AN INEQUALITY OF ERDÖS REVISITED

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ABSTRACT. In this short note we prove by analytic way an elementary geometric inequality of Erdös .

1. Introduction

In [1] it is reported an elementary geometric inequality due to Erdös and then proved by others as well. As far as I know, all the different proof are either synthetic or trigonometric. In this short paper an analytical proof is given.

2. The inequality

Le be AB a chord in a circle of center O which is not a diameter and let be OF the perpendicular radius which meets it at M. Let be P any point of the major arc AB different form G

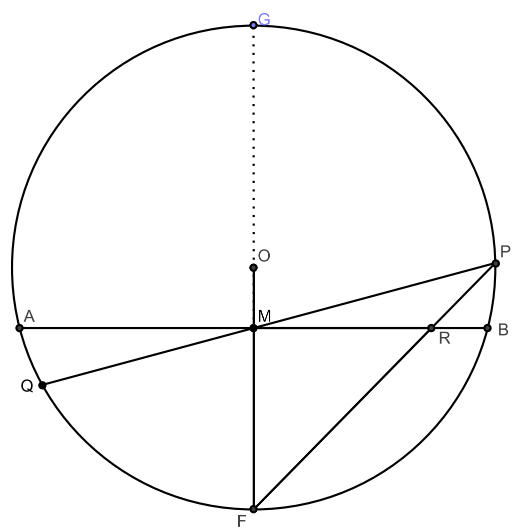

FIGURE 1

then

$$FR > QM. \tag{1}$$

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Let we choose a Cartesian coordinate system and we will prove that if $MRFC$ is a parallelogram then QC is perpendicular to QM. After that, the inequality 1 follows immediately, since $MC \cong FR$ is the hypothenuse of the triangle MQC.

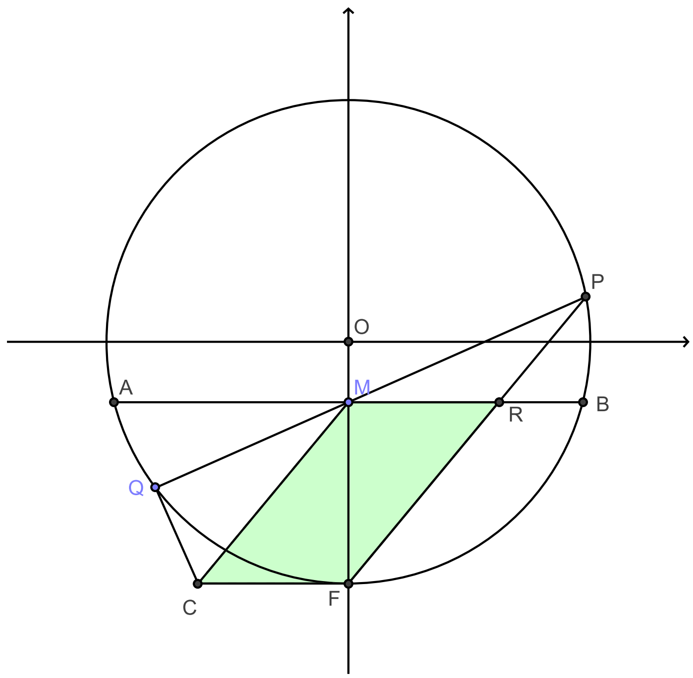

FIGURE 2

We have that

(1) $x^2 + y^2 = r^2$ is the equation of the circle;
(2) $M = (0, -h)$;
(3) $F = (0, -r)$;
(4) $y = mx - h$ is the equation of the line through P and Q.

By hypothesis on P we can assume that $m > 0$. Let us find the coordinates of P and Q. We must solve the system

$$\begin{cases} x^2 + y^2 = r^2 \\ y = mx - h \end{cases}$$

After some easy calculations, we find

$$\begin{cases} x_P = \dfrac{hm + \lambda}{1 + m^2} \\ y_P = -h + \dfrac{hm^2}{1 + m^2} + \dfrac{m\lambda}{1 + m^2} \end{cases}$$

and

$$\begin{cases} x_Q = \dfrac{hm-\lambda}{1+m^2} \\ y_Q = -h + \dfrac{hm^2}{1+m^2} - \dfrac{m\lambda}{1+m^2} \end{cases}$$

where

$$\lambda = \sqrt{-h^2 + r^2 + m^2 r^2}$$

If we call with m_1 the slope coefficient of the line through P and F, we have that

$$m_1 = \frac{y_P - y_F}{x_P - x_F} = \frac{-h + r + m^2 r + m\lambda}{m\,(h+\lambda)}.$$

Hence the equation of the line through C and M is

$$y = m_1 x - h.$$

The coordinates of the point C are given by the solution of the linear system

$$\begin{cases} y = m_1 x - h \\ y = -r \end{cases}$$

which is

$$\begin{cases} x_C = -\dfrac{(h-r)\,(hm+\lambda)}{h - r - m^2 r - m\lambda} \\ y_C = -r \end{cases}$$

Then

$$m_{QC} = \frac{y_Q - y_C}{x_Q - x_C}$$

and after some calculations we find

$$m_{QC} = -\frac{1}{m}$$

This means that $QC \perp QM$ as we want to prove.

References

[1] R. Honsberger "Mathematical Morsels" The Dolciani Mathematical Expositions n°3 1978 pag 96-98.

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