ON CERTAIN CONDITIONALLY CONVERGENT SERIES

LUCA GOLDONI

Abstract. In this paper we investigate the problem of the convergence of a very special kind of non absolutely convergent series which can not be solved by means of traditional tests as Dirichlet test.

1. Introduction

We investigate the behavior of the series

\[ \sum_{n=0}^{+\infty} (-1)^{n \mod p} a_n, \]

where \( p \) is an odd prime number and \( a_n \) is not negative for each \( n \). We could call ‘almost alternating series’ because the sequence of the signs is of the kind

+ \( \cdots \) + \( \cdots \) + \( \cdots \) + \( \cdots \)

\( p \)-terms  \( p \)-terms

We observe that the Dirichlet’s test is not applicable even in the case of further assumptions on \( a_n \) because the partial sums of the sequence \( b_n = (-1)^{n \mod p} \) are not bounded. Indeed, if we indicate with \( \sigma_n \) the sequence of this partial sums we have that \( \sigma_{pk} = k + 1 \).

2. The theorem

Lemma 1. Let be

\[ \sum_{n=0}^{+\infty} (-1)^{n \mod p} a_n, \]

where

(a): \( a_n \geq 0 \) for each \( n \in \mathbb{N} \).

(b): \( \sum_{n=0}^{+\infty} a_n = +\infty \).

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Dipartimento di Matematica. Università di Trento.
\( \lim_{n \to +\infty} a_n = 0. \)

and let be \((s_n)_n\) the sequence of the partial sums. If there exists the
\begin{equation}
\lim_{k \to +\infty} s_{pk} = s \in \mathbb{R}.
\end{equation}
then
\[ \lim_{k \to +\infty} s_{pk+1} = \lim_{k \to +\infty} s_{pk+2} \cdots \lim_{k \to +\infty} s_{p(k+1)-1} = s \]
so that the given series converges.

**Proof.** Since \((1)\) holds, it follows that
\[ \forall \varepsilon > 0 \exists k_1(\varepsilon) : \forall k > k_1(\varepsilon) \Rightarrow s - \frac{\varepsilon}{2} < s_{pk} < s + \frac{\varepsilon}{2}. \]

Let be \(1 \leq h \leq p-1\) then
\[ |s_{pk+h} - s_{pk}| = |a_{pk+1} + \cdots a_{pk+h}| \leq |a_{pk+1}| + \cdots |a_{pk+h}|. \]

Since hypothesis \((c)\) holds, it follows that
\[ \forall \varepsilon > 0 \exists \overline{k}(\varepsilon) : \forall n > \overline{k}(\varepsilon) \Rightarrow |a_n| \leq \frac{\varepsilon}{2h}. \]

Let be \(k\) such that \(pk + 1 > \overline{k}(\varepsilon)\) i.e.
\[ k > \frac{\overline{k}(\varepsilon) - 1}{p} = \overline{k_2}(\varepsilon). \]
then
\[ |a_{pk+1}| + \cdots |a_{pk+h}| \leq \frac{\varepsilon (h-1)}{2h} < \frac{\varepsilon}{2}. \]

thus
\[ |s_{pk+h} - s_{pk}| < \frac{\varepsilon}{2}. \]

If \(k > \max \{ k_1(\varepsilon), \overline{k_2}(\varepsilon) \}\) then
\[ \begin{cases} 
  s - \frac{\varepsilon}{2} < s_{pk} < s + \frac{\varepsilon}{2} \\
  s_{pk} - \frac{\varepsilon}{2} < s_{pk+h} < s_{pk} + \frac{\varepsilon}{2}
\end{cases} \]
so that \(s - \varepsilon < s_{pk+h} < s + \varepsilon.\) Hence
\[ \lim_{k \to +\infty} s_{pk+h} = s. \]

Since it holds for each \(1 \leq h \leq p\) the thesis follows. \(\square\)

**Lemma 2.** If
\[ \sum_{n=0}^{+\infty} (-1)^n (\mod p) a_n \]
satisfies the hypothesis of Lemma 1 and if
\begin{equation}
\text{(d)}: \quad d_k = a_{pk+p} + \sum_{h=1}^{p-1} (-1)^h a_{pk+h} \geq 0 \text{ for each } k \in \mathbb{N}.
\end{equation}
\begin{equation}
\text{(e)}: \sum_{k=0}^{+\infty} d_k < +\infty.
\end{equation}
then

\[ \exists \lim_{k \to \infty} s_{pk} = s < +\infty. \]

**Proof.** Since

\[ s_{pk+p} = s_{pk} + (-a_{pk+1} + a_{pk+2} - a_{pk+3} + \cdots - a_{pk+p-2} + a_{pk+p-1} + a_{pk+p}) \]

we have that

\[ s_{pk} = s_0 + \sum_{j=0}^{k-1} d_j. \]

from hypothesis \((d)\) it follows that the sequence \(s_{pk}\) in not decreasing so it has limit. Moreover, since

\[ \sum_{h=0}^{k-1} d_h \leq \sum_{h=0}^{+\infty} d_h < +\infty \]

the limit belongs to \(\mathbb{R}\). \(\square\)

So we have that

**Theorem 1.** If

\[ \sum_{n=0}^{+\infty} (-1)^n \binom{n}{\text{mod } p} a_n. \]

where

(a): \(a_n \geq 0\) for each \(n \in \mathbb{N}\).
(b): \(\sum_{n=0}^{+\infty} a_n = +\infty\).
(c): \(\lim_{n \to +\infty} a_n = 0\).
(d): \(d_k = a_{pk+p} + \sum_{h=1}^{p-1} (-1)^h a_{pk+h} \geq 0\) for each \(k \in \mathbb{N}\).
(e): \(\sum_{k=0}^{+\infty} d_k < +\infty\).
(f): \(p\) is an odd prime number.

then the given series is simply convergent.

In particular we have the following

**Corollary 1.** If there exist \(A > 0\) and \(\delta > 0\) so that

\[ 0 \leq d_k \leq \frac{A}{k^\delta}. \]

then the given series converges.
REFERENCES


UNIVERSITÀ DI TRENTO, DIPARTIMENTO DI MATEMATICA, V. SOMMARIVE 14, 56100 TRENTO, ITALY
E-mail address: goldoni@science.unitn.it