A SMALL TRIVIA ABOUT MONIC POLYNOMIALS OF SECOND DEGREE WITH POSITIVE INTEGER COEFFICIENTS

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Abstract. In this paper we investigate the problem of simultaneous factorization of second degree polynomials with positive integer coefficients.

1. Introduction

If we consider a second degree polynomial with integer coefficients
\[ p(x) = x^2 + qx + p \]
which is reducible in \( \mathbb{Z}[x] \) it is well possible that even the polynomial
\[ q(x) = x^2 + px + q \]
is reducible in \( \mathbb{Z}[x] \) as well. For instance, if \( p > 0 \) then
\[
\begin{align*}
p(x) &= x^2 - (p + 1)x + p \\
q(x) &= x^2 + px - (p + 1)
\end{align*}
\]
are both reducible in \( \mathbb{Z}[x] \). But what about a polynomial
\[ p(x) = x^2 + qx + p \]
where both \( p \) and \( q \) are positive and \( p < q \)? We will prove that, with this condition, only the polynomial
\[ p(x) = x^2 + 6x + 5 \]
has the required property.

2. The result

Theorem 1. If \( p, q \) are positive integers then
\[
\begin{align*}
p(x) &= x^2 + qx + p \\
q(x) &= x^2 + px + q
\end{align*}
\]
are both reducible in \( \mathbb{Z}[x] \) if and only if \( p = 5, q = 6 \).

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Proof. The “if” is, of course, trivial. For the “only if”, let us consider the polynomial
\[ p(x) = x^2 + qx + p \]
If \( p(x) \) is reducible, then for a suitable divisor \( d \) of \( p \) we must have that
\[ q = d + \frac{p}{d}. \]
We can always assume that \( 1 \leq d \leq \sqrt{p} \). We split the proof in two cases

(1) For first we consider the case \( d = 1 \). We have that
\[ p(x) = x^2 + (p + 1)x + p \]
\[ q(x) = x^2 + px + (p + 1) \]
thus, from \( q(x) \) we have that
\[ p = d' + \frac{p + 1}{d'} \]
where \( d' \) is a suitable divisor of \( p + 1 \) and we can suppose, with generality, that
\[ 2 \leq d' \leq \sqrt{p + 1}. \]
But, for each \( p \) and each \( d' \) it is
\[ d' + \frac{p + 1}{d'} \leq \sqrt{p + 1} + \frac{p + 1}{2} \]
and
\[ \sqrt{p + 1} + \frac{p + 1}{2} < p \]
is true as soon as \( p \geq 7 \). Hence, we have only to check the values \( p = 1, 2, 3, 4, 5, 6 \) and among them we find that the only acceptable value is \( p = 5 \).

(2) Now we assume that \( 2 \leq d \leq \sqrt{p} \). In this case, from \( p(x) \) we have that
\[ q = d + \frac{p}{d} \leq \sqrt{p} + \frac{p}{2} \]
while, from \( q(x) \), it must be
\[ p = d' + \frac{q}{d'} \]
for a suitable divisor of \( q \). Thus, it must be
\[ p = d' + \frac{q}{d'} \leq \sqrt{q} + q \leq \sqrt{\sqrt{p} + \frac{p}{2}} + \sqrt{p} + \frac{p}{2} \]
which is false as soon as \( p > 15 \). Hence, we have to check only the cases \( p = 1, \ldots, 14 \) and, among them, we cannot find any further polynomial. This proves the result.

\[ \square \]