UNIVERSITÀ DEGLI STUDI DI TRENTO

DOTTORATO DI RICERCA IN MATEMATICA

XIV CICLO

Tesi presentata per il conseguimento del titolo di Dottore di Ricerca

Erika Damian

Loop Lie algebras which are $\mathrm{sl}_2(\mathbb{C})$−modules.

Relatore

Prof. Andrea Caranti

Febbraio 2003

Acknowledgements

Firstly, I would like to thank Andrea Caranti for all his help. He has been always ready with new ideas and explanations. His suggestions and encouragement during my Ph.D. fellowship have made this thesis possible and a great adventure!

Then I am very grateful to Giuseppe Jurman who has been of great importance in helping me going through the computational part of this work. He has been always at my side and his support was invaluable when everything seemed to be hopeless.

Another mention is for Marina Avitabile who spent a lot of time with me discussing several algebraic problems looking for a solution.

Then I would like to thank Claretta with whom I spent a lot of time in preparing lessons and correcting papers. All this stuff has been made much more pleasant by her enthusiasm.

I am also indebted with Andrea Lucchini for his useful suggestions and his kindness when I bombarded him with many drafts of this thesis.

I also extend thanks to all the people I met during the three summer courses in Cortona, they shared with me the pleasure of doing maths and gave me renewed energy and enthusiasm in doing it.

During my time as a Ph.D. student in Trento I was lucky enough to meet many people that made the last four years a wonderful period. Firstly, I want to thank all my office-mates: Antonio, Vincenzo, Ale, Biba and Michela (this is a chronological order and I think that Michela deserves a special mention as she shared with me the last period of my staying in Trento which has been the most hectic). Then I extend thanks to Antonio & Cristina, Beppe & Elena, Fabio, Gippo, Luca, Velitchko, Mimmo, Nadia, Davide and Marcello.

My thanks also go to Amerigo, Miriam and Lucia for their kindness and for their assistence in the administrative and technical matters.

Finally, I want to thank all the people that play an important role in my almost-no-math world:
Marcus who is a great friend.
Andrea who has been able to keep his fingers crossed for almost a week while I was proving the main theorem of this thesis!
Maurizio for his great patience and support when I am feeling blue.

My parents and my sister who always backed me up.
Claudio, because he is always at my side, taking care of me with his immense love.

Contents

Introduction **1**

1 Basic concepts **9**
- 1.1 Lie algebras 9
- 1.2 Graphs 18

2 Graphs and finitely presented Lie algebras **25**
- 2.1 Presentations in class $\mathcal{C}(d)$ and graphs 26
- 2.2 Lie algebras presented by presentations in class $\mathcal{C}(d)$ 31

3 Periodic Lie algebras which are $\mathrm{sl}_2(\mathbb{C})-$modules **37**
- 3.1 Representations of $\mathrm{sl}_2(\mathbb{C})$ 37
- 3.2 Finitely presented Lie algebras which are $\mathrm{sl}_2(\mathbb{C})$-modules 43
- 3.3 Examples 44
  - 3.3.1 The loop algebra $\hat{\mathbf{sl}}^{(\mathbf{2})}(\mathbf{3})$ 44

4 Further examples **61**

Bibliography **69**

Introduction

In their famous paper [11], E. S. Golod and I. R. Shafarevich proved that, given any field k, if $\mathcal{P}$ is a presentation of an associative k-algebra with d generators and at most $(d-1)^2/4$ relations, the algebra presented by $\mathcal{P}$ must be infinite-dimensional. Later Vinberg, [22], proved that $(d-1)^2/4$ can be replaced by $d^2/4$ in the Golod-Shafarevich theorem.

In 1977 H. Koch, [15], showed that the same kind of result can be stated in the Lie algebras context; in particular, he proved that, if L is a finite-dimensional nilpotent Lie k-algebra with generation rank d, where $d > 2$, and relation rank r, then $r > d^2/4$.

Let us note that the concepts of generation rank and relation rank for finite-dimensional nilpotent Lie algebras are defined by using the usual procedures. Indeed, a generating system E of a finite-dimensional Lie algebra L is called minimal if there is no proper subset of E which is a generating system of L. The generation rank of L, $d(L)$, is the cardinality of a minimal generating system. Note that, if L has generation rank d, then L is a factor algebra of the free Lie algebra on d generators F_d relative to an ideal I. If R is a subset of I that generates I as an ideal of F_d, we call R a minimal relation system of L when I is not generated by any proper subset of R. The relation rank of L, $r(L)$, is the cardinality of a minimal relation system.

A problem which arises quite naturally in this context is the construction of examples of finite-dimensional nilpotent Lie algebras with "few" relations, in the sense that the quotient r/d^2 is the smallest possible. In his paper Koch describes a class $\mathcal{L}$ of Lie algebras with the property that, for each $L \in \mathcal{L}$, with generation rank d and relation rank r, there exists a series of finite-dimensional nilpotent Lie algebras L_s with $L_1 := L$ and $d(L_s) = d^s$, $r(L_s) = \dfrac{r}{d^2-1}(d^{2s}-1)$. Thus, given a Lie algebra L with small value of $r(L)/d(L)^2$ there is provided a whole series of Lie algebras L_s where
$r(L_s)/d(L_s)^2 < r/(d^2-1)$.

Starting from this result, J. Wisliceny, who intensively worked on the topic during the last twenty years, was able to build a series of finite-dimensional nilpotent Lie k-algebras, $\{L(n)\}_{n\in\mathbb{N}}$, such that $\lim_{n\to\infty} \dfrac{r(L(n))}{d(L(n))^2} = 1/4$ holds, thus obtaining the asymptotical exactness of the bound (see [23] as a reference). Later, in [25], Wisliceny was able to prove the same result in the associative algebras context.

Golod-Shafarevich theorem for both associative and Lie algebras has been restated in different terms by using the concept of entropy of an algebra in a paper by M. F. Newman, C. Schneider and A. Shalev, [16].

Let $A = \bigoplus_{n\geq 0} A_n$ be a graded k-algebra, we define the entropy of A by $H(A) = \limsup_{n\to\infty} \sqrt[n]{\dim_k A_n}$.

There is a connection between the entropy of a graded algebra A and the Hilbert series associated to it $A(t) = \sum_{n=0}^{\infty}(\dim_k A_n)t^n$: in fact $H(A) = 1/R$, where R is the radius of convergence of the formal power series $A(t)$. The Hilbert series associated to an algebra have been widely studied in commutative algebra and for graded associative algebras, it should be regarded as a "generalised dimension" and it carries all the information on asymptotic behaviour of an algebra, see [21] as a reference.

In their paper the authors study the entropy of associative algebras and Lie algebras, in particular they consider the entropy of free algebras and how it is related to the entropy of their quotients and their subalgebras. One of the key observations, which relies on a paper by A. E. Bereznyĭ, [4], is that, given a graded Lie algebra L, then its universal enveloping algebra $U(L)$ admits a natural grading induced from L and $H(L) = H(U(L))$. As a consequence they can extend the results they prove in the associative algebras context to Lie algebras. Let us state the result they obtain and that is related to the Golod-Shafarevich theorem: let $\mathcal{P}$ be an associative or a Lie algebra presentation containing d generators and r homogeneous relations of degree at least 2 with $r < d^2/4$. If A denotes the algebra presented by $\mathcal{P}$ then $H(A) > 1$.

Let us observe that a graded k-algebra has entropy zero if and only if it is finite-dimensional. Indeed, if a graded k-algebra $A := \bigoplus_{n=0}^{\infty} A_n$ is finite-dimensional, then the sequence $\{\dim_k(A_n)\}_{n\in\mathbb{N}}$ is definitely zero. Thus $H(A) = \lim_{n\to\infty} \sqrt[n]{\dim_k(A_n)} = 0$. On the other hand, if $H(A) = 0$, then for a sufficiently large $N \in \mathbb{N}$ we get $\dim_k(A_j) < 1$, for all $j > N$, thus $A = \bigoplus_{n=1}^{N} A_n$. Furthermore, by using the same argument, we obtain that infinite-dimensional k-algebras have entropy grater or equal to one. In their paper the authors highlight two examples of graded k-algebras with entropy one. In particular, finitely generated graded associative commutative algebras have clearly entropy one and this also holds for PI-algebras over a field of characteristic zero, [3]. Examples of graded Lie algebras of entropy one are thin Lie algebras, Lie algebras of maximal class and loop algebras of finite-dimensional simple Lie algebras.

Since loop Lie algebras are widely mentioned in this thesis, we take the chance here to remind how they are defined (see [9] as a reference). Let S denote a finite-dimensional simple Lie algebra over a field k, let $k[t, t^{-1}]$

(with t an indeterminate) denote the ring of polynomials in t and t^{-1} with coefficients in k. We call $\overline{S} := k[t,t^{-1}] \otimes_k S$, with a Lie bracket defined as follows: $[u \otimes x, v \otimes y] := uv \otimes [x,y]$ where $u,\ v \in k[t,t^{-1}]$ and $x,\ y \in S$, the loop algebra of S. Note that $\overline{S}$ is infinite-dimensional and we can consider the following graduation on it: $\overline{S} = \bigoplus_{n\in\mathbb{Z}}^{\infty} S \otimes kt^n$. As we will mainly work on finitely presented Lie algebras which have a natural graduation on $\mathbb{N}$, we will call loop Lie algebras also those which are obtained as a tensor product of a simple Lie algebra with the ring of polynomials $k[t]$. In this case we obtain an infinite-dimensional Lie algebra with a natural grading on $\mathbb{N}$,
$\hat{S} = \bigoplus_{n\in\mathbb{N}}^{\infty} S \otimes kt^n$, and we can calculate its entropy:

$$H(\hat{S}) = \limsup_{n\to\infty} \sqrt[n]{\dim_k S} = 1.$$

As we observed above, it is natural to search among k-algebras with entropy one those which are finitely presented with "few" relations. Using the theorem stated above, we can conclude that at least $d^2/4$ relations are required to present a k-algebra on d generators with entropy one. Moreover, as the authors observe, going through the proof of that theorem one obtains that if $d > 2$, then more than $d^2/4$ relations are needed.

The first problem we face in this thesis is the construction of examples of presentations of periodic Lie algebras with "few" relations (we call a Lie algebra $L = \bigoplus_{n=1}^{\infty} L_n$ periodic if the function $f_L(n) := \dim_k(L_n)$ defined on the positive integers is ultimately periodic, i.e. there exist $N,\ h \in \mathbb{N}$, such that $f_L(n+h) = f_L(n)$ for all $n > N$). Let us remark that periodic Lie algebras have entropy one. More precisely, we are looking for periodic Lie algebras presented by d generators and $d^2/4 + 1$ relations. Roughly speaking we are investigating a class of Lie algebras with entropy one which are close to be finite-dimensional. In order to tackle this problem, we get some hints from the classes of presentations Wisliceny investigated in his papers. In fact, in [24], he considered sets of relations he called *Erhöhungsysteme* and he showed that, if $L = \bigoplus_{n=1}^{\infty} L_n$ is a free Lie k-algebra freely generated by d generators and R is a *Erhöhungsystem* then the factor algebra $L/I(R)$ is nilpotent, being $I(R)$ the ideal of L generated by R. In a later paper, [25], he obtained the same result in the associative algebras context defining the *Erhöhungsysteme* of associative algebras in a similar way. In this paper, which deals with associative algebras, he introduced sets of relations, called *Quasierhöhungsysteme*, employing a set of defining rules similar to the defining rules of the previous case but weaker, in some sense. As he observed, these sets of relations do not always lead to nilpotent associative algebras. This second class of relations, translated in the Lie algebras context, seemed a good starting point for our investigations.

Let us call $\mathcal{C}(d)$, where $d > 2$, the class of presentations on d generators we are considering, see chapter 2 for a detailed description of this class. As we would like to automatically generate presentations of $\mathcal{C}(d)$ through an *ad hoc* designed software we should investigate this class in order to reduce the number of presentations which present isomorphic Lie algebras and thus easing the computational burden.

Given a presentation $\mathcal{P} = (X|R)$ in $\mathcal{C}(d)$, with $X := \{x_1, \dots, x_d\}$ as set of generators and R the set of relations, we have, by definition, that R is a disjoint union of two subsets $R_{\mathcal{M}_d}$ and $R_{\mathcal{H}_d}$. The first result we prove is that the cardinalities of these two subsets of R depend only on d, namely $|R_{\mathcal{M}_d}| = \lfloor d/2 \rfloor + 2$ and $R_{\mathcal{H}_d} = \lfloor d^2/4 \rfloor - \lfloor d/2 \rfloor - 1$.

Note that if S is the set of relations of another presentation $\mathcal{P}' \in \mathcal{C}(d)$ on the same set of generators X, such that

$$[x_i, x_j] \in S_{\mathcal{M}_d} \iff [x_i, x_j] \in R_{\mathcal{M}_d} \text{ or } [x_j, x_i] \in R_{\mathcal{M}_d},$$

then the Lie algebras presented by $\mathcal{P}$ and $\mathcal{P}'$ are isomorphic. This simple observation led us to consider the idea of associating a presentation $(X|R) \in \mathcal{C}(d)$ with a graph $\mathcal{G}(d, R)$, being X its vertex set and

$$E(G(d,R)) := \{\{x_i, x_j\} : [x_i, x_j] \in R_{\mathcal{M}_d}, \text{ or } [x_j, x_i] \in R_{\mathcal{M}_d}\}$$

its edges set.

The analysis of the graphs associated with the presentations in $\mathcal{C}(d)$ is the topic of the second chapter of this thesis. Actually, given a graph isomorphism ϕ between two graphs $\mathcal{G}(d, R)$ and $\mathcal{G}(d, S)$ associated with two presentations $\mathcal{P}$ and $\mathcal{P}'$, such that the canonical extension to a Lie algebra homomorphism $\overline{\phi}$, gives $\overline{\phi}(R_{\mathcal{H}_d}) = S_{\mathcal{H}_d}$, we are able to prove that the Lie algebras presented by $\mathcal{P}$ and $\mathcal{P}'$ are isomorphic as Lie algebras. This theorem allows us to restrict our attention to presentations of $\mathcal{C}(d)$ associated with a graph which belongs to the set A of all non-isomorphic graphs with d vertices and $\lfloor d/2 \rfloor + 2$ edges. This is the setting where we can prove some useful results about the connections between the properties of a presentation $\mathcal{P}$ with the entropy of the Lie algebra presented by $\mathcal{P}$. The key theorem of this section gives two as a lower bound to the entropy of the Lie algebra presented by $\mathcal{P}$, if the subset $R_{\mathcal{H}_d}$ of the set of relations satisfies a certain property. This result has two corollaries which allow us to see connections between some properties of the graph associated to a presentation $\mathcal{P}$ with the entropy of the Lie algebra presented by $\mathcal{P}$.

In the last section of the second chapter we use the results of the previous section to solve our problem for $d = 3$ and $d = 4$. In fact we show that every presentation in $\mathcal{C}(3)$ gives a 3-dimensional abelian Lie algebra which has entropy zero. For $d = 4$ we get that there are only two isomorphism classes of Lie algebras which can be presented by a presentation $\mathcal{P}$ in $\mathcal{C}(4)$. Here $\mathcal{P}$ can be the presentation of either a 5-dimensional metabelian Lie algebra or a central extension of a free Lie algebra of rank 2; thus neither of them with entropy one.

In addition, we prove that there are at most 16 isomorphism classes of Lie algebras with a presentation in $\mathcal{C}(5)$. There is computational evidence that one of these Lie algebras which have a presentation in $\mathcal{C}(5)$, say $\mathcal{L}$, is periodic. The second chapter of this thesis is devoted to prove this fact when $k = \mathbb{C}$, in particular we are able to show that $\mathcal{L}$ is a loop Lie algebra of $\mathrm{sl}_3(\mathbb{C})$.

In order to prove that $\mathcal{L}$ is a periodic Lie algebra we had to develop a non-standard approach to the problem. More precisely, we have a presentation $\mathcal{P}_{\mathcal{L}} \in \mathcal{C}(5)$ of a Lie algebra $\mathcal{L}$, which has a natural graduation $\mathcal{L} = \bigoplus_{n\in\mathbb{N}} \mathcal{L}_n$, with $\dim_{\mathbb{C}}\mathcal{L}_1 = 5$ and $\dim_{\mathbb{C}}\mathcal{L}_2 = 3$. There is computational evidence of the fact that the homogeneous components of even degree are 3-dimensional $\mathbb{C}$-modules and those of odd degree are 5-dimensional $\mathbb{C}$-modules. By using computational devices and software as `GAP` and `Anu-p-quotient`, we can work on finite-dimensional nilpotent quotients of $\mathcal{L}$, say $\mathcal{L}(h) = \mathcal{L}/\bigoplus_{n=h+1}^{\infty} \mathcal{L}_n$, and we gain computational evidence of the fact that $\mathcal{L}$ is isomorphic to a loop algebra of $\mathrm{sl}_3(\mathbb{C})$. At this stage, proving that these two Lie algebras are isomorphic appeared to be a difficult task. In fact, similar theorems concerning finitely presented thin Lie algebras or Lie algebras of maximal class are proved by induction on the degrees of the homogeneous components and considering the consequences of the relations given in the presentation. Even if this kind of approach would probably work in our context, it would certainly require a large amount of calculations, this is the reason why we decided to look for a different way to deal with this problem. Going through papers on finitely presented Lie algebras we realise that there are two kinds of results. There are papers dealing with the implementation of algorithms that allow to find basis of finitely presented finite-dimensional Lie algebras (see for example [13], [20], [10], [7]). The results we are interested in are those on the structure of finitely presented Lie algebras but they are useless in our context as they deal with finitely presented Lie algebras with some additional hypothesis on their structure. We find, for example, results on solvable finitely presented Lie algebras in [2], or on finitely presented Lie algebras which are finite-dimensional-by-abelian in [6].

We have been luckier when, looking for papers dealing with loop Lie algebras, we have found a series of papers by J. M. Osborn and K. Zhao where they work on the Kirillov's problem for Lie algebras, that is to say, the analysis of the structure of doubly $\mathbb{Z}$-graded Lie algebras on a field of characteristic zero, such that each graded component has dimension zero or one. In [17] they find a characterisation of a loop Lie algebra $\hat{sl}^{(2)}(3)$ graded on the integers such that the homogeneous components of even degree are 3-dimensional and those of odd degree are 5-dimensional. The subalgebra of $\hat{sl}^{(2)}(3)$ we obtain by considering the homogeneous components of positive degree can be seen as a loop algebra of $\mathrm{sl}_3(\mathbb{C})$, and it is exactly the algebra we are looking for.

Let us go into more details, in their paper Osborn and Zhao prove the following theorem: if $A = \bigoplus_{i,\ j \in \mathbb{Z}} A_{i,j}$ is a $\mathbb{Z} \times \mathbb{Z}$-graded Lie k-algebra over a field of characteristic zero, with $\dim A_{i,j} \leq 1$, for each i and j, such that the following conditions (I), (II) and (III) are satisfied, then A is isomorphic to a loop Lie algebra of $\mathrm{sl}_2(k)$, called $\hat{\mathrm{sl}}(2)$, or to a loop Lie algebra of $\mathrm{sl}_3(k)$, indicated as $\hat{sl}^{(2)}(3)$. The conditions considered in the statement are as follows, let $w_{i,j}$ be chosen such that $A_{i,j} = kw_{i,j}$ for all $(i.j) \in \mathbb{Z} \times \mathbb{Z}$, for any $r \in \mathbb{Z}$ we define $A_r = \bigoplus_{i \in \mathbb{Z}} A_{i,r}$ and $A'_r = \bigoplus_{i \in \mathbb{Z}} A_{r,i}$,

(I) $\dim_k A_{\pm 1,0} = \dim_k A_{0,\pm 1} = 1,$ and A is generated by $w_{\pm 1,0},\ w_{0,\pm 1}$;

(II) $A'_0 \simeq \mathrm{sl}_2(k)$;

(III) $[w_{-1,0}, w_{1,0}] = 0$ and $\mathrm{ad}(w_{\pm 1,0})$ act faithfully on A_1.

Going through the proof of this theorem we find out that if $\dim_k A'_1 = 3$, then $A \simeq \hat{\mathrm{sl}}(2)$ whereas, if $\dim_k A'_1 = 5$, then $A \simeq \hat{sl}^{(2)}(3)$. In their paper the authors consider a $\mathbb{Z}_2$-graduation of $\mathrm{sl}_3(k)$, so that it can be seen as a direct sum of an irreducible 3-dimensional $\mathrm{sl}_2(k)$-module, V_0, and an irreducible 5-dimensional $\mathrm{sl}_2(k)$-module, say V_1, thus giving $\hat{sl}^{(2)}(3)$ in the following way:

$$\hat{sl}^{(2)}(3) = \bigoplus_{i \in \mathbb{Z}} (k(t^{2i} \otimes V_0) \oplus k(t^{2i+1} \otimes V_1)).$$

Let us remark that, with respect to this graduation, $\hat{sl}^{(2)}(3)$ has 5-dimensional irreducible $\mathrm{sl}_2(k)$-modules as homogeneous components of odd degree, and 3-dimensional irreducible $\mathrm{sl}_2(k)$-modules as homogeneous components of even degree. This graduation reminds us of the Lie algebra $\mathcal{L}$ we are trying to describe.

This characterisation suggests us to give a $\mathrm{sl}_2(k)$-module structure to $\mathcal{L}$. Let us recall that $\mathcal{L}$ is a finitely presented Lie algebra with a presentation that belongs to $\mathcal{C}(5)$, we can use $\mathcal{L} = (X|R)$ as a notation to indicate the presentation we are considering, where X is the set of generators and R the set of relations. Let V be the free k-module freely generated by X, $L(V)$ the free Lie algebra generated by V, and S the ideal of $L(V)$ generated by R, then by definition we have that $\mathcal{L} = L(V)/S$. We are able to find a new free generating set of V, say Y, where $|Y| = |X| = 5$, such that, defining an irreducible sl_2-action on the basis Y of V in the canonical way, we obtain a sl_2-module structure on $L(V)$ and with respect to this action S becomes a sl_2-submodule of $L(V)$. Let us mention that if k is a field of characteristic zero, then for each $n \in \mathbb{N}$ there exists exactly one irreducible n-dimensional

$\mathrm{sl}_2(k)$-module, up to isomorphism. Thus we obtain a $\mathrm{sl}_2(k)$-module structure on $\mathcal{L}$ and we can explicitly build a $\mathrm{sl}_2(k)$-epimorphism of $\mathcal{L}$ onto $\hat{sl}^{(2)}(3)$ which is also a epimorphism of Lie algebras. In fact this $\mathrm{sl}_2(k)$-epimorphism, say ϕ, is a $\mathrm{sl}_2(k)$-isomorphism of Lie algebras. This $\mathrm{sl}_2(k)$-module structure on $\mathcal{L}$ is a key fact in proving that ϕ is a bijection.

Chapter 1

Basic concepts

1.1 Lie algebras

We use [14] as a reference for general background on Lie algebras.

A Lie algebra L is a k-module, where k is a field, with an operation $L \times L \to L$, denoted $(x, y) \mapsto [x, y]$, called Lie bracket, satisfying the following axioms:

1. The bracket operation is bilinear.

2. $[x, x] = 0$ for all x in L.

3. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x,\ y,\ z \in L$.

Axiom (3) is called the *Jacoby identity*. Note that axiom (2) implies antisymmetry, i.e. $[x, y] + [y, x] = 0$.

We will use a simpler notation for iterated commutators:

$$[x_1, \ldots, x_n] := [\ldots[[x_1, x_2], x_3], \ldots, x_n].$$

Using the Jacoby identity it can be proved the following formula:

$$[y, \underbrace{z, \ldots, z}_{\lambda}, v] = \sum_{i=0}^{\lambda} (-1)^{i+1} \binom{\lambda}{i} [v, \underbrace{z, \ldots, z}_{i}, y, \underbrace{z, \ldots, z}_{\lambda - i}],$$

called the *generalised Jacoby identity*.

Subalgebras of Lie algebras and homomorphisms between Lie algebras are defined as usual.

It is worth noting that, given an associative k-algebra A, it acquires a structure of Lie algebra by defining the Lie bracket of two elements $x,\ y \in A$ as $[x, y] = xy - yx$. We will use A^- to indicate that we are considering A as a Lie algebra.

In particular, let V be a k-module, we can define $\text{gl}(V) := \text{End}_k(V)^-$, where $\text{End}_k(V)$ denotes the algebra of k-endomorphisms of V. Subalgebras of $\text{gl}(V)$ are called *linear Lie algebras*. In addition, if V is a n-dimensional k-module, then we can consider the matrix representation of the elements of $\text{End}_k(V)$ obtaining the k-algebra $M(n,k)$. Then $\text{gl}(n,k) := M(n,k)^-$ is a Lie algebra.

Since, for all $x,\ y \in M(n,k)$,

$$Tr(xy) = Tr(yx),$$
$$Tr(x+y) = Tr(x) + Tr(y),$$

we get that the subset of $\text{gl}(n,k)$ of matrices with trace zero, is a Lie subalgebra of $\text{gl}(n,k)$. This subalgebra is denoted as $\text{sl}(n,k)$ and it is called *special linear algebra*, in analogy with the special linear group $\text{SL}(n,k)$ of endomorphisms with determinant one.

Any homomorphism between a Lie k-algebra L and $\text{gl}(V)$, say

$$\rho : L \to \text{gl}(V)$$

is called a *representation* of L.

The notion of representation is equivalent to that of L-module. Indeed, given a k-module V, we say that V is an L-module if there exists a k-bilinear map

$$\cdot : L \times V \to V$$

satisfying the following axiom:

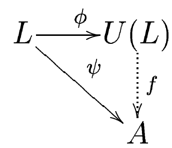
$$[x,y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v), .5\text{cmfor all} x,\ y \in L \text{ and for all } v \in V.$$

Moreover, if V is an L-module, we may define a mapping $\rho : L \to \text{gl}(V)$ by setting $\rho(x)(v) := x \cdot v$, where $x \in L,\ v \in V$. On the other hand, if $\alpha : L \to \text{gl}(M)$ is a representation of L then the k-module M becomes an L-module if we put $x \cdot v := \alpha(x)(v)$, for all $x \in L,\ v \in M$.

Submodules, quotient modules and homomorphisms between modules are defined as usual.

We can go a bit further in showing the connection between Lie algebras and associative algebras. Let L be a Lie algebra and consider a Lie algebra homomorphism between L and an associative algebra A, with its natural Lie algebra structure. Among all these associative algebras there exists one, $U(L)$, called the *universal enveloping algebra of* L which has a universal property, stated in the next result.

Proposition 1.1.1 *Let L be a Lie k-algebra. There exists an associative k-algebra $U(L)$ and a Lie algebra homomorphism $\phi : L \to U(L)^-$ with the* *following property: for any associative algebra A and any Lie algebra homomorphism $\psi : L \to A^-$, there is a unique associative algebra homomorphism $f : U(L) \to A$ making the following diagram commutative:*



We will not give a detailed proof of this proposition but we find it useful to show the construction of the universal enveloping algebra $U(L)$.

Let us consider the tensor algebra of L over k, $T(L) = \bigoplus_{n \geq 0} L^{\otimes n}$, where $L^{\otimes 0} := k$ and $L^{\otimes n} := \underbrace{L \otimes \ldots \otimes L}_{n}$ for $n > 0$. It has a structure of associative algebra with unit with respect to the following associative product, defined on homogeneous generators of $T(L)$:

$$(x_1 \otimes \ldots \otimes x_k)(y_1 \otimes \ldots \otimes y_m) := x_1 \otimes \ldots \otimes x_k \otimes y_1 \otimes \ldots \otimes y_m \in L^{\otimes(k+m)}.$$

Let I be the ideal of $T(L)$ generated by the elements $x \otimes y - y \otimes x - [x, y]$ where $x,\ y \in L$, then $U(L) = T(L)/I$. Let $\phi = p \circ i$, where i is the canonical injection $i : L \to \bigoplus_{n \geq 0} L^{\otimes n}$ and p the canonical surjective algebra homomorphism $p : T(L) \to T(L)/I$. It is easy to show that $\phi([x, y]) = [\phi(x), \phi(y)]$, thus ϕ is a Lie algebra homomorphism. In addition we get that ϕ is injective, thus L can be considered as a Lie subalgebra of $U(L)^-$. This last fact is a direct consequence of the Poincaré-Birkhoff-Witt theorem:

Theorem 1.1.2 (Poincaré-Birkhoff-Witt) *Let L be a Lie k-algebra with ordered basis $(x_j)_{j \in J}$. Suppose that $(U(L), \phi)$ is a universal enveloping algebra of L. Then $\{\phi(x_{j_1}) \ldots \phi(x_{j_n}) \ : \ n \in \mathbb{N},\ j_1 \leq j_2 \leq \ldots \leq j_n \in J\}$ is a basis of $U(L)$ over k.*

It follows from the universal property stated above that the universal enveloping algebra of a Lie algebra is unique up to isomorphism.

We find it useful to highlight the equivalence of representation theory of L and $U(L)$. It is obvious that any $U(L)$-module is naturally an L-module. Let us consider an L-module M, thus we have a representation of L in M, say $\rho : L \to \mathrm{gl}(M)$. Then, according to the universal property of the universal enveloping algebra, there exists a unique homomorphism $\psi : U(L) \to \mathrm{End}_k(M)$ extending ρ.

In what follows we shall recall some basic facts and definitions concerning different classes of Lie algebras such as graded Lie algebras, free Lie algebras, finitely presented Lie algebras and loop Lie algebras. We shall use [1] and [18] as references.

Graded algebras. Let R be a k-algebra and G a commutative semigroup (written additively). A collection of k-submodules of R, $\{R_g | g \in G\}$, is called a *G-grading* of R if

$$R = \bigoplus_{g \in G} R_g \text{ and } R_g R_h \subseteq R_{g+h} \text{ for all } g,\ h \in G.$$

We will call R_g the *homogeneous component of degree g* of R.

In this thesis the role of G will be usually played by $\mathbb{N}$ with ordinary addition, but we will also have $\mathbb{Z}$-gradings, $\mathbb{Z} \times \mathbb{Z}$-gradings and $\mathbb{Z}_2$-gradings.

An additive subgroup S of a graded algebra R with index semigroup G is called homogeneous if

$$S = \bigoplus_{g \in G} (S \cap R_g).$$

Note that if S is a homogeneous ideal of R then by putting $S_g := S \cap R_g$, for each $g \in G$, we may write:

$$R/S = \bigoplus_{g \in G} R_g / \bigoplus_{g \in G} S_g \simeq \bigoplus_{g \in G} (R_g/S_g).$$

Thus the quotient algebra R/S is endowed with a structure of G-graded algebra, its homogeneous component $(R/S)_g$ being the image of R_g under natural epimorphism is isomorphic to R_g/S_g as k-modules.

Free Lie algebras. Let L be a Lie algebra over k generated by a set X. We say L *free on* X if, for any Lie algebra M and any mapping $\alpha : X \to M$, there exists a unique homomorphism of Lie algebras $\beta : L \to M$ extending α.

The uniqueness (up to isomorphism) of such an algebra L is an immediate consequence of the definition. It is useful to recall how to construct the free Lie algebra with a set X of generators, indicated as $L(X)$. Let us consider the k-module freely generated by X, say V, and let $T(V)$ be the tensor algebra on V, thus $T(V) = \bigoplus_{n \in \mathbb{N}} T_n(V)$, where $T_0(V) = k$ and $T_n(V) = \underbrace{V \otimes \ldots \otimes V}_{n}$ for each $n > 0$. Let us observe that $T(V)$ is a $\mathbb{N}$-graded associative algebra. Let L be the Lie subalgebra of $T(V)^-$ generated by X. Given any map $\alpha : X \to M$, let α be extended first to a linear map $\overline{\alpha} : V \to M \subset U(M)$ then canonically to an associative algebra homomorphism $\hat{\alpha} : T(V) \to U(M)$, or a Lie algebra homomorphism $\hat{\alpha} : T(V)^- \to U(M)^-$ whose restriction to L is the desired β, since $\hat{\alpha}$ maps X into M. Thus $L(X)$ inherits a structure of $\mathbb{N}$-graded Lie algebra in the following way:

$$L(X) = \bigoplus_{n=1}^{\infty} L_n(X),$$

where $L_1(X)$ is the free $k-$module freely generated by the set X (we shall write $\langle X \rangle_k$ in the sequel) and $L_{n+1} := [L_1(X), L_n(X)]$, for all $n \geq 1$.

Let us note that, given a free k-module V with basis X, if Y is another basis of V then $L(Y) = L(X)$ and $L_n(Y) = L_n(X)$, for all $n \geq 1$, thus we may write $L(V)$ instead of $L(X)$.

Finitely presented Lie algebras. Let X be a nonempty set, $L(X)$ a free Lie k-algebra freely generated by X, and R a subset in $L(X)$ (possibly empty). By $(X|R)$ we denote the quotient Lie algebra $G := L(X)/S$ where S is the least ideal of $L(X)$ containing R. A Lie algebra G is called *finitely generated* (resp., *finitely related* or *finitely presented*) if G has a presentation $(X|R)$ in which $|X| < \infty$ (resp., $|R| < \infty$, or $|X|,\ |R| < \infty$). If $X := \{x_1, \dots, x_d\}$ and $R := \{r_1, \dots, r_l\}$ we may replace $(X|R)$ by $(x_1, \dots, x_d | r_1 = 0, \dots, r_l = 0)$. An obvious consequence of the defining property of a free Lie algebra and the definitions given above is the following.

Lemma 1.1.3 *Let $G = (X|R)$ and H be a Lie algebra over a field k. We consider a map $\phi : X \to H$ such that for each $r = r(x_1, \dots, x_d)$ in R we have $r(\phi(x_1), \dots, \phi(x_d)) = 0$. Then the map*

$$\begin{array}{cccc} \overline{\phi}: & G & \longrightarrow & H \\ & w(x_1, \dots, x_d) + S & \mapsto & w(\phi(x_1), \dots, \phi(x_d)) \end{array}$$

is a correctly defined unique homomorphism from G to H such that $\overline{\phi}(x+S) = \phi(x)$, for all $x \in X$.

Moreover, if R is a set of homogeneous elements of degree h, then S is a graded ideal of $L(X)$ and $G = (X|R)$ is a graded Lie algebra, in particular we have the following:

$$S := \bigoplus_{n=1}^{\infty} S_n,$$

where $S_n = (0),\ \forall n < h$ and $S_h := \langle R \rangle_k,\ S_{n+1} := [L_1(X), S_n],\ \forall n \geq h$;

$$G := \bigoplus_{n=1}^{\infty} L_n(X)/S_n.$$

Loop Lie algebras. We use [9] as a reference for this paragraph. Let M be a simple Lie algebra over a field k of characteristic zero. Let $k[t, t^{-1}]$ (with t an indeterminate) denote the ring of polynomials in t and t^{-1}, with coefficients in k. We let

$$\tilde{M} = k[t, t^{-1}] \otimes_k M,$$

and we define a Lie bracket on $\tilde{M}$ by

$$[u \otimes x, v \otimes y] := uv \otimes [x, y], \text{ where } u,\ v \in k[t, t^{-1}] \text{ and } x,\ y \in M.$$

Thus $\tilde{M}$ is an infinite-dimensional Lie algebra with a natural $\mathbb{Z}$-grading defined by requiring that elements $t^n \otimes x$ are homogeneous of degree n. We call $\tilde{M}$ the *loop algebra* of M.

The same kind of construction can be done by using $k[t]$ instead of $k[t, t^{-1}]$ and obtaining in this way an infinite-dimensional Lie algebra with a natural $\mathbb{N}$-grading and we shall use the same name, loop algebras, to indicate these algebras.

Hilbert series and the Golod-Shafarevich theorem. We will use [21] as a reference for this paragraph.

Let $A = \bigoplus_{n \geq 0} A_n$ be a graded algebra over a field k. Recall that the homogeneous components A_n are k-submodules of A such that $A_i A_j \leq A_{i+j}$. Throughout this section we will assume that graded algebras are *connected* that is to say the zero-component is one-dimensional and is generated by the unity of k.

We define the Hilbert series associated with A by

$$A(t) := \sum_{n=0}^{\infty} (\dim_k A_n) t^n.$$

The Hilbert series of a graded algebra A carries information on the asymptotic behaviour of A. In particular, the Golod-Shafarevich theorem has been proved with the aid of the Hilbert series. We find it interesting to show how these series are used in this proof.

Observe that the Hilbert series of a graded algebra A is, in particular, the generating function for A. Let us recall that, given a graded k-vector space $V = \bigoplus_{n=0}^{\infty} V_n$, the formal power series

$$H_V(t) := \sum_{n=0}^{\infty} (\dim_k V_n) t^n,$$

is called the *generating function* for V.

Note that if $U = \bigoplus_{n=0}^{\infty} U_n$ and $V = \bigoplus_{n=0}^{\infty} V_n$ are graded k-modules, then we may introduce natural gradings on the spaces $U \oplus V$ and $U \otimes V$:

$$(U \oplus V)_n = U_n \oplus V_n; \qquad (U \otimes V)_n = \sum_{i=0}^{n} (U_i \oplus V_{n-i}).$$

As a consequence we get that

$$H_{U \oplus V}(t) = H_U(t) + H_V(t)$$

and

$$H_{U\otimes V}(t) = H_U(t)H_V(t).$$

Let us recall that a k-submodule of a graded algebra A is homogeneous if $V_n = V \cap A_n$ gives a $\mathbb{N}$-grading on V.

Thus, if U and V are homogeneous k-submodules of an algebra A we get

$$H_{U+V}(t) \leq H_V(t) + H_U(t)$$

and

$$H_{UV}(t) \leq H_V(t)H_U(t).$$

The inequalities become equalities if the representations (either as the sum or a linear combination of the products respectively) is unique.

We need to extend the concept of generating function as follows, if M is a set of homogeneous elements, then the formal series

$$H_M(t) := \sum d_n t^n,$$

where d_n is the numbers of elements in M of degree n, is called the generating function of the set M.

Let us remark that if kM is the k-module generated by M, then $H_{kM}(t) \leq H_M(t)$, and the equality holds if the elements of M are linearly independent.

We are now ready to show some computations of Hilbert series in the associative algebras context.

Let $X = \{x_1, \dots, x_d\}$ be a set and let $A(X)$ the free associative algebra generated by X. If V is the free k-module freely generated by X, then we may consider the canonical $\mathbb{N}$-grading on $A(X) = \bigoplus_{n=0}^{\infty} A_n(X)$, where $A_1(X) := V$ and $A_{n+1}(X) = A_n(X)A_1(X)$ for all $n \geq 1$. Moreover, let us recall that $A(X) \simeq T(V)$ being $T(V)$ the tensor algebra on V, and $A_n(X) \simeq T_n(V)$ for all $n \geq 0$. Thus we get that $\dim_k(A_n(X)) = d^n$ and

$$H_{A(X)}(t) = (1 - dt)^{-1}.$$

Let I be a homogeneous ideal in $A(X)$, thus there exists a graded k-submodule of $A(X)$, say M, such that $A(X) = M \oplus I$ and $A(X)/I$ is isomorphic to M as graded k-modules. Thus

$$H_{A(X)/I}(t) = H_M(t).$$

Let us assume that the ideal I is generated by a finite set R of homogeneous relations and let r_n be the number of relations in $R \cap A_n(X)$. Let us remark that $A(X) = A(X)X \oplus k$, thus $H_{A(X)}(t) = H_{A(X)}(t)H_X(t) + 1$. Then $I = A(X)RA(X) = A(X)RA(X)X + A(X)R$, hence $I \subseteq IX + MR$ and we get

$$H_I(t) \leq H_I(t)H_X(t) + H_M(t)H_R(t).$$

thus

$$H_{A(X)}(t) - H_{A(X)/I}(t) \leq H_{A(X)}(t) - 1 - H_{A(X)/I}(t)H_X(t) + H_{A(X)/I}(t)H_R(t).$$

As a consequence we get that

$$H_{A(X)/I}(t)(1 - H_X(t) + H_R(t)) \geq 1.$$

Let us observe that if X is a finite set with $|X| = d$ and R is a homogeneous subset of $\bigoplus_{n\geq 2} A_n(X)$ such that $|R \cap A_n(X)| = r_n$ and $r_n \leq (d-1)^2/4$ for all $n \geq 2$, then the associative algebra presented by $(X|R)$ is infinite-dimensional. Indeed, in this case we get $H_X(t) = dt$ and $H_R(t) = \sum_{n\geq 2} r_n t^n$, hence $P(t) = (1 - dt + \sum_{n\geq 2} r_n t^n)^{-1}$. It can be shown that the terms of the series $P(t)$ are positive. Since $H_{(X|R)}(t) \geq P(t)$, we get that $(X|R)$ is an infinite-dimensional associative algebra. Thus we proved the Golod-Shafarevich theorem in the associative algebras context.

The entropy of graded algebras. We will use [16] as a reference for this paragraph since the entropy of graded algebras has been introduced in this paper.

Let $A = \bigoplus_{n\geq 0} A_n$ be a graded algebra over a field k.

We define the *entropy* of A by

$$H(A) := \limsup_{n\to\infty} \sqrt[n]{\dim_k A_n}.$$

It should be noted that $H(A) = 1/R$ being R the radius of convergence of the Hilbert series $A(t)$ associated with A.

It is clear that a free associative algebra of rank d has entropy d. Indeed, if A is a free associative algebra of rank d, then it has a canonical $\mathbb{N}$-grading $A = \bigoplus_{n\geq 1} A_n$ with $\dim_k(A_n) = d^n$.

As it has been said in the introduction, in [4] it has been shown that given a Lie algebra L we get $H(L) = H(U(L))$, being $U(L)$ the universal enveloping algebra of L. Thus, if L is a free Lie algebra of rank d with $d > 1$, then $U(L)$ is a free associative algebra of rank d, as a consequence we get $H(L) = d$.

We find it useful to recall some properties of the entropy function that has been studied in [16].

Theorem 1.1.4 *Let I be a nonzero ideal in a free associative algebra A of finite rank. Then $H(A/I) < H(A)$.*

This result can be extended to the Lie algebras context by noting that if J is a nonzero ideal in a free Lie algebra of rank d, then $U(L/J) \simeq A/J\sigma$, being A the free associative algebra of rank d and $\sigma : L \to A$ the canonical embedding. Thus

$$H(L/J) = H(U(L/J)) = H(A/J\sigma) < H(A) = H(L),$$

and we may state this result as a corollary.

Corollary 1.1.5 *Let J be a nonzero ideal in a free Lie algebra L of finite rank. Then $H(L/J) < H(L)$.*

Moreover, when dealing with finitely presented algebras we may say something more on the entropy of the factor algebra, in particular we may obtain a version of the Golod-Shafarevich theorem in terms of entropy for both associative algebras and Lie algebras.

Theorem 1.1.6 *Let $\mathcal{P}$ be an associative or a Lie presentation containing d generators and r homogeneous relations of degree at least 2 with $r < d^2/4$. If A denotes the algebra presented by $\mathcal{P}$, then $H(A) > 1$.*

1.2 Graphs

We use [5] and [12] as a reference for a general background on graph theory.

A graph G is an ordered pair of disjoint sets (V, E) such that E is a subset of the set $V^{(2)}$ of unordered pairs of V. We consider only finite graphs, that is, V and E are always finite sets. The set V is the set of *vertices* and E is the set of *edges*. If G is a graph, then $V = V(G)$ is the vertex set of G and $E(G)$ is the edge set of G.

An edge $\{u, v\}$ is said to *join* the vertices u and v. If $\{u, v\} \in E(G)$, then u and v are *adjacent* vertices of G. The *order* of G is the number of vertices and the *size* of G is the number of edges in G. We will call (n, m)-graph an arbitrary graph with order n and size m.

We say that $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$.

We say that a mapping $\phi : V \to V'$ is a graph homomorphism between G and G' if for any $\{u, v\} \in E$, we get that $\{\phi(u), \phi(v)\} \in E'$. Two graphs are *isomorphic* if there exists a bijective homomorphism between them.

The size of a graph of order n is at least zero and at most $\binom{n}{2}$. A graph of order n and size $\binom{n}{2}$ is called a *complete n-graph* and is denoted by K_n.

The set of vertices adjacent to a vertex $u \in V(G)$, the *neighbourhood* of u, is denoted by $\Gamma(u)$ and $d(u) := |\Gamma(u)|$ is the *degree* of u. If $V(G) = \{u_1, \dots, u_n\}$ then $d(G) := (d(u_1), \dots, d(u_n))$ is the *score* of G and we can easily prove the following:

$$\sum_{i=1}^{n} d(u_i) = 2|E(G)|.$$

It is easy to note that two isomorphic graphs have the same score, up to reordering.

As we said in the introduction, we are interested in considering all non-isomorphic graphs on d vertices and $\lfloor d/2 \rfloor + 2$ edges for $2 < d \leq 6$. We shall recall a result, namely a theorem by Pólya, that allows people to enumerate non-isomorphic graphs with fixed order and size.

Pólya's Theorem We begin this paragraph with showing how to count non-isomorphic graphs with n vertices.

Let us remark that a graph G on a set V of n vertices is completely determined by its edge set $E(G)$. Furthermore, let $Y := V^{(2)}$, the set of graphs on V may be identified with the set of functions from Y to $\{0, 1\}$, say $\{0, 1\}^Y$, where $\phi \in \{0, 1\}^Y$ corresponds to the graph whose set of edges consists of the elements of Y which ϕ maps onto one. The symmetric group $\Gamma = \mathrm{Sym}(V)$ acts on Y in a natural way, hence it acts on $\{0, 1\}^Y$ as follows: for all $\phi \in \{0, 1\}^Y$ $x \in Y$ $\alpha \in \Gamma$ we get $\phi^\alpha(x) := \phi(x^{\alpha^{-1}})$. We may write $\Gamma^{(2)}$ when we want to emphasise that we are considering the action of Γ on $V^{(2)}$. Thus two graphs with a common vertex set are isomorphic precisely when the corresponding functions lie in the same orbit of Γ. As a consequence enumerating non-isomorphic graphs with vertex set V is equivalent to enumerate the orbits of Γ acting on $\{0,1\}^Y$. This problem can be solved by using the well known Cauchy-Frobenius lemma:

Lemma 1.2.1 (Cauchy-Frobenius Lemma) *Let Γ be a finite group acting on a finite set Ω. Then Γ has m orbits on Ω where*

$$m|\Gamma| = \sum_{g\in\Gamma} |Fix(g)|,$$

where $Fix(g) := \{\omega \in \Omega | \omega^g = \omega\}$.

Let us observe that if $V = \{1, \dots, n\}$, then $\Gamma = \mathrm{Sym}(n)$ and each element $g \in \Gamma$ is an essentially unique product of disjoint cycles (cyclic permutation), let us write $g = \xi_1 \dots \xi_s$ such product, including cycles of length one. Let us remark that if $\phi \in Fix(g)$ then ϕ is constant on the cycles of g, thus $|Fix(g)| = |\{0,1\}|^s$. As a consequence, by using the Cauchy-Frobenius lemma, we obtain that the number of non-isomorphic graphs on n vertices is $1/n! \sum_{g\in\Gamma} 2^{c(x)}$ where $c(x)$ is the number of disjoint cycles in the decomposition of x.

We shall proceed in a similar way to enumerate the non-isomorphic (n,m)-graphs.

For each $\phi \in \{0,1\}^Y$ we may define the *weight* of ϕ as the number of edges of the graph associated to ϕ. Let us observe that for all $g \in \Gamma$ we get $w(\phi) = w(\phi^g)$ thus functions in the same orbit of Γ have the same weight. Therefore the weight $w(F)$ of an orbit F of Γ is the weight of any function ϕ in F. Let C_k be the number of orbits of weight k. Then the series in the indeterminate x, $C(x) = \sum_{k=0}^{\infty} C_k x^k$, is called the *function counting series*. We will use the following generalisation of the Cauchy-Frobenius lemma:

Lemma 1.2.2 *Let Γ be a finite group acting on a finite set Ω. Let w be a function from Ω to $\mathbb{C}$ which is constant on the orbits of Γ on Ω. Let $O_1, \dots O_l$ the orbits of Γ on Ω and denote by $w(O_i) = w(\alpha)$ for all $\alpha \in O_i$. Then*

$$|\Gamma| \sum_{i=1}^{l} w(O_i) = \sum_{g\in\Gamma} \sum_{\beta\in Fix(g)} w(\beta)$$

.

Let us consider $g \in \Gamma$ and its decomposition into disjoint cycles, $g = \xi_1 \dots \xi_s$, including cycles of length one. For each integer $k \in \{1, \dots, n\}$, let $j_k(g)$ be the number of cycles of length k in the disjoint cycle decomposition of g. Let us define the *cycle index* of Γ as the following polynomial in the variables $s_1, \dots s_n$:

$$Z(\Gamma) = Z(\Gamma; s_1, \dots s_n) = |\Gamma|^{-1} \sum_{h \in \Gamma} \prod_{k=1}^{n} s_k^{j_k(h)}.$$

We want to enumerate the non-isomorphic graphs on n vertices by weight.

For each $h \in \Gamma$ and $k \in \mathbb{N}$ let

$$\mathcal{F}(h,k) = Fix(h) \cap \{\phi \in \{0,1\}^Y : w(\phi) = k\} \text{ and } \psi(h,k) = |\mathcal{F}(h,k)|.$$

Let us apply the lemma above for Γ acting on $\{\phi \in \{0,1\}^Y : w(\phi) = k\}$, we obtain that $C_k = 1/|\Gamma| \sum_{h \in G} \psi(h,k)$. Therefore

$$C(x) = 1/|\Gamma| \sum_{h \in \Gamma} \sum_{k=0}^{\infty} \psi(h,k) x^k.$$

The Pólya's enumeration theorem gives us another way to express this series in terms of the cycle index of Γ and in our context it can be stated in the following way.

Theorem 1.2.3 *The polynomial $C(x)$ which enumerates graphs of order n by number of edges is given by*

$$C(x) = Z(S_n^{(2)}, 1+x)$$

where

$$Z(S_n^{(2)}) = \frac{1}{n!} \sum_{(j)} \frac{p!}{\prod k^{j_k} j_k!} \prod s_{2k+1}^{kj_{2k+1}} \prod_k (s_k s_{2k}^{k-1})^{j_{2k}} s_k^{k\binom{j_k}{2}} \prod_{r<t} s_{[r,t]}^{(r,t) j_r j_t},$$

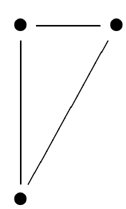
as usual $[r,t]$ denote the l.c.m. and (r,t) the g.c.d.
In this formula $Z(S_n^{(2)}, 1+x)$ is an abbreviation for
$Z(S_n^{(2)}; 1+x, 1+x^2, \dots, 1+x^n)$.

We will use this theorem to count $(d, \lfloor d/2 \rfloor + 2)$-graphs for $d < 7$. We find it interesting to write the cycle index formulas for $S_d^{(2)}$ with $d < 7$:

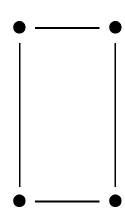
$$
\begin{aligned}
Z(S_2^{(2)}, 1+x) &= 1+x \\
Z(S_3^{(2)}, 1+x) &= 1+x+x^2+x^3 \\
Z(S_4^{(2)}, 1+x) &= 1+x+2x^2+3x^3+2x^4+x^5+x^6 \\
Z(S_5^{(2)}, 1+x) &= 1+x+2x^2+4x^3+6x^4+6x^5+6x^6+4x^7+2x^8+ \\
&\quad +x^9+x^{10} \\
Z(S_6^{(2)}, 1+x) &= 1+x+2x^2+5x^3+9x^4+15x^5+21x^6+24x^7+24x^8+ \\
&\quad +21x^9+15x^{10}+9x^{11}+5x^{12}+2x^{13}+x^{14}+x^{15}
\end{aligned}
$$

We find it useful for our purposes to draw all non-isomorphic $(d, \lfloor d/2 \rfloor + 2)$-graphs for $3 \le d \le 6$.

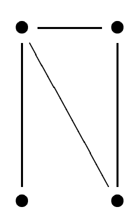
Non-isomorphic $(3,3)$-graphs



Non-isomorphic $(4,4)$-graphs

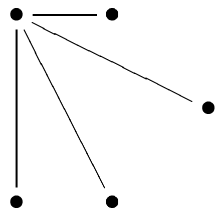


(a)

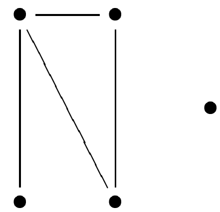


(b)

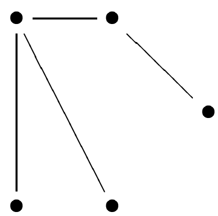
Non-isomorphic $(5, 4)$-graphs



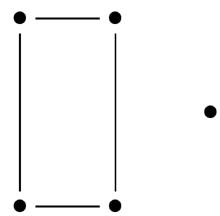
(a)



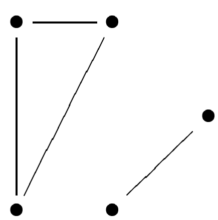
(b)



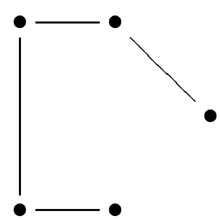
(c)



(d)

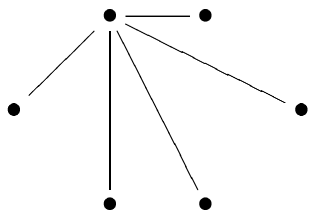


(e)

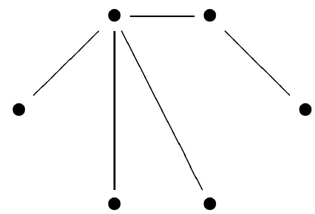


(f)

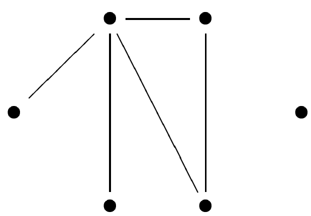
Non-isomorphic (6,5)-graphs



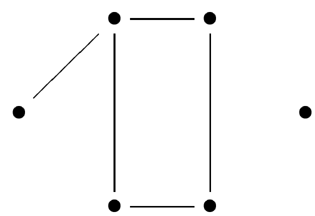
(a)



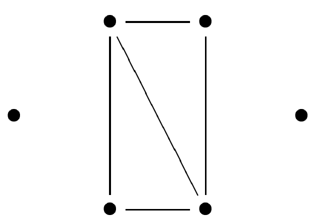
(b)



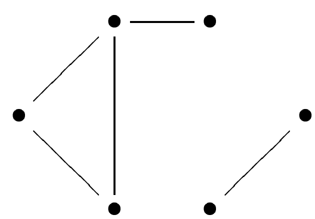
(c)



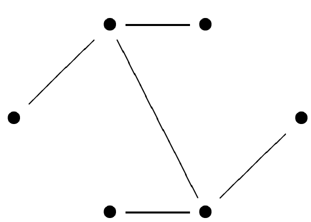
(d)



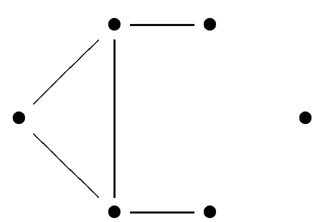
(e)



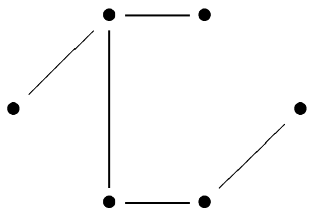
(f)



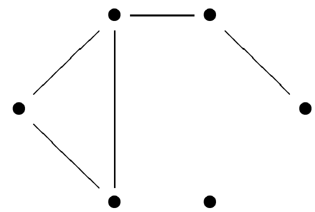
(g)



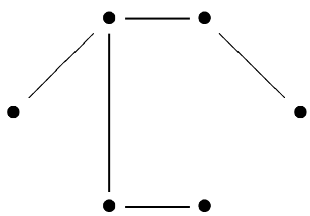
(h)



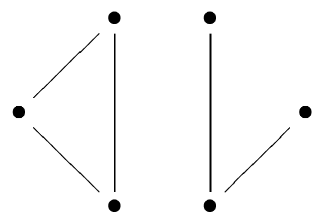
(i)



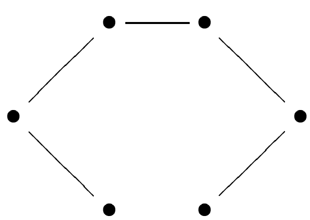
(l)



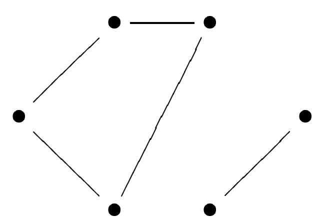
(m)



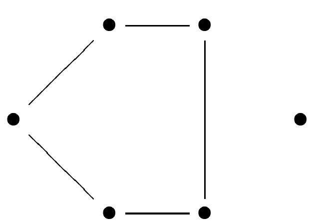
(n)



(o)



(p)



(q)

Chapter 2

Graphs and finitely presented Lie algebras

In his paper [24], Wisliceny shows the asymptotical exactness of the Golod-Shafarevich bound in the Lie algebras context by constructing a series of finite-dimensional nilpotent Lie algebras $L(d)$ with d generators, $d \in \mathbb{N}$, such that

$$\lim_{d \to \infty} \frac{r(L(d))}{d^2} = 1/4.$$

The key fact in this proof is the introduction of the concept of *Erhöhungsystem der Stufe* s, where $s \in \mathbb{N}$.

In particular, let $L(X)$ be the free Lie algebra freely generated by $X = \{x_1, \ldots, x_d\}$ with its canonical $\mathbb{N}$-grading, $L(X) = \bigoplus_{n=1}^{\infty} L_n(X)$.

For each $s \in \mathbb{N}$, let B_s the canonical basis of $L_s(X)$ whose elements are of the form $[x_{i_1}, \ldots, x_{i_s}]$, where $(i_1, \ldots, i_s) \in \{1, \ldots, d\}^s$. The set $R \subset L_s(X)$ is called *Erhöhungsystem der Stufe* s if the following holds.

1. Each element of R is a finite sum of elements of B_s, where in each summand, which is of the form $[x_{i_1}, \ldots, x_{i_s}]$, we have $i_1 \leq \ldots \leq i_s$.

2. For each $\alpha \in B_s$ there exists at most one $\beta \in R$ which has α as a summand.

In his paper Wisliceny proves that if R is a Erhöhungsystem then $(X|R)$ is the presentation of a finite dimensional nilpotent Lie algebra.

Moreover, in his paper [25], the author extends the concept of Erhöhungsystem in the associative algebras context showing as in the previous case that associative algebras presented employing Erhöhungsysteme are finite-dimensional nilpotent associative algebras.

In addition, in this latter paper Wisliceny introduces a concept which is weaker than that defined above. In particular, let $A(X)$ be the free associative algebra freely generated by a set $X = \{x_1, \ldots, x_d\}$ with its natural $\mathbb{N}$-grading, $A = \bigoplus_{n=1}^{\infty} A_n$. We say that the set $R \subset A_2$ is a *Quasierhöhungsystem* of A if the following holds.

1. Each element of R has one of the following forms, $x_{i_1}x_{i_2} - x_{i_3}x_{i_4}$ with $i_1 \leq i_2 \leq i_3 \leq i_4$ or $x_{i_1}x_{i_2}$ with $i_1 \leq i_2$.
2. Each product $x_i x_j$, with $i \leq j$ appears as a summand of exactly one element of R.

As the author observes, the Quasierhöhungsysteme do not always present a finite-dimensional associative algebra even if in many cases they work for this purpose.

Since we are trying to construct presentations of periodic Lie algebras, which are in some sense a borderline between finite-dimensional Lie algebras and infinite-dimensional Lie algebras, we shall use the ideas given by Wisliceny to construct, in the Lie algebras context, sets of relations which are close to the concept of Quasierhöhungsystem. The aim of this chapter is to investigate finitely presented Lie algebras, presented by these sets of relations.

2.1 Presentations in class $\mathcal{C}(d)$ and graphs

Let $d \in \mathbb{N}$, $X = \{x_1, \ldots x_d\}$, $R = \{r_1, \ldots, r_l\}$,

$$\mathcal{M}_d := \{[x_i, x_j] \ : \ i,\ j \ \in \{1, \ldots, d\},\ i \neq j,\ \},$$

$$\mathcal{H}_d := \{[x_i, x_j] - [x_s, x_t] \ : \ i,\ j,\ s,\ t \ \in \{1, \ldots, d\},\ |\{i,\ j, s,\ t\}| \geq 3\ \},$$

we say that the presentation $(X|R)$ belongs to $\mathcal{C}(d)$ if the following conditions hold:

A.1 $l = \lfloor d^2/4 \rfloor + 1$,

A.2 $R \subset \mathcal{M}_d \cup \mathcal{H}_d$,

A.3 for all $[x_i, x_j] \in \mathcal{M}_d$, one and only one of the following holds:

- (a) $[x_i, x_j] \in R$,
- (b) $[x_j, x_i] \in R$,
- (c) there exists one and only one $[x_s, x_t] \in \mathcal{M}_d$ such that $[x_i, x_j] - [x_s, x_t] \in R \cap \mathcal{H}_d$,
- (d) there exists one and only one $[x_s, x_t] \in \mathcal{M}_d$ such that $[x_j, x_i] - [x_s, x_t] \in R \cap \mathcal{H}_d$

Let us remark that we will use the notation $(X|R)$ to indicate either a presentation or the Lie algebra presented in this way, the meaning will be clear from context.

Note that $\mathcal{C}(1) = \emptyset$, in fact $\mathcal{M}_1 = \emptyset = \mathcal{H}_1$, so if $(X|R) \in \mathcal{C}(1)$ then (A.1) implies that $|R| = 1$ and, by using (A.2), we get that $R = \emptyset$.

In addition we get that $\mathcal{C}(2) = \emptyset$, as in this case $\mathcal{M}_2 = \{[x_1, x_2],\ [x_2, x_1]\ \}$ and $\mathcal{H}_2 = \emptyset$, thus, if $(X|R)$ is in $\mathcal{C}(2)$ then, by (A.1), we get that $|R| = 2$, and, by (A.2), $R = \mathcal{M}_2$ but this contradicts (A.3).

So in the sequel we will always assume $d > 2$.

We can prove the following lemma.

Lemma 2.1.1 *Let* $X = \{x_1, \ldots x_d\}$, *and* $R = \{r_1, \ldots, r_l\}$, *if* $(X|R) \in \mathcal{C}(d)$ *then* $|R \cap \mathcal{M}_d| = \lfloor d/2 \rfloor + 2$ *and* $|R \cap \mathcal{H}_d| = \lfloor d^2/4 \rfloor - \lfloor d/2 \rfloor - 1$.

Proof. Let $l_{\mathcal{M}_d} := |R \cap \mathcal{M}_d|$, $l_{\mathcal{H}_d} := |R \cap \mathcal{H}_d|$ and observe that $l = l_{\mathcal{M}_d} + l_{\mathcal{H}_d}$. Since $R \subset \mathcal{M}_d \cup \mathcal{H}_d$ and $\mathcal{M}_d \cap \mathcal{H}_d = \emptyset$, then by (A.1) we get

$$\lfloor d^2/4 \rfloor + 1 = l_{\mathcal{M}_d} + l_{\mathcal{H}_d}.$$

Let us consider $\mathcal{M}_d = \mathcal{M}_d^+ \cup \mathcal{M}_d^-$ where

$$\mathcal{M}_d^+ := \{[x_i, x_j] \in \mathcal{M}_d\ :\ i < j\ \}$$

$$\mathcal{M}_d^- := \{[x_i, x_j] \in \mathcal{M}_d\ :\ i > j\ \}.$$

By using (A.3) we get that $|\mathcal{M}_d^+| = l_{\mathcal{M}_d} + 2l_{\mathcal{H}_d}$. As $|\mathcal{M}_d^+| = d(d-1)/2$, it follows that $d(d-1)/2 = l_{\mathcal{M}_d} + 2l_{\mathcal{H}_d}$. Thus we should solve the following:

$$\begin{cases} l_{\mathcal{M}_d} + l_{\mathcal{H}_d} = \lfloor d^2/4 \rfloor + 1 \\ l_{\mathcal{M}_d} + 2l_{\mathcal{H}_d} = d(d-1)/2 \end{cases}$$

The statement comes easily solving this system. □

We would like to find a way to produce examples of finitely presented Lie algebras in $\mathcal{C}(d)$ with entropy one.

Let us remark that given $(X|R)$, let $R' := \{r \in L(X)\ :\ -r \in R\ \}$, then $(X|R)$, $(X|R \cup R')$, $(X|R')$ are all isomorphic Lie algebras. As a consequence, if $R_{\mathcal{M}} := R \cap \mathcal{M}$, there is no loss of generality assuming that $R_{\mathcal{M}} = R_{\mathcal{M}^+}$.

In order to allow a better understanding of our class of presentations we find it useful to associate with each presentation a graph in the following way.

Definition 2.1.2 *Let* $X = \{x_1, \ldots, x_d\}$ *be a set and* $R \subset L(X)$ *being* $L(X)$ *the free Lie algebra freely generated by* X. *Let* $\mathcal{G}(d, R)$ *be the graph with vertex set*

$$V(d, R) = \{x_1, \ldots, x_d\}$$

and edges set

$$E(d, R) = \{\{x_i, x_j\}\ :\ [x_i, x_j] \in R\ \text{or}\ [x_j, x_i] \in R\ \}.$$

Note that if $(X|R)$ is in $\mathcal{C}(d)$ then $\mathcal{G}(d,R) = \mathcal{G}(d,R_{\mathcal{M}})$ and it is a graph with $\lfloor d/2 \rfloor + 2$ edges.

We can prove the following useful result.

Proposition 2.1.3 *Let $(X|R)$ be in $\mathcal{C}(d)$ and $S \subset \mathcal{M}_d$ such that*

$$(S \cap \mathcal{M}_d^+) \cap \{[x_i, x_j] \ : \ [x_j, x_i] \in S \cap \mathcal{M}_d^-\} = \emptyset.$$

Assume there exists a graph isomorphism ϕ between $\mathcal{G}(d,R)$ and $\mathcal{G}(d,S)$. Let

$$T := \{[\phi(x_i), \phi(x_j)] - [\phi(x_s), \phi(x_t)] \ : [x_i, x_j] - [x_s, x_t] \in R_{\mathcal{H}_d} \},$$

then $(X|S \cup T)$ is in $\mathcal{C}(d)$ and it is isomorphic to $(X|R)$.

Proof. As ϕ is a graph isomorphism, we get that $\phi(V(d,R)) = V(d,S)$ and, for all $[x_i, x_j] \in R_{\mathcal{M}_d}$, $[\phi(x_i), \phi(x_j)] \neq 0$ in $L(X)$. in addition we have that

$$\begin{aligned} [x_i, x_j] \in R_{\mathcal{M}_d} \quad &\Longleftrightarrow \quad \{x_i, x_j\} \in E(d,R) \iff \\ &\Longleftrightarrow \quad \{\phi(x_i), \phi(x_j)\} \in E(d,S) \iff \\ &\Longleftrightarrow \quad [\phi(x_i), \phi(x_j)] \in S, \text{ or } [\phi(x_j), \phi(x_i)] \in S. \end{aligned}$$

Thus $|R_{\mathcal{M}_d}| \leq |S|$. Note that, by hypothesis, $[x_i, x_j] \in S$ implies $[x_j, x_i] \notin S$, as a consequence we get that $|R_{\mathcal{M}_d}| = |S|$. Note that $|T| = |R_{\mathcal{H}_d}|$, thus $|S \cup T| = |R| = \lfloor d^2/4 \rfloor + 1$ and (A.1) is satisfied by the presentation $(X|S \cup T)$.

Let us observe that (A.2) holds for $(X|S \cup T)$. Indeed, since ϕ is a graph isomorphism we get that $T \subset \mathcal{H}_d$, thus $S \cup T \subset \mathcal{M}_d \cup \mathcal{H}_d$.

We shall prove that (A.3) holds for the presentation $(X|S \cup T)$. Let $[x_i, x_j]$ be an arbitrary element of $\mathcal{M}_d$.

Let us assume that $[x_i, x_j]$ is in S, then by hypothesis we get that $[x_j, x_i] \notin S$. in addition, $[x_i, x_j] \in S$ implies that only one between (A.3)(a) and (A.3)(b) holds for $[\phi^{-1}(x_i), \phi^{-1}(x_j)]$ and $R_{\mathcal{M}_d}$. As a consequence neither condition (A.3)(c) nor (A.3)(d) holds for $[\phi^{-1}(x_i), \phi^{-1}(x_j)]$. It follows that neither condition (A.3)(c) nor (A.3)(d) holds for $[x_i, x_j]$ and T.

Furthermore, let us assume that $[x_i, x_j] \notin S$ and $[x_j, x_i] \notin S$, then we get that (A.3)(a) and (A.3)(b) do not hold for $[\phi^{-1}(x_i), \phi^{-1}(x_j)]$ and the set $R_{\mathcal{M}_d}$. As a consequence we get that one and only one between (A.3)(c) and (A.3)(d) holds for $[\phi^{-1}(x_i), \phi^{-1}(x_j)]$ and $R_{\mathcal{H}_d}$. Thus we may assume that there exists a unique $[\phi^{-1}(x_s), \phi^{-1}(t)] \in \mathcal{M}_d$ such that $[\phi^{-1}(x_i), \phi^{-1}(x_j)] - [\phi^{-1}(x_s), \phi^{-1}(t)] \in R_{\mathcal{H}_d}$. It follows that $[x_s, x_t] \in \mathcal{M}_d$ is the only element such that $[x_i, x_j] - [x_s, x_t] \in T$. Thus $(X|S \cup T) \in \mathcal{C}(d)$.

By using Lemma (1.1.3) we can conclude that $(X|R)$ and $(X|S \cup T)$ are isomorphic as Lie algebras. $\square$

Let $A := \{\mathcal{G}_1, \ldots, \mathcal{G}_n\}$ be the set of all non-isomorphic graphs with d vertices and $\lfloor d/2 \rfloor + 2$ edges. By using Proposition 2.1.3 we can restrict our analysis to the presentations $(X|R)$ in $\mathcal{C}(d)$ such that $\mathcal{G}(d, R_{\mathcal{M}_d}) \in A$. Since we may focus our attention on presentations, $(X|R)$, where $R_{\mathcal{M}_d} = R_{\mathcal{M}_d^+}$, we have exactly n possible choices for $R_{\mathcal{M}_d}$. Let $B := \{M_1, \ldots, M_n\}$ where $M_i \subseteq \mathcal{M}_d^+$ is the edges set of $\mathcal{G}_i$. Let us observe that if $(X|M_i \cup T)$, $T \subset \mathcal{H}_d$, and $(X|M_i \cup S)$, $S \subset \mathcal{H}_d$, are two Lie algebras in $\mathcal{C}(d)$ and there exists a graph automorphism, ϕ, of $\mathcal{G}_i$ such that $T = \{[\phi(x_i), \phi(x_j)] - [\phi(x_s), \phi(x_t)] : [x_i, x_j] - [x_s, x_t] \in S\}$, then by the previous proposition we get that these two Lie algebras are isomorphic.

We shall prove a result that gives us a lower bound to the entropy of a Lie algebra $(X|R)$ when $R_{\mathcal{H}}$ has a certain property.

Theorem 2.1.4 *Let $(X|R) \in \mathcal{C}(d)$ where $X := \{x_1, \ldots x_d\}$. Let us assume that there exists $[x_i, x_j] - [x_i, x_h] \in R_{\mathcal{H}_d}$ or $[x_i, x_j] - [x_h, x_i] \in R_{\mathcal{H}_d}$, then the entropy of $(X|R)$ is greater or equal to two.*

Proof. Let $L(X) = \bigoplus_{n=1}^{\infty} L_n(X)$ be the free Lie algebra freely generated by X with its canonical $\mathbb{N}$-grading. We may assume that $[x_i, x_j] - [x_i, x_h] \in R_{\mathcal{H}_d}$.

Let $Y := \{v, w\}$ be a set and let $L(Y)$ be the free Lie algebra freely generated by Y. Consider the k-homomorphism $\phi : X \to L(Y)$ defined in the following way.

$$\phi(x_n) : \begin{cases} 0 & if\ n \notin \{i, j, h\} \\ v & if\ n = i \\ w & if\ n \in \{j, h\} \end{cases}$$

By the universal property of free Lie algebras ϕ uniquely extends to a homomorphism of Lie algebras $\overline{\phi} : L(X) \to L(Y)$. Since $\phi(X) = Y$ and Y generates $L(Y)$ as a Lie algebra we have that $\overline{\phi}$ is an epimorphism of Lie algebras.

We shall show that $R \subset \ker(\overline{\phi})$ as in this case we get that the ideal generated by R, say S, is contained in $\ker(\overline{\phi})$ and we can build an epimorphism of Lie algebras between $(X|R) = L(X)/S$ and $L(Y)$.

Let us observe that $\mathcal{M}_d \setminus \ker(\overline{\phi}) = \{\pm[x_i, x_j], \pm[x_i, x_h]\}$. Since $(X|R) \in \mathcal{C}(d)$, by using (A.3) we get that $[x_i, x_j] - [x_i, x_h] \in R$ is the only element of R containing either $[x_i, x_j]$ or $[x_i, x_h]$. Furthermore, neither $[x_j, x_i]$ nor $[x_h, x_i]$ are cointained in any element of R. Let us observe that $\overline{\phi}([x_i, x_j] - [x_i, x_h]) = [v, w] - [v, w] = 0$. Thus $R \subset \ker(\overline{\phi})$.

As a consequence we can build the following epimorphism of Lie algebras.

$$\begin{array}{rccc} \psi : & L(X)/S & \to & L(Y) \\ & g + S & \mapsto & \overline{\phi}(g) \end{array}$$

Since $L(Y)$ has entropy two (see Corollary 3.2. in [16]) and observing that it is isomorphic to a quotient (not necessarily a proper quotient) of $L(X)/S$ we get that $H(L(Y)) \leq H(L(X)/S)$ (see Corollary 3.1. in [16]). Thus $(X|R)$ has entropy at least two. $\square$

Since we are looking for examples of presentations $(X|R) \in \mathcal{C}(d)$ of Lie algebras with entropy one, this theorem allows us to consider $R_{\mathcal{H}_d} \subset \overline{\mathcal{H}_d} := \{ [x_i, x_j] - [x_s, x_t] \ : \ i,\ j,\ s,\ t\ \in \{1, \ldots, d\},\ |\{i,\ j, s,\ t\}| = 4 \}$.

We can also prove a corollary that establishes a connection between the graph associated with a Lie algebra $(X|R)$ in $\mathcal{C}(d)$, and the entropy of $(X|R)$ for $d < 6$.

Corollary 2.1.5 *Let $d \in \{3,\ 4,\ 5\}$, $(X|R) \in \mathcal{C}(d)$ and let q_i the degree of the vertex x_i in $\mathcal{G}(d, R)$. If there exists $i \in \{1, \ldots, d\}$, such that $q_i < d - \lfloor d^2/4 \rfloor + \lfloor d/2 \rfloor$, then $H((X|R)) \geq 2$.*

Proof. There is no loss of generality assuming that $i = 1$ and let us recall that we may assume $R_{\mathcal{M}_d} = R_{\mathcal{M}_d^+}$.

By hypothesis there exist $x_{n_1}, \ldots, x_{n_{q_1}}$ distinct elements of X such that $A := \{[x_1, x_{n_1}], \ldots, [x_1, x_{n_{q_1}}]\} \subset R_{\mathcal{M}_d}$. Moreover, for all $[x_1, x_j] \notin A$ (A3)(c) or (A3)(d) holds.

We want to show that there exists $[x_1, x_j] - [x_1, x_k]$ in $R_{\mathcal{H}_d}$ and use the previous theorem to reach the conclusion.

Let us assume that there does not exist $[x_1, x_j] - [x_1, x_k] \in R_{\mathcal{H}_d}$ for $j, k \in \{1, \ldots, d\}$. Thus $|\{[x_1, x_j] - [x_s, x_t] \ : \ [x_1, x_j] \notin A\} \cap R_{\mathcal{H}_d}| = (d-1) - q_1 > \lfloor d^2/4 \rfloor - \lfloor d/2 \rfloor - 1 = |R_{\mathcal{H}_d}|$ but this is a contradiction.

Let us observe that we need the restriction on d. Indeed, if $d > 5$ we have $d - \lfloor d^2/4 \rfloor + \lfloor d/2 \rfloor \leq 0$ so in this case the upper bound on q_i does not make any sense. $\square$

Let us recall that we indicate with $K_{1,d-1}$ the graph on d vertices, $\{x_1, \ldots, x_d\}$, with the following edges set: $\{\{x_1, x_j\} \ : \ j = 2, \ldots, d\}$. We can prove another corollary that links the graph associated with the presentation $(X|R)$ in $\mathcal{C}(d)$ and the entropy of the Lie algebra $(X|R)$.

Corollary 2.1.6 *Let $d \in \{4,\ 5\}$, $(X|R) \in \mathcal{C}(d)$ and let us assume that $\mathcal{G}(d, R)$ has a subgraph isomorphic to $K_{1,d-1}$, then $H((X|R)) \geq 2$.*

Proof. Let us observe that $\lfloor d/2 \rfloor + 2 - (d-1) \geq 0$ if and only if $d \leq 6$. As a consequence, if $d > 6$ then $\mathcal{G}(d, R)$ cannot have a subgraph isomorphic to $K_{1,d-1}$.

There is no loss of generality assuming that $K_{1,d-1}$ is a subgraph of $\mathcal{G}(d, R)$. Thus $[x_1, x_j] \in R_{\mathcal{M}_d}$ for all $j \in \{2, \ldots d\}$, as a consequence we get that $R_{\mathcal{H}_d} \subset \{[x_i, x_j] - [x_s, x_t] \ : \ i,\ j,\ s,\ t \in \{2, \ldots, d\},\ |\{i,\ j,\ s,\ t\}| = 4\}$.

Let us observe that $|R_{\mathcal{M}_d}| - (d-1) \leq 1$, so there exists at most one edge in $\mathcal{G}(d, R_{\mathcal{M}_d}) \backslash K_{1,d-1}$.

Since $d \geq 4$, there exists a vertex $x_k \in X$ such that $\{x_1, x_k\} \in E(\mathcal{G}(d, R))$ and $E(\mathcal{G}(d, R)) \cap \{\{x_k, x_j\} \ : \ j \in \{2, \ldots, d\}\} = \emptyset$. Indeed, if $E(\mathcal{G}(d, R)) \cap \{\{x_k, x_j\} \ : \ j \in \{2, \ldots, d\}\} \neq \emptyset$, for all $x_k \in X$, then we have $d(x_k) \geq 2$ for all $k \in \{2, \ldots, d\}$ being $d(x_k)$ the degree of the vertex x_k in the graph $\mathcal{G}(d, R)$. By using the well known formula $2|E(\mathcal{G}(d, R))| = \sum_{k=1}^{d} d(x_k)$ we get that:

$$\begin{array}{ll} 2(\lfloor d/2 \rfloor + 2) \geq (d-1) + 2(d-1) & \Longleftrightarrow \\ 6 \geq 3d - 2\lfloor d/2 \rfloor \geq 2d & \Longleftrightarrow \\ 3 \geq d. & \end{array}$$

We want to show that there exist $i, j \in \{n \ : n \neq k, \ 2 \leq n \leq d\}$ such that $[x_k, x_j] - [x_k, x_i] \in R_{\mathcal{H}_d}$ or $[x_k, x_j] - [x_i, x_k] \in R_{\mathcal{H}_d}$. Let us assume that $R_{\mathcal{H}_d}$ does not contain any element of that kind, so we have that $|R_{\mathcal{H}_d}| \geq (d-2)$. Indeed, $[x_k, x_j] \notin R_{\mathcal{M}_d}$ for all $j \in \{2, \ldots d\}$ such that $j \neq k$. Thus by (A.3), $[x_k, x_j]$ should be a summand in a polynomial of $R_{\mathcal{H}_d}$, for each $j \neq 1, \ k$. By our hypothesis there exist $(d-2)$ distinct polynomials of $R_{\mathcal{H}_d}$ containing these elements.

If $d = 4$, then we find a contradiction as $|R_{\mathcal{H}_d}| = 1$ and $(d-2) = 2$.

If $d = 5$ then we have that $|R_{\mathcal{M}_d}| = 4 = (d-1)$ so $\mathcal{G}(5, R) = K_{1,4}$ and $|R_{\mathcal{H}_d}| = 3 = (d-2)$. Note that $R_{\mathcal{H}_d} \subset \{[x_i, x_j] - [x_s, x_t] : i, \ j, \ s, \ t \in \{2, 3, 4, 5\}, |\{i, j, s, t\}| = 4\}$ thus $|R_{\mathcal{H}_d}| < 1$, and we get a contradiction.

As a consequence there exist $i, j \in \{n \ : n \neq k, \ 2 \leq n \leq d\}$ such that $[x_k, x_j] - [x_k, x_i] \in R_{\mathcal{H}_d}$ or $[x_k, x_j] - [x_i, x_k] \in R_{\mathcal{H}_d}$. By using Theorem (2.1.4) we reach the conclusion. □

2.2 Lie algebras presented by presentations in class $\mathcal{C}(d)$

We are now ready to prove some results on Lie algebras $(X|R)$ in $\mathcal{C}(d)$ for $d < 6$.

Theorem 2.2.1 *If $(X|R)$ is a presentation in $\mathcal{C}(3)$, then the Lie algebra presented by $(X|R)$ is the 3-dimensional abelian Lie algebra.*

Proof. Let $(X|R) \in \mathcal{C}(3)$ and assume $R_{\mathcal{M}_3} = R_{\mathcal{M}_3^+}$ then by Lemma (2.1.1) we get that $|R_{\mathcal{M}_3}| = 3$ and $|R_{\mathcal{H}_3}| = 0$. As $|{\mathcal{M}_3}^+| = 3$ we get that $R = R_{\mathcal{M}_3} = \mathcal{M}_3^+ = \{[x_1, x_2], [x_1, x_3], [x_2, x_3]\}$ and we are done. □

Theorem 2.2.2 *If $(X|R)$ is a presentation in $\mathcal{C}(4)$, then $(X|R)$ is either a 5-dimensional metabelian Lie algebra or a central extension of a free Lie algebra with two free generators.*

Proof. Let us note that if $(X|R) \in \mathcal{C}(4)$ then $|R^+_{\mathcal{M}_4}| = 4$ and $|R_{\mathcal{H}_4}| = 1$. Let us recall that there are two non-isomorphic $(4,4)$-graphs, $\mathcal{G}_1 := \mathcal{G}(X, E_1)$ and $\mathcal{G}_2 := \mathcal{G}(X, E_2)$, where $E_1 := \{\{x_1, x_2\}, \ \{x_1, x_3\}, \ \{x_1, x_4\}, \ \{x_2, x_3\}\}$ and $E_2 := \{\{x_1, x_2\}, \ \{x_1, x_3\}, \ \{x_2, x_4\}, \ \{x_3, x_4\}\}$.

Let $(X|R_1)$ be in $\mathcal{C}(4)$ with $\mathcal{G}(X, R_1) = \mathcal{G}_1$ then we may assume that $R_{1\mathcal{H}_4} = \{[x_2, x_4] - [x_3, x_4]\}$.

Let us remark that from Theorem (2.1.4) it follows that $(X|R_1)$ has entropy at least two. in addition, going through the proof of that theorem we can construct a free Lie algebra of rank two which is an epimorphic image of $(X|R_1)$. Indeed, let us consider the following basis on $L_1(X)$, $Y := \{v_1 := x_1, v_2 := x_2 - x_3, v_3 := x_3, v_4 := x_4\}$. Thus $R = \{[v_1, v_2],\ [v_1, v_3],\ [v_1, v_4],\ [v_2, v_3],\ [v_2, v_4]\}$ and $v_1,\ v_2$ are in the center of $(X|R_1)$. Since the ideal of $L(X)$ generated by R is homogeneous with respect to the canonical $\mathbb{N}$-grading of $L(X)$, then $(X|R)$ can be seen as a $\mathbb{N}$-graded Lie algebra $(X|R) = \bigoplus_{n=1}^{\infty} A_n$. In addition let us note that $(X|R)$ is generated by the first homogeneous component, hence we get $A_{n+1} = [A_n, A_1]$ for all $n \in \mathbb{N}$. Let $Y_1 := \langle\{v_1, v_2\}\rangle_k$ and $Y_2 := \langle\{v_3, v_4\}\rangle_k$. Notice that

$$\begin{aligned} A_2 &= [A_1, A_1] = [Y_1, A_1] + [Y_2, A_2] = [Y_2, A_2] = \\ &= [Y_2, Y_1] + [Y_2, Y_2] = [Y_2, Y_2] = A_2, \end{aligned}$$

and for all $n \in \mathbb{N}$ we get

$$A_{n+1} = [A_n, A_1] = [A_n, Y_2].$$

Thus we have that $(X|R) = Y_1 \oplus B$ being B the Lie subalgebra of $(X|R)$ generated by Y_2. We shall show that B is a free Lie algebra freely generated by $\{v_3, v_4\}$.

Let $\phi : X \to L(\{v_3, v_4\})$ be the map that acts as the zero map on Y_1 and as the identity map on Y_2. As we saw in the proof of Theorem (2.1.4), ϕ uniquely extends to an epimorphism of Lie algebras $\overline{\phi} : (X|R) \to L(\{v_3, v_4\})$. Since $Y_1 \subset \ker(\overline{\phi})$ then the restriction of $\overline{\phi}$ to B is an epimorphism of Lie algebras $\overline{\phi}_{|_B} : B \to L(\{v_3, v_4\})$. By using the universal property of free Lie algebras and observing that B is generated by $\{v_3, v_4\}$ we get that $\overline{\phi}_{|_B}$ is an isomorphism of Lie algebras.

Let $(X|R_2)$ be in $\mathcal{C}(4)$ with $\mathcal{G}(X, R_2) = \mathcal{G}_2$ then we may assume that $R_2 \cap \mathcal{H}_4 = \{[x_1, x_4] - [x_3, x_2]\}$. Since $R_2 \subset L_2(X)$, the ideal S in $L(X)$ generated by R is homogeneous with respect to the canonical $\mathbb{N}$-grading of $L(X)$. Thus we can consider the induced $\mathbb{N}$-grading on $(X|R_2)$, say $(X|R_2) = \bigoplus_{n\in\mathbb{N}} A_n$. Let us observe that $\{[x_1, x_4]\}$ is a basis of A_3 and by using the Jacoby identity together with the relation $[x_1, x_4] - [x_3, x_2]$ we get that $A_3 = (0)$. Thus $(X|R_2)$ is a 5-dimensional metabelian Lie algebra. □

Theorem 2.2.3 *Let $(X|R)$ be a presentation in $\mathcal{C}(5)$ and let us assume that the Lie algebra presented by $(X|R)$ has entropy strictly less than two. Let $\mathcal{G}(5, R)$ the graph associated with $(X|R)$. Then $\mathcal{G}(5, R)$ belongs to one of the three isomorphism classes which have the graphs labelled by (c), (e) and (f) as* *representatives in the list of non-isomorphic* $(5,4)$*-graphs we gave in section 1.2.*

Moreover, for each isomorphism class of $(5,4)$*-graphs there are at most 8 choices for the set* $(X|R)$ *in* $\mathcal{C}(5)$*.*

Proof. Since $(X|R)$ is in $\mathcal{C}(5)$, by Lemma (2.1.1) we get that $|R_{\mathcal{M}_5}| = 4$ and $|R_{\mathcal{H}_5}| = 3$

As we showed in section 1.2 there are six non-isomorphic $(5,4)$-graphs.

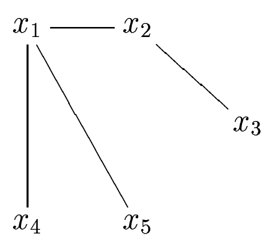
We shall use all the results proved so far in order to exclude from our analysis graphs associated to presentations of Lie algebras with entropy greater or equal to two.

By using Corollary (2.1.5) we can exclude the $(5,4)$-graphs labelled as (b) and (d) in section 1.2 from our analysis. Indeed, in both cases there is a vertex with degree zero thus if $(X|R)$ is associated with one of these graphs, then Corollary (2.1.5) ensures that $(X|R)$ has entropy at least two.

We can also exclude the $(5,4)$-graph labelled by (a). Indeed, it has a subgraph isomorphic to $K_{1,4}$, thus Corollary (2.1.6) says that if $(X|R)$ is associated with this graph, then the Lie algebra presented by $(X|R)$ has entropy at least two.

Thus we obtain the first part of our statement, let us consider the last part.

Assume that $(X|R)$ is associated with the $(5,4)$-graph labelled by (c) and let us draw it in the following way:



where $R \cap \mathcal{M}_5 = \{[x_1, x_2],\ [x_1, x_4],\ [x_1, x_5],\ [x_2, x_3]\}$.

Let us observe that $\{[x_3, x_1],\ [x_3, x_4],\ [x_3, x_5]\}$ is a set of monomials that appear as summands of polynomials belonging to $R \cap \mathcal{H}_5$. If there exist $i,\ j \in \{1, 4, 5\}$ such that $[x_3, x_i] - [x_3, x_j] \in R \cap \mathcal{H}_5$, then by Theorem (2.1.4) we get that the Lie algebra presented by $(X|R)$ has entropy at least two. Thus, if $(X|R)$ is a Lie algebra with entropy less than two, then the three monomials containing the generator x_3 should be summands of distinct polynomials of $R \cap \mathcal{H}_5$. By using this argument also on x_4 and x_5 we find out that $x_3,\ x_4$ and x_5 should appear in each polynomial of $R \cap \mathcal{H}_5$ and none of them should appear in two monomials which are summands of the same polynomial.

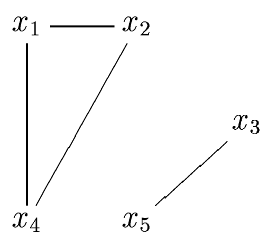
Let us consider the following equivalence relation on the elements of $R \cap \mathcal{H}_5$: $[x_i, x_j] - [x_s, x_t]$ is equivalent to $[x_a, x_b] - [x_c, x_d]$ if and only if $(\{i, j\}, \{s, t\}) = (\{a, b\}, \{c, d\})$ as elements of $\binom{X}{2} \times \binom{X}{2}$.

By the observations above we get that the three elements of $R \cap \mathcal{H}_5$ belong to the following set of distinct equivalence classes:
$\{(\{1,3\},\{4,5\}),\ (\{4,3\},\{5,2\}),\ (\{5,3\},\{4,2\})\}$.

Notice that each equivalence class contains four elements of $\mathcal{H}_5$. Moreover, as we have already observed, nothing change if we replace an element $r \in R$ with $-r$. Thus for each equivalence class C we may restrict our choice of $r \in C \cap R$ to two possibilities. Therefore, given $C = (\{a,b\},\{c,d\})$ let us consider the following two possibilities: either $[a,b] - [c,d]$ or $[b,a] - [c,d]$, where $a < b$ end $c < d$.

In addition, we may fix $[x_1, x_3] - [x_4, x_5]$ as an element belonging to R, obtaining in this way four choices for the other two elements of $R \cap \mathcal{H}_5$. Indeed, consider the Lie algebra automorphism of $L(X)$ which maps x_1 in $-x_1$, then R is mapped in a set R' which contains $[x_3, x_1] - [x_4, x_5]$ and $L(X)/R$ is isomorphic to $L(X)/R'$ as a Lie algebra. Therefore we obtain at most four choices for $(X|R) \in \mathcal{C}(5)$ associated with the graph (c).

Assume that $(X|R)$ is associated with the graph (e). Let us draw it as follows.



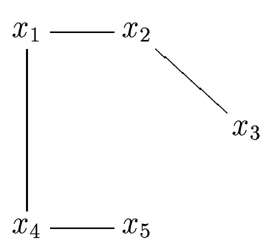
By using the same arguments used in the previous case we find that the three elements of $R \cap \mathcal{H}_5$ belong to the following set of distinct equivalence classes:
$\{(\{1,3\},\{a,5\}),\ (\{2,3\},\{b,5\}),\ (\{4,3\},\{c,5\})\}$, where $a \in \{2,4\}$, $b \in \{1,4\}$, $c \in \{1,2\}$.

Let us assume that $a = 2$ thus we get that the three elements of $R \cap \mathcal{H}_5$ belong to
$\{(\{1,3\},\{2,5\}),\ (\{2,3\},\{4,5\}),\ (\{4,3\},\{1,5\})\}$.

On the other hand, if $a = 4$, then the set is
$\{(\{1,3\},\{4,5\}),\ (\{2,3\},\{1,5\}),\ (\{4,3\},\{2,5\})\}$.

Thus we have 8 possible choices for the $(X|R) \in \mathcal{C}(5)$ associated with the graph (e).

Let us now consider the last possible case, when $(X|R)$ is associated with the graph (f) and let us draw it as follows.



As above we obtain that the three elements of $R \cap \mathcal{H}_5$ belong to the following set of distinct equivalence classes:
$\{(\{1,3\},\{2,5\}),\ (\{4,3\},\{1,5\}),\ (\{5,3\},\{4,2\})\}$. Thus we have 4 choices for $(X|R) \in \mathcal{C}(5)$ associated with (f). $\square$

Even if we are not able to describe completely $\mathcal{C}(5)$, this theorem allows us to restrict our analysis to 16 presentations. By using computational devices we are able to have some hints that indicate that in $\mathcal{C}(5)$ there exists a presentation of a loop Lie algebra. In particular, let us consider the following set of relations associated with the graph labelled by (f): $R = \{[x_1,x_2],[x_1,x_4],[x_2,x_3],[x_4,x_5],[x_3,x_5]-[x_2,x_4],[x_3,x_1]-[x_5,x_2],[x_3,x_4]-[x_5,x_1]\}$.

In the next chapter we shall show that if $k = \mathbb{C}$ then this is the presentation of the loop Lie algebra of $\mathrm{sl}_3(\mathbb{C})$.

Chapter 3

Periodic Lie algebras which are $\mathrm{sl}_2(\mathbb{C})$−modules

Our goal in this chapter is showing that the finitely presented Lie $\mathbb{C}$-algebra $(X|R)$ being $X := \{x_1, \ldots x_5\}$ and $R := \{[x_1, x_2], [x_1, x_4], [x_2, x_3], [x_4, x_5], [x_3, x_5] - [x_2, x_4], [x_3, x_1] - [x_5, x_2], [x_3, x_4] - [x_5, x_1]\}$, is isomorphic to a loop Lie algebra of $\mathrm{sl}_3(\mathbb{C})$.

In order to prove this isomorphism we shall use heavily the representation theory of Lie algebras and in particular the representation theory of the special linear algebra $\mathrm{sl}_2(\mathbb{C})$.

In the sequel we will use $\mathcal{L}$ to indicate the finitely presented Lie algebra $(X|R)$ described above.

3.1 Representations of $\mathrm{sl}_2(\mathbb{C})$

In this section we shall recall some well known results about the representations of $\mathrm{sl}_2(\mathbb{C})$ and we will point out some technical facts that will be useful in the sequel. The main references for this section are [8] and [14].

Let us recall that $\mathrm{sl}_2(\mathbb{C})$ is the Lie algebra of 2×2-matrices with trace zero, the Lie bracket is defined as follows: $[x, y] := xy - yx$ being $x, y \in \mathrm{sl}_2(\mathbb{C})$.

Let us begin with giving a basis of $\mathrm{sl}_2(\mathbb{C})$:

$$W_0 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad W_1 := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad W_{-1} := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

satisfying

$$[W_0, W_1] = 2W_1, \quad [W_0, W_{-1}] = -2W_{-1}, \quad [W_1, W_{-1}] = W_0.$$

Let us note that every finite-dimensional $\mathrm{sl}_2(\mathbb{C})$-module is completely reducible (see [8], Th. 9.19) and, given $m \geq 1$, there is only one isomorphism class of irreducible $\mathrm{sl}_2(\mathbb{C})$-modules of dimension m.

It is useful to describe in details the irreducible representations of $\mathrm{sl}_2(\mathbb{C})$.

Let V be a finite-dimensional irreducible $\mathrm{sl}_2(\mathbb{C})$-module. Since W_0 is semisimple, we get that its action on V is diagonalisable, (see [8], Th. 9.20). Hence we get the following decomposition of V, $V = \bigoplus_{\alpha\in\Phi} V_\alpha$, where Φ is a set of complex numbers such that for any vector, $v \in V_\alpha$, we have $W_0(v) = \alpha\cdot v$. We call $\alpha \in \Phi$ a *weight* of W_0 (or an *eigenvalue* for the action of W_0) in V and we call V_α a *weight space* (or the *eigenspace* associated to the eigenvalue α).

An easy calculation shows that $W_1(V_\alpha) \subset V_{\alpha+2}$ and $W_{-1}(V_\alpha) \subset V_{\alpha-2}$. As a consequence, by using the irreducibility of V, we get that all the complex numbers in Φ must be congruent to one another $\mod 2$.

Moreover, there exists a weight of V, $\beta \in \mathbb{C}$, called the *highest weight* of V, such that $\Phi = \{\beta,\ \beta-2,\ \dots\beta-2t\}$ and $V_\alpha \neq 0$ for all $\alpha \in \Phi$.

Furthermore, given $v \in V_\beta$ we get that $W_1(v) = 0$ and $\{v, W_{-1}(v), \dots, W_{-1}^t(v)\}$ is a basis of V; v is called the *highest weight vector* of V.

It is useful to write down the matrices that describe the action of $W_1,\ W_0,\ W_{-1}$ on V with respect to this basis. Let $\rho : \mathrm{sl}_2(\mathbb{C}) \to \mathrm{gl}(V)$ be the irreducible representation we are considering, then:

$$\rho(W_1) = (a_{i,j}) \in M(t+1,\mathbb{C}) \quad \text{where} \quad a_{i,j} = \begin{cases} 0 & i \neq j-1, \\ i(\beta-i+1) & i = j-1, \end{cases}$$

$$\rho(W_0) = (a_{i,j}) \in M(t+1,\mathbb{C}) \quad \text{where} \quad a_{i,j} = \begin{cases} 0 & i \neq j, \\ (\beta-2(i-1)) & i = j, \end{cases}$$

Moreover, by using the fact that V is an irreducible $\mathrm{sl}_2(\mathbb{C})$-module we may show that β is a non-negative integer and that the eigenvalues α of W_0 on V form a string of integers differing by 2 and symmetric about the origin in $\mathbb{Z}$. Let us remark that by symmetry we can construct a basis of V choosing a vector $v \in V$ such that $W_{-1}(v) = 0$ and the set $\{v,\ W_1(v), \dots W_1^k(v)\}$, where k is the smallest integer such that $W_1^{k+1}(v) = 0$, is a basis of V.

As a consequence we get that there is a unique irreducible representation, $V^{(n)}$, for each non-negative integer n. The $\mathrm{sl}_2(\mathbb{C})$-module $V^{(n)}$ has dimension $(n+1)$ with W_0 having eigenvalues $n, n-2, \dots, -n+2, -n$.

We can point out some more useful facts as corollaries of what we have just said. In particular, any representation V of $\mathrm{sl}_2(\mathbb{C})$ such that the eigenvalues of W_0 all have the same parity and occur with multiplicity one (i.e. $\dim(V_\alpha) = 1$ for all eigenvalues α of W_0) is necessarily irreducible. Moreover, the number of irreducible factors in an arbitrary representation V of $\mathrm{sl}_2(\mathbb{C})$ is exactly the sum of the multiplicities of 0 and 1 as eigenvalues of W_0.

Let us remark that knowing the eigenspace decomposition of given representations allows us to construct the eigenspace decomposition of all their tensor and alternating products.

Lemma 3.1.1 (Clebsch-Gordan formula)

$$V^{(n)} \otimes V^{(m)} = \bigoplus_{j=0}^{q} V^{(n+m-2j)}, \quad \textit{with} \quad q = \min\{m,\ n\}.$$

The standard proof of this formula uses the following combinatorial identity $(\sum_{\mu=0}^{n} x^{n-2\mu})(\sum_{\nu=0}^{m} x^{n-2\nu}) = \sum_{j=0}^{q}(\sum_{\nu=0}^{n+m-2j} x^{n+m-2j-2\nu})$ in connection with formal characters associated to representations of semisimple Lie algebras (see [14], p.126). We shall give a different proof that will be useful in the sequel.

Proof. [Clebsch-Gordan formula] There is no loss of generality assuming $n \geq m$ thus $q = m$.

Let us consider the standard basis of $V^{(n)}$, $\{x_{n-2t} \ : \ t \in \{0, \dots, n\}\}$, where $W_0(x_{n-2t}) = (n-2t)x_{n-2t}$ for all $t \in \{0, \dots, n\}$. Let $\{y_{m-2t} \ : \ t \in \{0, \dots, m\}\}$ be the standard basis of $V^{(m)}$.

Let us recall that for all $a \in \mathrm{sl}_2(\mathbb{C})$ and $v \otimes w \in V^{(n)} \otimes V^{(m)}$ we have $a(v \otimes w) := a(v) \otimes w + v \otimes a(w)$.

For all $s \in \{0, \dots, m\}$ let us consider the following element of $V^{(n)} \otimes V^{(m)}$:

$$z(s) := \sum_{j=0}^{s} (-1)^j x_{-n+2j} \otimes y_{-m+2s-2j}.$$

Let us observe that for $s = 0$ we get $W_{-1}(z(0)) = W_{-1}(x_{-n} \otimes x_{-m}) = 0$, and $W_0(z(0)) = (-n-m)z(0)$, thus there exists an irreducible sl_2-submodule of $V^{(n)} \otimes V^{(m)}$ isomorphic to $V^{(n+m)}$. Let us assume that $s \neq 0$ then

$$\begin{aligned}
W_{-1}(z(s)) &= \textstyle\sum_{j=0}^{s}(-1)^j W_{-1}(x_{-n+2j} \otimes y_{-m+2s-2j}) = \\
&= \textstyle\sum_{j=1}^{s}(-1)^j (x_{-n-2+2j} \otimes y_{-m+2s-2j}) + \\
&\quad + \textstyle\sum_{j=0}^{s-1}(-1)^j (x_{-n+2j} \otimes y_{-m+2s-2-2j})] = \\
&= \textstyle\sum_{j=1}^{s}(-1)^j (x_{-n-2+2j} \otimes y_{-m+2s-2j}) + \\
&\quad + \textstyle\sum_{k=1}^{s}(-1)^{k-1} (x_{-n+2(k-1)} \otimes y_{-m+2s-2k}) = \\
&= \textstyle\sum_{j=1}^{s}(-1)^{j-1}(-1+1)(x_{-n-2+2j} \otimes y_{-m+2s-2j}) = 0.
\end{aligned}$$

In addition we get that

$$\begin{aligned}
W_0(z(s)) &= \textstyle\sum_{j=0}^{s}(-1)^j W_0(x_{-n+2j} \otimes y_{-m+2s-2j}) = \\
&= \textstyle\sum_{j=0}^{s}(-1)^j(-n-m+2s)(x_{-n+2j} \otimes y_{-m+2s-2j}) = \\
&= (-n-m+2s)z(s).
\end{aligned}$$

As a consequence we get that for all $s \in \{0, \dots, m\}$ there exists an irreducible $\mathrm{sl}_2(\mathbb{C})$-submodule of $V^{(n)} \otimes V^{(m)}$ isomorphic to $V^{(m+n-2s)}$ so $\bigoplus_{s=0}^{m} V^{(m+n-2s)}$ is a $\mathrm{sl}_2(\mathbb{C})$-submodule of $V^{(n)} \otimes V^{(m)}$.

We reach the conclusion by observing that

$$\begin{aligned}
\dim_{\mathbb{C}} \left(\bigoplus_{s=0}^{m} V^{(m+n-2s)}\right) &= \\
& \sum_{s=0}^{m}(m+n-2s+1) = \\
&= (m+1)(m+n+1) - 2\sum_{s=1}^{m} s = \\
&= (m+1)(n+1) = \\
&= \dim_{\mathbb{C}} \left(V^{(n)} \otimes V^{(m)}\right).
\end{aligned}$$

□

In the proof of the previous formula we build basis of the irreducible $\mathrm{sl}_2(\mathbb{C})$-submodules of $V^{(n)} \otimes V^{(m)}$. Indeed for each $s \in \{0, \dots, \min\{n, m\}\}$ we consider a vector $z(s) \in V^{(n)} \otimes V^{(m)}$ with weight $-(n+m-2s)$ which is the lowest weight vector of the irreducible component of $V^{(n)} \otimes V^{(m)}$ $\mathrm{sl}_2(\mathbb{C})$-isomorphic to $V^{(n+m-2s)}$. We find it useful to build explicitly highest weight vectors of the irreducible factors of $V^{(n)} \otimes V^{(m)}$ for all $n,\ m \in \mathbb{N}$.

Lemma 3.1.2 *Let $B := \{y_n, \dots, y_{-n}\}$ be the standard basis of $V^{(n)}$ and $C := \{z_m, \dots, z_{-m}\}$ the standard basis of $V^{(m)}$. Then, if $n \geq m$, for $k = 0, \dots, m$, $s_{(n+m-2k)}$ is a highest weight vector of the irreducible $\mathrm{sl}_2(\mathbb{C})$-submodule of $V^{(n)} \otimes V^{(m)}$ isomorphic to $V^{(n+m-2k)}$, where:*

$$s_{(n+m-2k)} = \sum_{j=0}^{k} \lambda_j (y_{n-2j} \otimes z_{m-2k+2j}),$$

where $\lambda_0 := 1$ and for each $j > 0$ we put $\lambda_j = -\lambda_{j-1} \left(\dfrac{(m-k+j)(k-j+1)}{(n-j+1)j}\right)$.

Proof. Let us observe that $W_0(s_{n+m-2k}) = (n+m-2k)s_{(n+m-2k)}$. In addition, by observing that B and C are standard basis we get the following equalities.

$$W_1(s_{(n+m-2k)}) = \sum_{j=0}^{k} \lambda_j(n-j+1)j(y_{n-2j+2} \otimes z_{m-2k+2j})+$$

$$+\sum_{j=0}^{k} \lambda_j(m-k+j+1)(k-j)(y_{n-2j} \otimes z_{m-2k+2j+2}) =$$

$$= \sum_{j=1}^{k} \lambda_j(n-j+1)j(y_{n-2j+2} \otimes z_{m-2k+2j})+$$

$$+\sum_{j=0}^{k-1} \lambda_j(m-k+j+1)(k-j)(y_{n-2j+2} \otimes z_{m-2k+2j+2}) =$$

$$= \sum_{j=1}^{k}(\lambda_j(n-j+1)j)(y_{n-2j+2} \otimes z_{m-2k+2j})+$$

$$+\sum_{j=1}^{k} \lambda_{j-1}(m-k+j)(k-j+1))(y_{n-2j+2} \otimes z_{m-2k+2j}) =$$

$$= \sum_{j=1}^{k}(\lambda_j(n-j+1)j + \lambda_{j-1}(m-k+j)(k-j+1))(y_{n-2j+2} \otimes z_{m-2k+2j}) =$$

$$= 0.$$

Thus for all $k \in \{0, \dots, m\}$ we proved that $s_{(n+m-2k)}$ is a highest weight vector of the irreducible $\mathrm{sl}_2(k)$-submodule of $V^{(n)} \otimes V^{(m)}$ which is isomorphic to $V^{(n+m-2k)}$. □

In the following lemma we show the decomposition of the wedge product of an irreducible $\mathrm{sl}_2(\mathbb{C})$-module as a sum of irreducible $\mathrm{sl}_2(\mathbb{C})$-modules.

Lemma 3.1.3

$$V^{(n)} \wedge V^{(n)} = \bigoplus_{j=0}^{q} V^{(2n-2-4j)},$$

where $q+1 = n/2$ if n is even and $q+1 = (n+1)/2$ if n is odd.

Proof. Let us first note that $\dim_{\mathbb{C}}\left(V^{(n)} \wedge V^{(n)}\right) = \binom{n+1}{2}$ and

$$\begin{aligned} \dim_{\mathbb{C}}\left(\bigoplus_{j=0}^{q} V^{(2n-2-4j)}\right) &= \\ &= \sum_{j=0}^{q}(2n-1-4j) = \\ &= (q+1)(2n-1) - 4\frac{(q+1)q}{2} = \\ &= (q+1)(2n-1-2q) = \binom{n+1}{2}. \end{aligned}$$

Thus it is sufficient to show that there exists a $\mathrm{sl}_2(\mathbb{C})$-submodule of $V^{(n)} \wedge V^{(n)}$ isomorphic to $\bigoplus_{j=0}^{q} V^{(2n-2-4j)}$.

Let $\{x_n, x_{n-2}, \ldots, x_{-n}\}$ be a standard basis of $V^{(n)}$. For $j \in \{0, \ldots, q\}$ let us consider the following element in $V^{(n)} \wedge V^{(n)}$, $z(j) := \sum_{k=0}^{j} (-1)^k x_{-n+2k} \otimes y_{-n+2+4j-2k}$. Observe that for all $j \in \{0, \ldots, q\}$ we get: $W_0(z(j)) = (-2n + 2 + 4j)z(j)$ and $W_{-1}(z(j)) = 0$.

The conclusion comes easily as in the proof of the previous lemma.

It is worth noting that a basis of the weight space $\left(V^{(n)} \wedge V^{(n)}\right)_0$ is $\{x_{n-2j} \wedge x_{-n+2j} : 0 \leq 2j < n\}\}$. Thus the dimension of $\left(V^{(n)} \wedge V^{(n)}\right)_0$ as a $\mathbb{C}$-vector space is $n/2$ if n is even and $(n+1)/2$ if n is odd. Since the weights that appear in the decomposition of $V^{(n)}$ have all the same parity we get that the weights that appear in the decomposition of $V^{(n)} \wedge V^{(n)}$ are all even. As a consequence we get that $\left(V^{(n)} \wedge V^{(n)}\right)_1 = (0)$ and $\dim_{\mathbb{C}} \left(V^{(n)} \wedge V^{(n)}\right)_0$ gives the number of irreducible factors of $V^{(n)} \wedge V^{(n)}$. $\square$

Let us remark that given a $\mathrm{sl}_2(\mathbb{C})$-module V (not necessarily irreducible) we can find the decomposition of $V \wedge V$ as a direct sum of irreducible $\mathrm{sl}_2(\mathbb{C})$-modules using the two lemmas above together with the following well known fact (see [8], p. 474):

Lemma 3.1.4

$$\bigwedge^2 (V^{(n)} \oplus V^{(m)}) \simeq \left(\bigwedge^2 (V^{(n)}\right) \oplus \left(V^{(n)} \otimes V^{(m)}\right) \oplus \left(\bigwedge^2 (V^{(m)}\right).$$

3.2 Finitely presented Lie algebras which are $\mathrm{sl}_2(\mathbb{C})$-modules

Let $k := \mathbb{C}$. Let $M := (X|R)$ be a finitely presented Lie algebra such that $R \subseteq L_2(X)$ being $L_2(X)$ the second homogeneous component of the free Lie algebra $L(X)$ with respect to the canonical $\mathbb{N}$-grading on $L(X)$. Let V be the k-module generated by the set X, as we saw in section 1.1 we may write $L(V)$ instead of $L(X)$. Let S be the ideal generated by R in $L(X)$. Since $R \subseteq L_2(X)$ we get that S is a graded ideal, $S = \bigoplus_{i=1} S_i$ where $S_1 = (0)$, $S_2 = \langle R \rangle_k$ and $S_{i+1} = [L_1(V), S_i]$. Thus M is a graded Lie algebra, $M := \bigoplus_{i=1} M_i$, where $M_i = L_i(V)/S_i$. Let us observe that $[M_1, M_i] = M_{i+1}$.

Let $\rho : \mathrm{sl}_2(k) \to \mathrm{gl}(V)$ be a representation of $\mathrm{sl}_2(k)$. As we have seen in 1.1, $L(V)$ becomes a sl_2-module with respect to the action defined by ρ. Let us assume that S_2 is a $\mathrm{sl}_2(k)$-submodule of $L_2(V)$. As a consequence we get that S is a $\mathrm{sl}_2(k)$-submodule of $L(V)$ and each homogeneous component of S is a $\mathrm{sl}_2(k)$-submodule of $L(V)$.

Since k is a field of zero characteristic we have that every $\mathrm{sl}_2(k)$-module is completely reducible. Thus we get that $L(V)$ is completely reducible and for all $i \in \mathbb{N}$ there exists a $\mathrm{sl}_2(k)$-complement G_i of S_i in $L_i(V)$. Thus M is a $\mathrm{sl}_2(k)$-module and $M_i \simeq_{\mathrm{sl}_2(k)} G_i$ for all $i \in \mathbb{N}$. In addition we get that $M_{i+1} = [M_1, M_i] \simeq_{\mathrm{sl}_2(k)} \dfrac{[M_1, G_i]}{S_{i+1} \cap [M_1, G_i]}$.

It is useful to prove the following lemma.

Lemma 3.2.1 *Let $L = \bigoplus_{i=1}^{\infty} L_i$ be a graded Lie algebra. Let us assume that L is a $\mathrm{sl}_2(k)$-module where the homogeneous components are finite-dimensional $\mathrm{sl}_2(k)$-modules. Then $[L_i, L_j]$ is a $\mathrm{sl}_2(k)$-submodule of L_{i+j} and $[L_i, L_j] \leq_{\mathrm{sl}_2} L_i \otimes L_j$, $\forall i, j$.*

Proof. Let us consider the following map:

$$\begin{array}{rccc} \eta : & L_i \otimes L_j & \to & [L_i, L_j] \\ & x \otimes y & \mapsto & [x, y] \end{array}$$

Let us note that it is a $\mathrm{sl}_2(k)$-epimorphism as for all $x \otimes y \in L_i \otimes L_j$ and for all $w \in \mathrm{sl}_2(k)$ we get:

$$\begin{aligned} \eta(w(x \otimes y)) &= \eta(w(x) \otimes y + x \otimes w(y)) = \\ &= [w(x), y] + [x, w(y)] = \\ &= w([x, y]) = w(\eta(x \otimes y)). \end{aligned}$$

As every representation of $\mathrm{sl}_2(k)$ is completely reducible we are done. □

3.3 Examples

Let us consider $k := \mathbb{C}$

3.3.1 The loop algebra $\hat{\mathrm{sl}}^{(2)}(3)$

In $M(3,k)$, the algebra of 3×3 matrices, let

$$w_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad w_1 = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$w_{-1} = -\sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$v_2 = 4\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad v_1 = -\sqrt{2}\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$v_0 = \frac{2}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad v_{-1} = -\sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$v_{-2} = 4\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

If we define the bracket on $M(3,k)$ in the canonical way, then we find out that $\{w_1, w_0, w_{-1}, v_i : \ -2 \geq i \geq 2\}$ forms a basis of the split simple Lie algebra $\mathrm{sl}_3(k)$. In addition we get that $V_0 = \bigoplus_{-1 \leq i \leq 1} kw_i$ is isomorphic to $\mathrm{sl}_2(k)$ as

$$[w_i, w_j] = (j-i)w_{i+j} \qquad \forall i,j \in \{-1,0,1\}.$$

Moreover $V_1 = \bigoplus_{i=-2}^{2} kv_i$ is an irreducible 5-dimensional V_0-module, indeed

$$[w_i, v_j] = (j-2i)v_{i+j}, \qquad \forall i \in \{0, \pm 1\}, \quad \forall j \in \{0, \pm 1, \pm 2\}.$$

The vector space

$$\hat{sl}^{(2)}(3) := \bigoplus_{i=1}^{\infty}(k(t^{2i} \otimes V_0) \oplus k(t^{2i-1} \otimes V_1))$$

becomes a Lie algebra with the bracket:

$$[t^i \otimes u, t^j \otimes v] = t^{i+j} \otimes [u,v].$$

Moreover, $\hat{sl}^{(2)}(3)$ becomes a V_0-module by defining the action as follows:

$$\begin{array}{l} \forall v \in V_0,\ \forall t^j \otimes z \in \hat{sl}^{(2)}(3) \\ v(t^j \otimes z) := t^j \otimes [v, z]. \end{array}$$

In the sequel we will use the bracket to indicate this action of V_0 on $\hat{sl}^{(2)}(3)$.

Furthermore, let γ the Lie algebras isomorphism between $\mathrm{sl}_2(k)$ and V_0, we may consider $\hat{sl}^{(2)}(3)$ as a $\mathrm{sl}_2(k)$ by defining the action as follows:

$$\begin{array}{l} \forall x \in \mathrm{sl}_2(k),\ \forall z \in \hat{sl}^{(2)}(3) \\ x(z) := \gamma(x)(z). \end{array}$$

Note that $[v_i, v_j] \in kw_{i+j}$ so $\hat{sl}^{(2)}(3)$ is a $\mathbb{N}$-graded Lie algebra where the homogeneous components of even degree are $k(t^{2i} \otimes V_0)$ and those of odd degree are $k(t^{2i-1} \otimes V_1)$. Observe that $[v_0, v_1] = 2w_1,\ [v_0, v_{-1}] = -2w_{-1}$, so $\hat{sl}^{(2)}(3)$ is generated as a Lie algebra by the first homogeneous component, $k(t \otimes V_1)$.

It is useful to write down the multiplication table of $\hat{sl}^{(2)}(3)$:

[,]	v_2	v_1	v_0	v_{-1}	v_{-2}
v_2	0	0	0	$4w_1$	$16w_0$
v_1	0	0	$-2w_1$	$-2w_0$	$4w_{-1}$
v_0	0	$2w_1$	0	$-2w_{-1}$	0
v_{-1}	$-4w_1$	$2w_0$	$2w_{-1}$	0	0
v_{-2}	$-16w_0$	$-4w_{-1}$	0	0	0

[,]	w_1	w_0	w_{-1}
w_1	0	$-w_1$	$-2w_0$
w_0	w_1	0	$-w_{-1}$
w_{-1}	$2w_0$	w_{-1}	0

[,]	v_2	v_1	v_0	v_{-1}	v_{-2}
w_1	0	$-v_2$	$-2v_1$	$-3v_0$	$-4v_{-1}$
w_0	$2v_2$	v_1	0	$-v_{-1}$	$-2v_{-2}$
w_{-1}	$4v_1$	$3v_0$	$2v_{-1}$	v_{-2}	0

We shall show that our finitely presented Lie algebra $\mathcal{L}$ is isomorphic to $\hat{sl}^{(2)}(3)$. Let us begin with a theorem that shows that $\hat{sl}^{(2)}(3)$ is an epimorphic image of $\mathcal{L}$.

Theorem 3.3.1 *Let V be a free k-module with basis $X = \{x_1, \ldots, x_5\}$. Let $L(X) = \bigoplus_{n=1}^{\infty} L_n(X)$ be the free Lie algebra freely generated by X with its* *canonical* $\mathbb{N}$*-grading. Consider*

$$\begin{aligned} Rel = \{ \quad & [x_1, x_2],\ [x_1, x_4],\ [x_2, x_3],\ [x_4, x_5], \\ & [x_3, x_5] - [x_2, x_4],\ [x_3, x_1] - [x_5, x_2],\ [x_3, x_4] - [x_5, x_1]\} \end{aligned}$$

a subset of $L_2(X)$. *Let* $\mathcal{L} = (X|Rel)$ *the factor algebra* $L(X)/S$ *where* S *is the least ideal of* $L(X)$ *containing* Rel. *Then* $\hat{sl}^{(2)}(3)$ *is an epimorphic image of* $\mathcal{L}$.

Proof. As we observed we may write $L(V) = L(X)$. We consider the following new free generating set for the k-module V, $Y = \{y_4,\ y_2,\ y_0,\ y_{-2},\ y_{-4}\}$ where

$$\begin{cases} y_4 = x_4 \\ y_2 = 2x_5 \\ y_0 = -12x_1 \\ y_{-2} = -24x_3 \\ y_{-4} = 96x_2 \end{cases}$$

Note that with this new basis on V we get
$Rel = \{[y_0, y_{-4}], [y_0, y_4], [y_{-4}, y_{-2}], [y_2, y_4], -1/96(2[y_{-2}, y_2]+[y_{-4}, y_4]), 1/288([y_{-2}, y_0]-3/2[y_2, y_{-4}]), -1/24([y_{-2}, y_4] - [y_2, y_0])\}$. Since $Rel \subset L_2(V)$, it follows that S is a graded ideal in $L(V)$, $S = \bigoplus_{i=1}^{\infty} S_i$ where $S_1 = (0)$ and S_2 is the k-module generated by Rel.

Let us consider V as an irreducible 5-dimensional $\mathrm{sl}_2(k)$-module with Y as a standard basis. Thus the action of $\mathrm{sl}_2(k)$ on V with respect to the basis Y is described by the following matrices.

$$W_0 = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix}$$

$$W_1 = \begin{bmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$W_{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Since V generates $L(V)$ as a Lie algebra, we can extend this action on V to $L(V)$. As a consequence $L(V)$ becomes a $\mathrm{sl}_2(k)$-module and every homogeneous component of $L(V)$ is a finite-dimensional $\mathrm{sl}_2(k)$-module.

Note that we can consider the following Lie algebras isomorphism between V_0 and sl_2:

$$\begin{array}{rlcl} \alpha: & \mathrm{sl}_2(k) & \to & V_0 \\ & W_0 & \mapsto & 2w_0 \\ & W_{-1} & \mapsto & w_{-1} \\ & W_1 & \mapsto & -w_1 \end{array}$$

As we observed above, this isomorphism of Lie algebras induces an action of $\mathrm{sl}_2(k)$ on $\hat{sl}^{(2)}(3)$ in the following way:

$$\begin{aligned} &\forall x \in \mathrm{sl}_2(k),\ \forall z \in \hat{sl}^{(2)}(3), \\ &\qquad x(z) := [\alpha(x), z]. \end{aligned}$$

In the sequel we will use the bracket to indicate this action of $\mathrm{sl}_2(k)$ on $\hat{sl}^{(2)}(3)$.

Notice that V_1 is a V_0-module so we may define the action of $\mathrm{sl}_2(k)$ on V_1 induced by α in the obvious way. Since $\alpha(W_0) = 2w_0$, we may observe that V_1 can be decomposed as a direct sum of weight spaces as follows:

$$V_1 = (V_1)_4 \oplus (V_1)_2 \oplus (V_1)_0 \oplus (V_1)_{-2} \oplus (V_1)_{-4}.$$

Indeed, $\{v_i : i \in \{\pm 1, 0, \pm 2\}\}$ is a basis of V_1 and $W_0(v_i) = 2iv_i$ for all $i \in \{\pm 1, 0, \pm 2\}$. Thus each eigenspace is 1-dimensional, in particular, the 0-eigenspace is 1-dimensional hence V_1 is an irreducible 5-dimensional $\mathrm{sl}_2(k)$-module. Since four is the highest weight and v_2 is the highest weight vector of V_1 with respect to the action induced by α, we may construct a standard basis of V_1 as follows:

$$\begin{array}{ll} f_4 & := v_2, \\ f_2 & := W_{-1}(v_2) = 4v_1, \\ f_0 & := W_{-1}^2(v_2) = 12v_0, \\ f_{-2} & := W_{-1}^3(v_2) = 24v_{-1}, \\ f_{-4} & := W_{-1}^4(v_2) = 24v_{-2}. \end{array}$$

Hence we have two standard basis for the two $\mathrm{sl}_2(k)$-modules V and $k(t \otimes V_1)$ and we are ready to construct a $\mathrm{sl}_2(k)$-isomorphism between V and $k(t \otimes V_1)$:

$$\begin{array}{rlcl} \phi: & V & \to & k(t \otimes V_1) \\ & y_4 & \mapsto & t \otimes v_2 \\ & y_2 & \mapsto & 4(t \otimes v_1) \\ & y_0 & \mapsto & 12(t \otimes v_0) \\ & y_{-2} & \mapsto & 24(t \otimes v_{-1}) \\ & y_{-4} & \mapsto & 24(t \otimes v_{-2}) \end{array}$$

Note that the action of $\mathrm{sl}_2(k)$ on $k(t \otimes V_1)$ with respect to the standard basis $\{\phi(y_4), \phi(y_2), \phi(y_0), \phi(y_{-2}), \phi(y_{-4})\}$ is described by the same matrices which describe the action of $\mathrm{sl}_2(k)$ on V with respect to the standard basis $\{y_4, y_2, y_0, y_{-2}, y_{-4}\}$. Thus ϕ is a $\mathrm{sl}_2(k)$-isomorphism.

Since $k(t \otimes V_1)$ generates $\hat{sl}^{(2)}(3)$ as a Lie algebra, it follows that ϕ uniquely extends to an epimorphism of Lie algebras $\overline{\phi} : L(V) \to \hat{sl}^{(2)}(3)$ such that $\overline{\phi}_{|V} = \phi$.

We shall show, by using induction over the degrees of the homogeneous components, that $\overline{\phi}$ is a $\mathrm{sl}_2(k)$-epimorphism.

Since $\overline{\phi}_{|_{L_1(V)}} = \phi$ we have that the restriction of $\overline{\phi}$ to $L_1(V)$ is a $\mathrm{sl}_2(k)$-homomorphism. Let us assume that for all $k < n$ the restriction of $\overline{\phi}$ to $\bigoplus_{s=1}^{k} L_s(V)$ is a $\mathrm{sl}_2(k)$-homomorphism, we shall show that this is true also for $\bigoplus_{s=1}^{n} L_s(V)$. Since $L(V)$ is generated by $L_1(V)$ we get that $L_n(V) = [L_1(V), L_{n-1}(V)]$. Thus it is sufficient to show that for all $v \in L_1(V)$ and for all $u \in L_{n-1}(V)$ we get that for all $x \in \mathrm{sl}_2(k)$, $\overline{\phi}(x([u,v])) = x(\overline{\phi}([u,v]))$. Let us observe that, by using the induction hypothesis and the fact that $\overline{\phi}$ is a Lie algebras homomorphism, we get the following equalities:

$$\begin{aligned}
\overline{\phi}(x([u,v])) &= \overline{\phi}([x(u),v] + [u,x(v)]) = \\
&= \overline{\phi}([x(u),v]) + \overline{\phi}([u,x(v)]) = \\
&= [\overline{\phi}(x(u)), \overline{\phi}(v)] + [\overline{\phi}(u), \overline{\phi}(x(v))] = \\
&= [x(\overline{\phi}(u)), \overline{\phi}(v)] + [\overline{\phi}(u), x(\overline{\phi}(v)] = \\
&= x([\overline{\phi}(u), \overline{\phi}(v)]).
\end{aligned}$$

Thus $\overline{\phi}$ is a $\mathrm{sl}_2(k)$-epimorphism.

We shall show that the ideal S of $L(V)$ generated by Rel is contained in $\ker(\overline{\phi})$. Therefore we will be able to construct a Lie algebra epimorphism between $L(V)/S$ and $\hat{sl}^{(2)}(3)$ obtaining the desired result.

Since $L_1(V)$ is an irreducible 5-dimensional $\mathrm{sl}_2(k)$-module and $L_2(V) \simeq_{\mathrm{sl}_2(k)} L_1(V) \wedge L_1(V)$, by using Lemma (3.1.4) we get that $L_2(V) \simeq_{\mathrm{sl}_2(k)} V^{(6)} \oplus V^{(2)}$.

Therefore, $k(t^2 \otimes V_0)$ is an irreducible 3-dimensional $\mathrm{sl}_2(k)$-module and $L_2(V)$ is a direct sum of an irreducible 7-dimensional $\mathrm{sl}_2(k)$-module and an irreducible 3-dimensional $\mathrm{sl}_2(k)$-module. As $\overline{\phi}$ is a $\mathrm{sl}_2(k)$-epimorphism, it follows that $\ker(\overline{\phi}) \cap L_2(V)$ is an irreducible 7-dimensional $\mathrm{sl}_2(k)$-module.

As a consequence, if we show that S_2 is an irreducible 7-dimensional $\mathrm{sl}_2(k)$-module, then we get that $S_2 \subseteq \ker \overline{\phi}$. Furthermore, if $S_2 \subseteq \ker \overline{\phi}$, then, by using the fact that $\overline{\phi}$ is a Lie algebras homomorphism, we get that the ideal S of $L(V)$ generated by S_2 is contained in $\ker \overline{\phi}$.

In order to see that S_2 is a $\mathrm{sl}_2(k)$-module it is sufficient to show that the irreducible 7-dimensional $\mathrm{sl}_2(k)$-module generated by $[y_2, y_4] \in S_2$ is contained in S_2.

Observe that $W_0([y_4, y_2]) = 6[y_4, y_2]$ and $W_1([y_4, y_2]) = 0$, so $[y_2, y_4]$ generates an irreducible 7-dimensional sl_2-submodule of $L_2(V)$.

Let us consider the standard basis for this $\mathrm{sl}_2(k)$-module obtained applying W_{-1}:

$$\begin{aligned}
a_1 &:= [y_2, y_4] \in S_2, \\
a_2 &:= [W_{-1}, [y_2, y_4]] = [y_0, y_4] + [y_2, y_2] = \\
&= [y_0, y_4] \in S_2, \\
a_3 &:= [W_{-1}, [y_0, y_4]] = [y_{-2}, y_4] + [y_0, y_2] \in S_2 \\
a_4 &:= [W_{-1}, [y_{-2}, y_4]] + [W_{-1}, [y_0, y_2]] = \\
&= [y_{-4}, y_4] + [y_{-2}, y_2] + [y_{-2}, y_2] + [y_0, y_0] = \\
&= [y_{-4}, y_4] + 2[y_{-2}, y_2] \in S_2 \\
a_5 &:= [W_{-1}, [y_{-4}, y_4]] + 2[W_{-1}, [y_{-2}, y_2]] = \\
&= [y_{-4}, y_2] + 2([y_{-4}, y_2] + [y_{-2}, y_0]) = \\
&= 3[y_{-4}, y_2] + 2[y_{-2}, y_0] \in S_2 \\
a_6 &:= 3[W_{-1}, [y_{-4}, y_2]] + 2[W_{-1}, [y_{-2}, y_0]] = \\
&= 3[y_{-4}, y_0] + 2[y_{-4}, y_0] = 5[y_{-4}, y_0] \in S_2 \\
a_7 &:= 5[W_{-1}, [y_{-4}, y_0]] = 5[y_{-4}, y_{-2}] \in S_2
\end{aligned}$$

As a consequence S_2 is an irreducible 7-dimensional $\mathrm{sl}_2(k)$-module contained in $\ker \overline{\phi}$ so $S \subset \ker \overline{\phi}$ and we get that $\hat{sl}^{(2)}(3)$ is an epimorphic image of $\mathcal{L}$. □

Let us remark that in the proof of this theorem we used only the fact that V is an irreducible 5-dimensional $\mathrm{sl}_2(k)$-module and that S_2 is an irreducible 7-dimensional $\mathrm{sl}_2(k)$-submodule of $L_2(V)$.

We shall show that $\mathcal{L}$ is isomorphic as a Lie algebra to $\hat{sl}^{(2)}(3)$.

Theorem 3.3.2 *Let V be a 5-dimensional irreducible* $\mathrm{sl}_2(k)$*-module and*

$$L(V) := \bigoplus_{i=1}^{\infty} L_i(V),$$

the free Lie algebra generated by V with its canonical $\mathbb{N}$-grading. Let Rel be a 7-dimensional $\mathrm{sl}_2(k)$*-submodule of $L_2(V)$. Consider $\mathcal{L}(V) = (V|Rel)$ the factor algebra $L(V)/S$ being S the least ideal of $L(V)$ containing Rel. Then*

$$\mathcal{L}(V) = \bigoplus_{i=1}^{\infty} \mathcal{L}_i(V)$$

is $\mathrm{sl}_2(k)$*-isomorphic to $\hat{sl}^{(2)}(3)$ as a Lie algebra.*

Proof. Let us remark that as $Rel \subset L_2(V)$, it follows that S is a homogeneous ideal of $L(V)$ and the $\mathbb{N}$-grading of $L(V)$ induces a $\mathbb{N}$-grading on $\mathcal{L}(V)$. Thus we may write $\mathcal{L}(V) = \bigoplus_{n=1}^{\infty} \mathcal{L}_n(V)$.

Let us consider a standard basis on V, $Y := \{y_4,\ y_2,\ y_0,\ y_{-2},\ y_{-4}\}$, where $y_i \in Y$ has weight i with respect to W_0.

By using Lemma (3.1.4) we obtain that

$$L_2(V) \simeq_{\mathrm{sl}_2(k)} L_1(V) \wedge L_1(V) \simeq_{\mathrm{sl}_2(k)} V^{(6)} \oplus V^{(2)}.$$

Since Rel is a 7-dimensional $\mathrm{sl}_2(k)$-submodule of $L_2(V)$ it is irreducible.

Moreover, by using Lemma (3.1.2) we may construct a standard basis of Rel:

$$\begin{aligned}
r^6_{(2,6)} &= [y_2, y_4],\\
r^6_{(2,4)} &= [y_0, y_4],\\
r^6_{(2,2)} &= [y_0, y_2] + [y_{-2}, y_4],\\
r^6_{(2,0)} &= 2[y_{-2}, y_2] + [y_{-4}, y_4],\\
r^6_{(2,-2)} &= 3[y_{-4}, y_2] + 2[y_{-2}, y_0],\\
r^6_{(2,-4)} &= 5[y_{-4}, y_2],\\
r^6_{(2,-6)} &= 5[y_{-4}, y_4].
\end{aligned}$$

Note that $r^s_{(2,j)}$ is the element of weight j that belongs to the standard basis of the irreducible s-dimensional $\mathrm{sl}_2(k)$-submodule of $L_2(V)$.

As we showed in the proof of the previous theorem, the following $\mathrm{sl}_2(k)$-isomorphism

$$\begin{array}{llll}
\phi: & V & \rightarrow & k(t \otimes V_1)\\
& y_4 & \mapsto & t \otimes v_2\\
& y_2 & \mapsto & 4(t \otimes v_1)\\
& y_0 & \mapsto & 12(t \otimes v_0)\\
& y_{-2} & \mapsto & 24(t \otimes v_{-1})\\
& y_{-4} & \mapsto & 24(t \otimes v_{-2})
\end{array}$$

uniquely extends to a $\mathrm{sl}_2(k)$-epimorphism of Lie algebras $\overline{\phi} : L(V) \to \hat{sl}^{(2)}(3)$.

Furthermore, as we observed above, we get the following $\mathrm{sl}_2(k)$-epimorphism of Lie algebras.

$$\begin{array}{rccc} \rho: & \mathcal{L}(V) & \to & \hat{sl}^{(2)}(3) \\ & x+S & \mapsto & \overline{\phi}(x) \end{array}$$

We shall show that ρ is a bijection.

We will divide our proof in the following steps.

Step 1 For each homogeneous component $L_i(V)$ of $L(V)$ we construct the standard basis of an irreducible $\mathrm{sl}_2(k)$-submodule G_i, which is 3-dimensional if i is even and 5-dimensional otherwise.

Step 2 By induction on the degrees of the homogeneous components we will show that the restriction of $\overline{\phi}$ to G_i is injective for all $i > 0$.

Step 3 We shall show that $\mathcal{L}_n(V) = G_i + S_i$, for all $i > 0$. In particular, this means that $\mathcal{L}_i(V) \simeq_{\mathrm{sl}_2(k)} G_i$ and ρ is an isomorphism.

Step 1.

We shall construct recursively a sequence of k-submodules of $L(V)$, $\{G_i\}_{i\in\mathbb{N}}$, with basis Y_i.

Let $Y_1 = Y$, assume we constructed Y_s for all $s < i$.

If i is even, then $Y_i := \{y_{(i,2)},\ y_{(i,0)},\ y_{(i,-2)}\}$ where

$$\begin{aligned} y_{(i,2)} &= 2([y_4, y_{(i-1,-2)}] - [y_{-2}, y_{(i-1,4)}]) - 3([y_2, y_{(i-1,0)}] - [y_0, y_{(i-1,2)}]), \\ y_{(i,0)} &= 2([y_4, y_{(i-1,-4)}] - [y_{-4}, y_{(i-1,4)}]) - ([y_2, y_{(i-1,-2)}] - [y_{-2}, y_{(i-1,2)}]), \\ y_{(i,-2)} &= ([y_2, y_{(i-1,-4)}] - [y_{-4}, y_{(i-1,2)}]) - ([y_0, y_{(i-1,-2)}] - [y_{-2}, y_{(i-1,0)}]). \end{aligned}$$

If i is odd, then $Y_i := \{y_{(i,4)},\ y_{(i,2)},\ y_{(i,0)},\ y_{(i,-2)}, y_{(i,-4)}\}$, where:

$$\begin{aligned} y_{(i,4)} &= [y_2, y_{(i-1,2)}] - 2[y_4, y_{(i-1,0)}], \\ y_{(i,2)} &= [y_0, y_{(i-1,2)}] - [y_2, y_{(i-1,0)}] - 2[y_4, y_{(i-1,-2)}], \\ y_{(i,0)} &= [y_{-2}, y_{(i-1,2)}] - 3[y_2, y_{(i-1,-2)}], \\ y_{(i,-2)} &= [y_{-4}, y_{(i-1,2)}] + [y_{-2}, y_{(i-1,0)}] - 3[y_0, y_{(i-1,-2)}], \\ y_{(i,-4)} &= 2[y_{-4}, y_{(i-1,0)}] - 2[y_{-2}, y_{(i-1,-2)}]. \end{aligned}$$

Note that we are considering $y_{(1,i)} := y_i$ for $i \in \{4,\ 2,\ 0,\ -2,\ -4\}$.

We shall show by induction on the degree of the homogeneous components of $L(V)$ that for all $i > 0$, G_i is an irreducible $\mathrm{sl}_2(k)$-submodule of $L_i(V)$. Furthermore, we shall prove that the basis given above is a standard basis of G_i, for all $i > 0$.

If $i = 1$ then $G_1 = V$ so it is an irreducible 5-dimensional $\mathrm{sl}_2(k)$-module and Y_1 is a standard basis by construction.

Let us assume the claim true for all $s < i$, we shall prove it for $s = i$.

If i is even then $i-1$ is odd and, by induction hypothesis, the basis given for G_{i-1} is the standard basis for $V^{(4)}$. Thus we get

$$\begin{aligned}
&W_1(y_{(i,2)}) = 0,\\
&W_{-1}(y_{(i,2)}) = y_{(i,0)},\\
&W_{-1}(y_{(i,0)}) = y_{(i,-2)},\\
&W_{-1}(y_{(i,-2)}) = 0,
\end{aligned}$$

and $G_i \simeq_{\mathrm{sl}_2(k)} V^{(2)}$.

If i is odd then $i-1$ is even and we get that:

$$\begin{aligned}
&W_1(y_{(i,4)}) = 0,\\
&W_{-1}(y_{(i,4)}) = y_{(i,2)},\\
&W_{-1}(y_{(i,2)}) = y_{(i,0)},\\
&W_{-1}(y_{(i,0)}) = y_{(i,-2)},\\
&W_{-1}(y_{(i,-2)}) = y_{(i,-4)},\\
&W_{-1}(y_{(i,-4)}) = 0,
\end{aligned}$$

and $G_i \simeq_{\mathrm{sl}_2(k)} V^{(4)}$.

Step 2. We shall show, by induction on the degree of the homogeneous components of $L(V)$, that for all $i \geq 1$ we have:

$$\begin{aligned}
\overline{\phi}(y_{(i,j)}) = &\quad (960)^{\frac{i-1}{2}}(12)^{\frac{i-1}{2}}t^i \otimes f_j \quad \text{if } i \text{ is odd,}\\
\overline{\phi}(y_{(i,j)}) = &\quad (960)^{\frac{i}{2}}(12)^{\frac{i-2}{2}}t^i \otimes g_j \quad \text{if } i \text{ is even,}
\end{aligned}$$

where

$$\begin{aligned}
f_4 &:= v_2,\\
f_2 &:= W_{-1}(v_2) = 4v_1,\\
f_0 &:= W_{-1}^2(v_2) = 12v_0,\\
f_{-2} &:= W_{-1}^3(v_2) = 24v_{-1},\\
f_{-4} &:= W_{-1}^4(v_2) = 24v_{-2};
\end{aligned}$$

$$\begin{aligned}
g_2 &:= w_1,\\
g_0 &:= W_{-1}(g_2) = 2w_0,\\
g_{-2} &:= W_{-1}^2(g_2) = 2w_{-1};
\end{aligned}$$

We find it useful to write down the multiplication table of these elements.

$[\ ,\ ]$	f_4	f_2	f_0	f_{-2}	f_{-4}	g_2	g_0	g_{-2}
f_4	0	0	0	$96g_2$	$192g_0$	0	$-4f_4$	$-2f_2$
f_2		0	$-96g_2$	$-96g_0$	$96g_{-2}$	$4f_4$	$-2f_2$	$-2f_0$
f_0			0	$-288g_{-2}$	0	$6f_2$	0	$-2f_{-2}$
f_{-2}				0	0	$6f_0$	$2f_{-2}$	$-2f_{-4}$
f_{-4}					0	$4f_{-2}$	$4f_{-4}$	0
g_2						0	$-2g_2$	$-2g_0$
g_0							0	$-2g_{-2}$
g_{-2}								0

For $i = 1$ we proved the statement in the previous theorem.

Let us assume the claim true for all $0 < s < i$, we should show that it is true for $s = i$.

If $i > 1$ is odd, then $i - 1$ is even thus we get that:

$$\begin{aligned}
\overline{\phi}(y_{(i,4)}) &= \overline{\phi}([y_2, y_{(i-1,2)}] - 2[y_4, y_{(i-1,0)}]) = \\
&= \overline{\phi}([y_2, y_{(i-1,2)}]) - 2\overline{\phi}([y_4, y_{(i-1,0)}]) = \\
&= (960)^{\frac{i-1}{2}}(12)^{\frac{i-3}{2}}(t^i \otimes [f_2, g_2] - 2t^i \otimes [f_4, g_0]) = \\
&= (960)^{\frac{i-1}{2}}(12)^{\frac{i-1}{2}} t^i \otimes f_4
\end{aligned}$$

As $\overline{\phi}$ is a $\mathrm{sl}_2(k)$-homomorphism we get that:

$$\begin{aligned}
\overline{\phi}(y_{(i,4-2j)}) &= \overline{\phi}(W_{-1}^j(y_{(i,4)})) = W_{-1}^j \overline{\phi}(y_{(i,4)}) = \\
&= W_{-1}^j((960)^{\frac{i-1}{2}}(12)^{\frac{i-1}{2}} f_4) = \\
&= (960)^{\frac{i-1}{2}}(12)^{\frac{i-1}{2}} t^i \otimes f_{4-2j},
\end{aligned}$$

for $j = 0,\ 1,\ 2,\ 3,\ 4$, and we are done.

If $i > 1$ is even, then:

$$\begin{aligned}
\overline{\phi}(y_{(i,2)}) &= \overline{\phi}(2([y_4, y_{(i-1,-2)}] - [y_{-2}, y_{(i-1,4)}]) - 3([y_2, y_{(i-1,0)}] - [y_0, y_{(i-1,2)}])) = \\
&= (960)^{\frac{i-2}{2}}(12)^{\frac{i-2}{2}} t^i \otimes (4[f_4, f_{-2}] - 6[f_2, f_0]) = \\
&= (960)^{\frac{i-2}{2}}(12)^{\frac{i-2}{2}} t^i \otimes (960g_2) = \\
&= (960)^{\frac{i}{2}}(12)^{\frac{i-2}{2}} t^i \otimes g_2.
\end{aligned}$$

As in the previous case this fact allows us to reach the conclusion.

Step 3. Let us observe that if $i = 1$ then $\mathcal{L}_1(V) = G_1$.

If $i = 2$ then $L_2(V) \simeq_{\mathrm{sl}_2(k)} V \wedge V \simeq_{\mathrm{sl}_2(k)} V^{(6)} \oplus V^{(2)}$. Since $S_2 \simeq_{\mathrm{sl}_2(k)} V^{(6)}$, then $\mathcal{L}_2(V) = L_2(V)/S_2 \simeq_{\mathrm{sl}_2(k)} V^{(2)}$, hence $\mathcal{L}_2(V) = G_2 + S_2$.

We will assume the claim true for all $n < i$ and we shall prove it for $n = i$ with $n > 2$.

If i is odd then $i-1$ is even and, by the induction hypothesis, we have that $\mathcal{L}_{i-1}(V) = [\mathcal{L}_1(V), \mathcal{L}_{i-2}(V)] = G_{i-1} \simeq_{\mathrm{sl}_2(k)} V^{(2)}$.

As we showed in the Lemma (3.2.1), $\mathcal{L}_i(V) = [\mathcal{L}_1(V), \mathcal{L}_{i-1}(V)]$ is a $\mathrm{sl}_2(k)$-submodule of $\mathcal{L}_1(V) \otimes \mathcal{L}_{i-1}(V) \simeq_{\mathrm{sl}_2(k)} V^{(6)} \oplus V^{(4)} \oplus V^{(2)}$.

We shall show that $\mathcal{L}_i(V) \simeq_{\mathrm{sl}_2(k)} V^{(4)}$.

Let us consider

$$\begin{array}{rlcl} \eta_i: & \mathcal{L}_1 \otimes \mathcal{L}_{i-1} & \to & [\mathcal{L}_1, \mathcal{L}_{i-1}] \\ & x \otimes y & \mapsto & [x, y] \end{array}$$

the $\mathrm{sl}_2(k)$-epimorphism we defined in Lemma 3.2.1.

We shall show that $\ker(\eta_i) \simeq_{\mathrm{sl}_2(k)} V^{(6)} \oplus V^{(2)}$.

Note that $v := y_0 \otimes y_{(i-1,0)}$ generates the $\mathrm{sl}_2(k)$-submodule of $\mathcal{L}_1 \otimes \mathcal{L}_{i-1}$ isomorphic to $V^{(6)} \oplus V^{(2)}$. Indeed, by using the Lemma (3.1.2) we may obtain the three highest weight vectors that generate the three irreducible $\mathrm{sl}_2(k)$-submodules of $\mathcal{L}_1(V) \otimes \mathcal{L}_{i-1}(V)$. In particular we get that:

$$\begin{aligned} s_{(6)} &= y_4 \otimes y_{(i-1,2)}, \\ s_{(2)} &= y_4 \otimes y_{(i-1,-2)} - 1/2 y_2 \otimes y_{(i-1,0)} + 1/6 y_0 \otimes y_{(i-1,2)}. \end{aligned}$$

Notice that we are using the same notation we used in the Lemma (3.1.2) and $s_{(j)}$ is the highest weight vector of weight j that generates the irreducible $\mathrm{sl}_2(k)$-submodule of $\mathcal{L}_1(V) \otimes \mathcal{L}_{i-1}(V)$ isomorphic to $V^{(j)}$.

Let us remark that

$$\begin{aligned} W_{-1}^3(s_{(6)}) &= W_{-1}^2(y_4 \otimes y_{(i-1,0)} + y_2 \otimes y_{(i-1,2)}) = \\ &= W_{-1}(y_4 \otimes y_{(i-1,-2)} + 2y_2 \otimes y_{(i-1,0)} + y_0 \otimes y_{(i-1,2)}) = \\ &= 3y_2 \otimes y_{(i-1,-2)} + 3y_0 \otimes y_{(i-1,0)} + y_{-2} \otimes y_{(i-1,2)}, \end{aligned}$$

and

$$W_{-1}(s_{(2)}) = 1/2 y_2 \otimes y_{(i-1,-2)} - 1/3 y_0 \otimes y_{(i-1,0)} + 1/6 y_{-2} \otimes y_{(i-1,2)}.$$

Hence $v := y_0 \otimes y_{(i-1,0)} = 1/5(W_{-1}^3(s_{(6)}) - 3W_{-1}(s_{(2)}))$, thus v generates the $\mathrm{sl}_2(k)$-submodule of $\mathcal{L}_1(V) \otimes \mathcal{L}_{i-1}(V)$ isomorphic to $V^{(6)} \oplus V^{(2)}$.

Thus, in order to prove that $\ker(\eta_i) \simeq_{\mathrm{sl}_2(k)} V^{(6)} \oplus V^{(2)}$, it is sufficient to show that $\eta_i(v) = 0$.

Let us observe that $\rho(y_{(i-1,0)}) = 5\rho([y_4, y_{(i-2,-4)}])$, indeed:

$$\begin{aligned}\rho([y_4, y_{(i-2,-4)}]) &= [\rho(y_4), \rho(y_{(i-2,-4)})] = \\ &= (960)^{\frac{i-3}{2}}(12)^{\frac{i-3}{2}} t^{i-1} \otimes [f_4, f_{-4}] = \\ &= (960)^{\frac{i-3}{2}}(12)^{\frac{i-3}{2}}(192) t^{i-1} \otimes g_0\end{aligned}$$

and

$$\rho(y_{(i-1,0)})) = (960)^{\frac{i-1}{2}}(12)^{\frac{i-3}{2}} t^{i-1} \otimes g_0.$$

Thus, by the induction hypothesis, we get that $y_{(i-1,0)} = 5[y_4, y_{(i-2,-4)}]$. As a consequence we obtain that:

$$\begin{aligned}[y_0, y_{(i-1,0)}] =& \ 5[y_0, [y_4, y_{(i-2,-4)}]] = \\ =& \ 5[y_4, [y_0, y_{(i-2,-4)}]] + 5[y_{(i-2,-4)}, [y_4, y_0]] = 0.\end{aligned}$$

If i is even, then $\mathcal{L}_1(V) \otimes \mathcal{L}_{i-1}(V) \simeq_{\mathrm{sl}_2(k)} V^{(8)} \oplus V^{(6)} \oplus V^{(4)} \oplus V^{(2)} \oplus V^{(0)}$ and we shall show that $\ker(\eta_i) \simeq_{\mathrm{sl}_2(k)} V^{(8)} \oplus V^{(6)} \oplus V^{(4)} \oplus V^{(0)}$.

Let us recall the generalised Jacoby identity:

$$[v, [y, \underbrace{z, \dots, z}_{\lambda}]] = \sum_{j=0}^{\lambda} (-1)^j \binom{\lambda}{j} [v, \underbrace{z, \dots, z}_{j}, y, \underbrace{z, \dots, z}_{\lambda-j}],$$

where iterated commutators are left normed so $[v, y, z] = [[v, y], z]$.

Let us observe that $a := y_0 \otimes y_{(i-1,4)}$ generates the $\mathrm{sl}_2(k)$-submodule of $\mathcal{L}_1(V) \otimes \mathcal{L}_{i-1}(V)$ isomorphic to $V^{(8)} \oplus V^{(6)} \oplus V^{(4)}$. Indeed,

$$\begin{aligned}&b := W_{-1}(W_1(a)) = 6(y_0 \otimes y_{(i-1,4)} + y_2 \otimes y_{(i-1,2)}) \\ &c := W_{-1}^2(W_1^2(a)) = 24(y_0 \otimes y_{(i-1,4)} + 2y_2 \otimes y_{(i-1,2)} + y_4 \otimes y_{(i-1,0)})\end{aligned}$$

and $\{a,\ b,\ c\}$ is a basis for $(\mathcal{L}_1(V) \otimes \mathcal{L}_{i-1}(V))_4$, the weight subspace of $\mathcal{L}_1(V) \otimes \mathcal{L}_{i-1}(V)$ relative to the weight 4.

As a consequence, in order to prove that the $\mathrm{sl}_2(k)$-submodule of $\mathcal{L}_1(V) \otimes \mathcal{L}_{i-1}(V)$ isomorphic to $V^{(8)} \oplus V^{(6)} \oplus V^{(4)}$ is contained in $\ker(\eta_i)$, it is sufficient to show that $\eta_i(a) = 0$.

Let us note that, by using the generalised Jacobi identity, we get the following equation.

$$0 = [y_4, \underbrace{y_{(2,0)}, \dots, y_{(2,0)}}_{\lambda}, y_0] + \sum_{j=0}^{\lambda} (-1)^j \binom{\lambda}{j} [y_0, \underbrace{y_{(2,0)}, \dots, y_{(2,0)}}_{j}, y_4, \underbrace{y_{(2,0)}, \dots, y_{(2,0)}}_{\lambda-j}], \tag{3.1}$$

Observe that, if $\lambda = \frac{i-2}{2}$, then we get:

$$\rho([y_4, \underbrace{y_{(2,0)}, \dots, y_{(2,0)}}_{\lambda}]) = (960)\lambda t^{i-1} \otimes [f_4, \underbrace{g_0, \dots, g_0}_{\lambda}] = (-4)^{\lambda} t^{i-1} \otimes f_4.$$

Thus, by the induction hypothesis, there exists $\beta \neq 0$ such that

$$[y_4, \underbrace{y_{(2,0)}, \dots, y_{(2,0)}}_{\lambda}] = \beta y_{(i-1,4)}.$$

In addition let us point out that $[y_0, y_{(2,0)}] = 0 = [y_0, y_4]$, thus the equation (3.1) becomes $0 = [y_{(i-1,4)}, y_0]$.

It remains to show that the 1-dimensional irreducible $\mathrm{sl}_2(k)$-submodule of $\mathcal{L}_1(V) \otimes \mathcal{L}_{i-1}(V)$ is contained in $\ker(\eta_i)$.

By using the Lemma (3.1.2) we obtain that the k-module generated by $v := y_{-4} \otimes y_{(i-1,4)} - y_{-2} \otimes y_{(i-1,2)} + y_0 \otimes y_{(i-1,0)} - y_2 \otimes y_{(i-1,-2)} + y_4 \otimes y_{(i-1,-4)}$ is the 1-dimensional irreducible $\mathrm{sl}_2(k)$-submodule of $\mathcal{L}_1(V) \otimes \mathcal{L}_{i-1}(V)$.

Note that:

$$\begin{aligned}
\eta_i(v) &= [y_{-4},[y_2,y_{(i-2,2)}]]-2[y_{-4},[y_4,y_{(i-2,0)}]]+\\
&-[y_{-2},[y_0,y_{(i-2,2)}]]+[y_{-2},[y_2,y_{(i-2,0)}]]+\\
&+2[y_{-2},[y_4,y_{(i-2,-2)}]]+[y_0,[y_{-2},y_{(i-2,2)}]]+\\
&-3[y_0,[y_2,y_{(i-2,-2)}]]-[y_2,[y_{-4},y_{(i-2,2)}]]+\\
&-[y_2,[y_{-2},y_{(i-2,0)}]]+3[y_2,[y_0,y_{(i-2,-2)}]]+\\
&+2[y_4,[y_{-4},y_{(i-2,0)}]]-2[y_4,[y_{-2},y_{(i-2,-2)}]]=\\
&= [y_{-4},[y_2,y_{(i-2,2)}]]-[y_2,[y_{-4},y_{(i-2,2)}]]+\\
&-2[y_{-4},[y_4,y_{(i-2,0)}]]+2[y_4,[y_{-4},y_{(i-2,0)}]]+\\
&-[y_{-2},[y_0,y_{(i-2,2)}]]+[y_0,[y_{-2},y_{(i-2,2)}]]+\\
&+[y_{-2},[y_2,y_{(i-2,0)}]]-[y_2,[y_{-2},y_{(i-2,0)}]]+\\
&+2[y_{-2},[y_4,y_{(i-2,-2)}]]-2[y_4,[y_{-2},y_{(i-2,-2)}]]+\\
&-3[y_0,[y_2,y_{(i-2,-2)}]]+3[y_2,[y_0,y_{(i-2,-2)}]]=\\
&= [y_{(i-2,2)},[y_2,y_{-4}]]-2[y_{(i-2,0)},[y_4,y_{-4}]]+\\
&-[y_{(i-2,2)}[y_0,y_{-2}]]+[y_{(i-2,0)}[y_2,y_{-2}]]+\\
&+2[y_{(i-2,-2)}[y_4,y_{-2}]]-3[y_{(i-2,-2)}[y_2,y_0]]=\\
&= 1/2([y_{(i-2,2)},y_{(2,-2)}]-[y_{(i-2,0)},y_{(2,0)}]+[y_{(i-2,-2)},y_{(2,2)}]).
\end{aligned}$$

Let us point out that if $i=4$, then

$$\eta_4(v)=1/2([y_{(2,2)},y_{(2,-2)}]-[y_{(2,0)},y_{(2,0)}]+[y_{(2,-2)},y_{(2,2)}])=0.$$

Let us assume that $i\geq 6$ and consider $\lambda=\dfrac{i-4}{2}\geq 1$, then we have the following equation.

$$0 = [y_{(2,2)}, \underbrace{y_{(2,0)}, \cdots, y_{(2,0)}}_{\lambda}, y_{(2,-2)}] +$$

$$+ \sum_{j=0}^{\lambda} (-1)^j \binom{\lambda}{j} [y_{(2,-2)}, \underbrace{y_{(2,0)}, \cdots, y_{(2,0)}}_{j}, y_{(2,2)}, \underbrace{y_{(2,0)}, \cdots, y_{(2,0)}}_{\lambda-j}].$$

Let us observe that:

$$\rho([y_{(2,-2)}, \underbrace{y_{(2,0)}, \cdots, y_{(2,0)}}_{j}, y_{(2,2)}, y_{(2,0)}]) =$$

$$= (960)^{j+3} t^{2j+6} \otimes [g_{-2}, \underbrace{g_0, \cdots, g_0}_{j}, g_2, g_0] =$$

$$= (960)^{j+3} t^{2j+6} \otimes (2)^j [g_{-2}, g_2, g_0] = 0.$$

Note that, if $2j + 6 < i$, then by using the induction hypothesis we get the following equation.

$$[y_{(2,-2)}, \underbrace{y_{(2,0)}, \cdots, y_{(2,0)}}_{j}, y_{(2,2)}, y_{(2,0)}]) = 0.$$

As a consequence we get:

$$0 = [y_{(2,2)}, \underbrace{y_{(2,0)}, \cdots, y_{(2,0)}}_{\lambda}, y_{(2,-2)}] +$$

$$+(-1)^{\lambda-1} \binom{\lambda}{\lambda-1} [y_{(2,-2)}, \underbrace{y_{(2,0)}, \cdots, y_{(2,0)}}_{\lambda-1}, y_{(2,2)}, y_{(2,0)}] +$$

$$+(-1)^{\lambda} \binom{\lambda}{\lambda} [y_{(2,-2)}, \underbrace{y_{(2,0)}, \cdots, y_{(2,0)}}_{\lambda}, y_{(2,2)}] =$$

$$= [y_{(2,2)}, \underbrace{y_{(2,0)}, \cdots, y_{(2,0)}}_{\lambda}, y_{(2,-2)}] +$$

$$+(-1)^{\lambda-1} \lambda [y_{(2,-2)}, \underbrace{y_{(2,0)}, \cdots, y_{(2,0)}}_{\lambda-1}, y_{(2,2)}, y_{(2,0)}] +$$

$$+(-1)^{\lambda} [y_{(2,-2)}, \underbrace{y_{(2,0)}, \cdots, y_{(2,0)}}_{\lambda}, y_{(2,2)}].$$

In addition:

$$\begin{aligned}
\rho([y_{(2,2)},\underbrace{y_{(2,0)},\dots,y_{(2,0)}}_{\lambda}]) &= (960)^{\lambda+1}t^{2\lambda+2}\otimes[g_2,\underbrace{g_0,\dots,g_0}_{\lambda}] = \\
&= (960)^{\lambda+1}t^{2\lambda+2}\otimes(-2)^{\lambda}g_2 = \\
&= (-1)^{\lambda}\rho((6)^{-\lambda}y_{(2\lambda+2,2)});
\end{aligned}$$

$$\begin{aligned}
\rho([y_{(2,-2)},\underbrace{y_{(2,0)},\dots,y_{(2,0)}}_{\lambda-1},y_{(2,2)}]) &= (960)^{\lambda+1}t^{2\lambda+2}\otimes[g_{-2},\underbrace{g_0,\dots,g_0}_{\lambda-1},g_2] = \\
&= (960)^{\lambda+1}t^{2\lambda+2}\otimes(2)^{\lambda-1}[g_{-2},g_2] = \\
&= (960)^{\lambda+1}t^{2\lambda+2}\otimes(2)^{\lambda}g_0 = \\
&= \rho((6)^{-\lambda}y_{(2\lambda+2,0)});
\end{aligned}$$

$$\begin{aligned}
\rho([y_{(2,-2)},\underbrace{y_{(2,0)},\dots,y_{(2,0)}}_{\lambda}] &= (960)^{\lambda+1}t^{2\lambda+2}\otimes[g_{-2},\underbrace{g_0,\dots,g_0}_{\lambda}] = \\
&= (960)^{\lambda+1}t^{2\lambda+2}\otimes(2)^{\lambda}g_{-2} = \\
&= \rho((6)^{-\lambda}y_{(2\lambda+2,-2)});
\end{aligned}$$

Thus by the induction hypothesis we get that

$$\begin{aligned}
0 = \; & (-1)^{\lambda}(6)^{-\lambda}[y_{(2\lambda+2,2)})y_{(2,-2)}]+ \\
& +(-1)^{\lambda-1}\lambda(6)^{-\lambda}[y_{(2\lambda+2,0)},y_{(2,0)}]+ \\
& +(-1)^{\lambda}(6)^{-\lambda}[y_{(2\lambda+2,-2)},y_{(2,2)}],
\end{aligned}$$

as a consequence:

$$0 = \; [y_{(2\lambda+2,2)},y_{(2,-2)}]-\lambda[y_{(2\lambda+2,0)},y_{(2,0)}]+[y_{(2\lambda+2,-2)},y_{(2,2)}]. \tag{3.2}$$

Let us recall that we showed $[y_{(i-2,-2)},y_{(2,-2)}]=0$, thus we get that:

$$\begin{aligned}
0 &= W_1^2([y_{(i-2,-2)},y_{(2,-2)}]) = \qquad (3.3)\\
&= 4([y_{(i-2,2)},y_{(2,-2)}]+2[y_{(i-2,0)},y_{(2,0)}]+[y_{(i-2,-2)},y_{(2,2)}]).
\end{aligned}$$

Note that $2\lambda + 2 = i - 2$ so by (3.2) and (3.3) we get that

$$0 = (-2 - \lambda)[y_{(2\lambda+2,0)}, y_{(2,0)}].$$

Since $\lambda \geq 1$, we get

$$[y_{(2\lambda+2,0)}, y_{(2,0)}] = 0,$$

thus

$$[y_{(i-2,2)}, y_{(2,-2)}] - [y_{(i-2,0)}, y_{(2,0)}] + [y_{(i-2,-2)}, y_{(2,2)}] = 0.$$

□

Chapter 4

Further examples

In this final chapter we shall give some examples of finitely presented Lie algebras with six generators which seem to be periodic.

As we observed in the introduction, our first problem in this work was the construction of examples of finitely presented periodic Lie algebras with d generators and $d^2/4 + 1$ relations.

In order to produce presentations which belong to $\mathcal{C}(d)$ we used the results we showed in the second chapter to write down an algorithm which works under `GAP`, [19]. We used this algorithm to produces all the presentations in $\mathcal{C}(d)$ for $d = 6, 7$. By using `Anu-p-Quotient`, [13], we were able to process all these presentations, then, by employing *ad hoc* designed Perl scripts, we were able to single out those Lie algebras that had some probability to be periodic. It should be noticed that there was no chance to scan by hand the finitely presented Lie algebras obtained by using `Anu-p-Quotient`. In fact we obtained almost 250 Lie algebras for $d = 6$ and more than 3000 Lie algebras for $d = 7$.

It should be mentioned that all the softwares written in Perl have been obtained in collaboration with Giuseppe Jurman.

It should be noted that in general it is hard to prove that a finitely presented Lie algebra is periodic. Thus in this final chapter we will show examples of Lie algebras with six generators for which we have computational evidence of the fact that they are periodic. We shall consider only Lie algebras with six generators since in this case we have the complete list of the Lie algebras with a presentation which belongs to $\mathcal{C}(6)$. On the other hand, the list of 7-generated Lie algebras with a presentation in $\mathcal{C}(7)$ is not complete because of memory limits in the hardware we used and probably because the algorithms we produced are not as efficient as possible.

By using `Anu-p-Quotient` we processed all the 6-generated Lie algebras which are likely to be periodic until the 50th homogeneous component, that is to say we are considering the nilpotent quotient of each Lie algebras L obtained by imposing $L_{51} = (0)$.

For each Lie algebra L in this list let us consider the following function.

$$\begin{array}{rccl} f_L: & \{1, \ldots, 50\} & \to & \mathbb{N} \\ & n & \mapsto & \dim_k(L_n) \end{array}$$

While scanning this list of 6-generated Lie algebras we may observe that for each Lie algebra L, f_L is ultimately periodic and the periodic behaviour begins at least at the 6th step. In addition we may observe that in each case we have that $f_L(2k) = f_L(2k+2)$ and $f_L(2k+1) = f_L(2k+3)$ for all $k \geq 4$. Moreover we may notice four different kinds of periodicity. In fact, for each L in our list, f_L is ultimately equal to one of the following functions defined on the natural numbers.

$$g_1(n) := \begin{cases} 3 & \text{if } n \text{ is even} \\ 5 & \text{if } n \text{ is odd} \end{cases}$$

$$g_2(n) := \begin{cases} 6 & \text{if } n \text{ is even} \\ 10 & \text{if } n \text{ is odd} \end{cases}$$

$$g_3(n) := \begin{cases} 7 & \text{if } n \text{ is even} \\ 10 & \text{if } n \text{ is odd} \end{cases}$$

$$g_4(n) := \begin{cases} 14 & \text{if } n \text{ is even} \\ 20 & \text{if } n \text{ is odd} \end{cases}$$

Even if we have many different examples of presentations of $\mathcal{C}(6)$ for each different periodic behaviour we shall give only one example for each case.

Let us consider the following list of presentations $\mathcal{P}_j = (X|R_j) \in \mathcal{C}(6)$ with $X := \{x_1, \ldots, x_6\}$ as generating set and R_j as set of relations:

$$R_1 = \{ \quad [x_1, x_2], [x_1, x_3], [x_1, x_4], [x_2, x_5], [x_3, x_6],$$

$$[x_1, x_5] - [x_2, x_4], [x_1, x_6] - [x_3, x_5], [x_2, x_3] - [x_4, x_6],$$

$$[x_2, x_6] - [x_4, x_5], [x_3, x_4] - [x_5, x_6]\};$$

$$R_2 = \{ \quad [x_1, x_2], [x_1, x_3], [x_1, x_4], [x_2, x_5], [x_2, x_6],$$

$$[x_1, x_5] - [x_3, x_6], [x_1, x_6] - [x_4, x_5], [x_2, x_3] - [x_4, x_6],$$

$$[x_2, x_4] - [x_3, x_5], [x_3, x_4] - [x_5, x_6]\};$$

$$R_3 = \{ \quad [x_1, x_2], [x_1, x_3], [x_1, x_4], [x_2, x_5], [x_3, x_6],$$

$$[x_1, x_5] - [x_2, x_4], [x_1, x_6] - [x_3, x_4], [x_2, x_3] - [x_5, x_6],$$

$$[x_2, x_6] - [x_4, x_5], [x_3, x_5] - [x_4, x_6]\};$$

$$R_4 = \{ \quad [x_1, x_2], [x_1, x_3], [x_1, x_4], [x_2, x_3], [x_2, x_5],$$

$$[x_1, x_5] - [x_4, x_6], [x_1, x_6] - [x_3, x_4], [x_2, x_4] - [x_5, x_6],$$

$$[x_2, x_6] - [x_3, x_5], [x_3, x_6] - [x_4, x_5]\}.$$

We may gain computational evidence that each presentation $\mathcal{P}_j$ gives a Lie algebra $G(j)$ such that $f_{G(j)}$ is definitively equal to g_j, for $j = 1, \dots, 4$.

We may obtain this almost-periodic behaviour in another way. In particular we may consider the k-module W generated by X as a $\mathrm{sl}_2(\mathbb{C})$-module isomorphic to $V^{(4)} \oplus V^{(0)}$. Let us consider $Y := \{y_4, y_2, y_0, y_{-2}, y_{-4}\}$ a standard basis for the $\mathrm{sl}_2(\mathbb{C})$-submodule of W which is $\mathrm{sl}_2(\mathbb{C})$-isomorphic to $V^{(4)}$ and let $Z := \{z_0\}$ be the basis of the $\mathrm{sl}_2(\mathbb{C})$-submodule of W which is $\mathrm{sl}_2(\mathbb{C})$-isomorphic to $V^{(0)}$. Let us consider the following presentations $(Y \cup Z | D_j)$, for $j = 1, 3$.

$$D_1 := \{ \quad [y_2, y_4] - [z_0, y_0], [y_0, y_4], [y_2, y_0] + [y_4, y_{-2}],$$
$$2[y_2, y_{-2}] + [y_4, y_{-4}], 3[y_2, y_{-4}] + 2[y_0, y_{-2}],$$
$$[y_0, y_{-4}] - [y_{-2}, z_0], [y_{-2}, y_{-4}],$$
$$[y_4, z_0], [y_2, z_0], [z_0, y_{-4}]\};$$

$$D_3 := \{ \quad [y_2, y_4], [y_0, y_4], [y_2, z_0] + [y_4, y_{-2}],$$
$$2[y_2, y_{-2}] + [y_4, y_{-4}], 3[y_2, y_{-4}] + 2[z_0, y_{-2}],$$
$$[y_0, y_{-4}], [y_{-2}, y_{-4}], [y_2, y_0] - [y_4, z_0],$$
$$[y_0, z_0], [y_{-2}, y_0] - [y_{-4}, z_0]\};$$

We have computational evidence that $f_{(Y \cup Z|D_j)}$ is definitively equal to g_j. Even if the k-module generated by D_j is not a sl_2-submodule of the second homogeneous component of the free Lie algebra generated by W, $L(W)$, we think that it is more useful to consider presentations in this form than in the previous one. At this stage we are not able to describe completely the two algebras $(Y \cup Z|D_j)$, $j = 1, 3$, but we may prove a partial result for $(Y \cup Z|D_1)$. In particular we prove that the finitely presented loop Lie algebra $\mathcal{L}(V)$ that we considered in the previous chapter is a homomorphic image of $(Y \cup Z|D_1)$.

Theorem 4.0.3 *Let $M := (Y \cup Z|D_1)$ be the finitely presented Lie algebra we described above. There exists a Lie algebras epimorphism of M onto $\mathcal{L}(V)$.*

Proof. Let us recall that $\mathcal{L}(V) = (Y|R)$ where

$$R = \{ \quad [y_0, y_{-4}], [y_0, y_4], [y_{-4}, y_{-2}], [y_2, y_4],$$
$$2[y_{-2}, y_2] + [y_{-4}, y_4], 2[y_{-2}, y_0] - 3[y_2, y_{-4}], [y_{-2}, y_4] - [y_2, y_0]\}.$$

Let us consider the k-module epimorphism of W onto V which is the identity map on Y and send z_0 to zero. This map may be extended to a Lie algebra epimorphism $\chi : L(W) \to \mathcal{L}(V)$. Let us observe that $D_1 \subseteq \ker(\chi)$, indeed

$$\chi([y_2, y_4] - [z_0, y_0]) = [y_2, y_4] = 0,$$

$$\chi([y_0, y_4]) = [y_0, y_4] = 0,$$

$$\chi([y_2, y_0] + [y_4, y_{-2}]) = 0,$$

$$\chi(2[y_2, y_{-2}] + [y_4, y_{-4}]) = 0,$$

$$\chi(3[y_2, y_{-4}] + 2[y_0, y_{-2}]) = 0,$$

$$\chi([y_0, y_{-4}] - [y_{-2}, z_0]) = [y_0, y_{-4}] = 0,$$

$$\chi([y_{-2}, y_{-4}]) = 0,$$

$$\chi([y_4, z_0]) = \chi([y_2, z_0]) = \chi([z_0, y_{-4}]) = 0.$$

Thus the ideal of $L(W)$ generated by D_1, say D, is contained in $\ker(\chi)$ and there exists a Lie algebra epimorphism of $M = L(W)/D$ onto $\mathcal{L}(V)$ which is induced by χ. $\square$

The last fact we want to point out here is that we have an example of finitely presented Lie algebra with six generators and ten homogeneous relations which is finite-dimensional. Indeed, let us consider the presentation with $Y \cup Z$ as set of generators and the following set of relations:

$$\begin{aligned} F := \{ \quad & [y_2, y_4], [z_0, y_4], [y_0, z_0], [y_0, y_{-4}], [y_{-2}, y_{-4}], \\ & [y_2, y_0] + [y_4, y_{-2}], 2[y_2, y_{-2}] + [y_4, y_{-4}], \\ & 3[y_2, y_{-4}] + 2[z_0, y_{-2}], [y_2, z_0] - [y_4, y_0], [y_{-2}, y_0] - [y_{-4}, z_0]\}. \end{aligned}$$

This is a nilpotent Lie algebra of dimension 39 as k-module.

Bibliography

[1] Yu. A. Bahturin, *Identical relations in Lie algebras*, VNU Science Press b.v., Utrecht, 1987, Translated from the Russian by Bakhturin. MR 88f:17032

[2] Gilbert Baumslag, *Subalgebras of finitely presented solvable Lie algebras*, J. Algebra **45** (1977), no. 2, 295–305. MR 55 #8128

[3] Allan Berele, *Homogeneous polynomial identities*, Israel J. Math. **42** (1982), no. 3, 258–272. MR 84b:16018

[4] A. E. Bereznyĭ, *Discrete groups with subexponential growth of coefficients of the Golod-Šafarevič series*, Uspekhi Mat. Nauk **35** (1980), no. 5(215), 225–226. MR 82k:20064

[5] Béla Bollobás, *Modern graph theory*, Springer-Verlag, New York, 1998. MR 99h:05001

[6] R. M. Bryant and J. R. J. Groves, *Finitely presented Lie algebras*, J. Algebra **218** (1999), no. 1, 1–25. MR 2001a:17013

[7] W. A. de Graaf and J. Wisliceny, *Constructing bases of finitely presented Lie algebras using Gröbner bases in free algebras*, Proceedings of the 1999 International Symposium on Symbolic and Algebraic Computation (Vancouver, BC) (New York), ACM, 1999, pp. 37–43 (electronic). MR 2002e:17007

[8] William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics. MR 93a:20069

[9] Howard Garland, *The arithmetic theory of loop groups*, Inst. Hautes Études Sci. Publ. Math. (1980), no. 52, 5–136. MR 83a:20057

[10] V. Gerdt and V. Kornyak, *An implementation in C of an algorithm for construction of finitely presented Lie superalgebras*, Comput. Sci. J. Moldova **4** (1996), no. 3, 399–427. MR 99m:17001

[11] E. S. Golod and I. R. Šafarevič, *On the class field tower*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 261–272. MR 28 #5056

[12] Frank Harary and Edgar M. Palmer, *Graphical enumeration*, Academic Press, New York, 1973. MR 50 #9682

[13] G. Havas, M. F. Newman, and E. A. O'Brien, *ANU p-Quotient Program (version 1.4), written in* `C`*, available as a share library with* `GAP` *and as part of* `Magma`*, or from* `http://wwwmaths.anu.edu.au/services/ftp.html`*, School of Mathematical Sciences, Canberra*, 1997.

[14] James E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York, 1978, Second printing, revised. MR 81b:17007

[15] H. Koch, *Erzeugenden- und Relationenrang für endlich dimensionale nilpotente Liesche Algebren*, Algebra i Logika **16** (1977), no. 3, 364–374, 378. MR 80i:17010

[16] M. F. Newman, Csaba Schneider, and Aner Shalev, *The entropy of graded algebras*, J. Algebra **223** (2000), no. 1, 85–100. MR 2001c:16078

[17] J. Marshall Osborn and Kaiming Zhao, *A characterization of certain loop Lie algebras*, J. Algebra **221** (1999), no. 1, 345–359. MR 1 722 916

[18] Christophe Reutenauer, *Free Lie algebras*, London Mathematical Society Monographs. New Series, vol. 7, The Clarendon Press Oxford University Press, New York, 1993, Oxford Science Publications. MR 94j:17002

[19] M. Schönert et al., *GAP – Groups, Algorithms, and Programming*, Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, fifth ed., 1995.

[20] Csaba Schneider, *Computing nilpotent quotients in finitely presented Lie rings*, Discrete Math. Theor. Comput. Sci. **1** (1997), no. 1, 1–16 (electronic). MR 98m:17003

[21] I. R. Shafarevich (ed.), *Algebra. VI*, Encyclopaedia of Mathematical Sciences, vol. 57, Springer-Verlag, Berlin, 1995, Combinatorial and asymptotic methods of algebra. Non-associative structures, A translation of *Current problems in mathematics. Fundamental directions. Vol. 57* (Russian), Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1990 [MR 91a:00016], Translation by R. M. Dimitrić, Translation edited by A. I. Kostrikin and I. R. Shafarevich. MR 96f:00042

[22] È. B. Vinberg, *On the theorem concerning the infinite-dimensionality of an associative algebra*, Izv. Akad. Nauk SSSR Ser. Mat. **29** (1965), 209–214. MR 30 #3108

[23] J. Wisliceny, *Zur Darstellung von Pro-p-Gruppen und Lieschen Algebren durch Erzeugende und Relationen*, Math. Nachr. **102** (1981), 57–78. MR 84i:17016

[24] ———, *Eine Methode zur Konstruktion nilpotenter Liescher Algebren*, Math. Nachr. **118** (1984), 209–214. MR 86k:17006

[25] ———, *Konstruktion nilpotenter assoziativer Algebren mit wenig Relationen*, Math. Nachr. **147** (1990), 75–82. MR 93b:16030