\}$ is entirely contained in the strip $a < x < b$. Using $V(x,y)$ as a Liapunov function, one has

$$\dot{V}(x,y) = -F(x,y)g(x) > 0$$

on the strip $a < x < b$, hence also on $V_{G(a)}$. This shows that no cycles can be contained in $V_{G(a)}$, and that every cycle has to meet both the lines $x = a$ and $x = b$. ♣

The theorem (1) only gives an upper bound on the number of cycles of (2), without guaranteeing the existence of such a cycle. In order to get an existence and uniqueness result one can impose a set of hypotheses which ensure the ultimate boundedness of the orbits of (2), together with the negative asymptotic stability of the critical point. The boundedness of the solutions of systems of type (2) has been studied in [1]. The conditions considered in [1] are compatible with those ones of this paper.

As an example, let us consider the system

$$x' = y + y^3 - (x^3 - x)(y^2 + 1), \qquad y' = -x. \tag{4}$$

The hypotheses of theorem 4 in [1] hold, so that the solutions of (4) are ultimately bounded. Moreover the origin is a repellor, hence there exists at least a limit cycle. Also, the hypotheses of 2 hold, hence the limit cycle is unique. Its orbital asymptotic stability comes from standard arguments. Such a system is equivalent to the second order equation

$$y'' + (y'^3 + y')(y^2 + 1) + y + y^3 = 0.$$

References

[1] M. L. Cartwright, H. P. F. Swinnerton-Dyer, *Boundedness theorems for some second-order differential equations*, Differential Equations **14** (1978), no. 11, 1378–1406 (1979).

[2] A. Gasull, A. Guillamon, *Non-existence, uniqueness of limit cycles and center problem in a system that includes predator-prey systems and generalized Liénard equations*, Diff. Eq. Dyn. Syst., **3** (1995), 345–366.

[3] T. Carletti, G. Villari, *A note on existence and uniqueness of limit cycles for Liénard systems*, preprint, Univ. di Firenze (2003), 16.

[4] G. Villari, *Some remarks on the uniqueness of the periodic solutions for Liénard equation*, Bollettino U.M.I., Serie VI, **IV** (1985), 173–182.

[5] D. Xiao, Z. Zhang, *On the uniqueness and nonexistence of limit cycles for predator-prey systems*, Nonlinearity **16** (2003), 1185–1201.

Marco Sabatini
Dip. di Matematica, Univ. di Trento, I-38050 Povo, (TN) - Italy.
E-mail: sabatini@science.unitn.it
Phone: ++39(0461)881670, Fax: ++39(0461)881624