ON CERTAIN CONDITIONALLY CONVERGENT SERIES

LUCA GOLDONI

Abstract. In this paper we investigate the problem of the convergence of a very special kind of non absolutely convergent series which can not be solved by means of traditional tests as Dirichlet test.

1. Introduction

We investigate the behavior of the series

$$\sum_{n=0}^{+\infty} (-1)^{n(\bmod p)} a_n.$$

where p is an odd prime number and a_n is not negative for each n. We could call '**almost alternating series**' because the sequence of the signs is of the kind

$$\underbrace{+ - \cdots - \boxed{+}}_{p-terms}\underbrace{\boxed{+} - \cdots - +}_{p-terms}\cdots$$

We observe that the Dirichlet's test is not applicable even in the case of further assumptions on a_n because the partial sums of the sequence $b_n = (-1)^{n \ (\bmod p)}$ are not bounded. Indeed, if we indicate with σ_n the sequence of this partial sums we have that $\sigma_{pk} = k + 1$.

2. The theorem

Lemma 1. *Let be*

$$\sum_{n=0}^{+\infty} (-1)^{n \ (\bmod p)} a_n.$$

where

- **(a):** $a_n \geq 0$ *for each* $n \in \mathbb{N}$.
- **(b):** $\sum\limits_{n=0}^{+\infty} a_n = +\infty$.

Date: April 7, 2013.

2000 *Mathematics Subject Classification.* 40A05, 30B50.

Key words and phrases. Infinite series, Conditionally convergence tests, Elementary proof.

Dipartimento di Matematica. Università di Trento.

(c): $\lim\limits_{n\to+\infty} a_n = 0.$

and let be $(s_n)_n$ *the sequence of the partial sums. If there exists the*

(1) $$\lim_{k\to+\infty} s_{pk} = s \in \mathbb{R}.$$

then

$$\lim_{k\to+\infty} s_{pk+1} = \lim_{k\to+\infty} s_{pk+2} \cdots \lim_{k\to+\infty} s_{p(k+1)-1} = s$$

so that the given series converges.

Proof. Since (1) holds, it follows that

$$\forall \varepsilon > 0\ \exists \overline{k_1}(\varepsilon) : \forall k > \overline{k_1}(\varepsilon) \Rightarrow s - \frac{\varepsilon}{2} < s_{pk} < s + \frac{\varepsilon}{2}.$$

Let be $1 \leq h \leq p-1$ then

$$|s_{pk+h} - s_{pk}| = |a_{pk+1} + \cdots a_{pk+h}| \leqslant |a_{pk+1}| + \cdots |a_{pk+h}|.$$

Since hypothesis (c) holds, it follows that

$$\forall \varepsilon > 0\ \exists \overline{n}(\varepsilon) : \forall n > \overline{n}(\varepsilon) \Rightarrow |a_n| \leqslant \frac{\varepsilon}{2h}.$$

Let be k such that $pk + 1 > \overline{n}(\varepsilon)$ i.e.

$$k > \frac{\overline{n}(\varepsilon) - 1}{p} = \overline{k_2}(\varepsilon).$$

then

$$|a_{pk+1}| + \cdots |a_{pk+h}| \leqslant \frac{\varepsilon(h-1)}{2h} < \frac{\varepsilon}{2}.$$

thus

$$|s_{pk+h} - s_{pk}| < \frac{\varepsilon}{2}.$$

If $k > \max\left\{\overline{k_1}(\varepsilon), \overline{k_2}(\varepsilon)\right\}$ then

$$\begin{cases} s - \dfrac{\varepsilon}{2} < s_{pk} < s + \dfrac{\varepsilon}{2} \\ s_{pk} - \dfrac{\varepsilon}{2} < s_{pk+h} < s_{pk} + \dfrac{\varepsilon}{2} \end{cases}$$

so that $s - \varepsilon < s_{pk+h} < s + \varepsilon$. Hence

$$\lim_{k\to\infty} s_{pk+h} = s.$$

Since it holds for each $1 \leq h \leq p$ the thesis follows. □

Lemma 2. *If*

$$\sum_{n=0}^{+\infty} (-1)^{n \pmod p} a_n$$

satisfies the hypothesis of Lemma 1 and if

(d): $d_k = a_{pk+p} + \sum\limits_{h=1}^{p-1} (-1)^h a_{pk+h} \geqslant 0$ *for each* $k \in \mathbb{N}$.

(e): $\sum\limits_{k=0}^{+\infty} d_k < +\infty.$

then
$$\exists \lim_{k\to\infty} s_{pk} = s < +\infty.$$

Proof. Since

$$s_{pk+p} = s_{pk} + (-a_{pk+1} + a_{pk+2} - a_{pk+3} + \cdots - a_{pk+p-2} + a_{pk+p-1} + a_{pk+p})$$

we have that
$$s_{pk} = s_0 + \sum_{j=0}^{k-1} d_j.$$

from hypothesis (d) it follows that the sequence s_{pk} in not decreasing so it has limit. Moreover, since
$$\sum_{h=0}^{k-1} d_h \leqslant \sum_{h=0}^{+\infty} d_h < +\infty$$

the limit belongs to $\mathbb{R}$. □

So we have that

Theorem 1. *If*
$$\sum_{n=0}^{+\infty} (-1)^{n \pmod p} a_n.$$

where

- **(a):** $a_n \geq 0$ *for each* $n \in \mathbb{N}$.
- **(b):** $\sum\limits_{n=0}^{+\infty} a_n = +\infty$.
- **(c):** $\lim\limits_{n\to+\infty} a_n = 0$.
- **(d):** $d_k = a_{pk+p} + \sum\limits_{h=1}^{p-1} (-1)^h a_{pk+h} \geqslant 0$ *for each* $k \in \mathbb{N}$.
- **(e):** $\sum\limits_{k=0}^{+\infty} d_k < +\infty$.
- **(f):** p *is an odd prime number.*

then the given series is simply convergent.

In particular we have the following

Corollary 1. *If there exist* $A > 0$ *and* $\delta > 0$ *so that*
$$0 \leqslant d_k \leqslant \frac{A}{k^\delta}.$$

then the given series converges.

References

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Università di Trento, Dipartimento di Matematica, v. Sommarive 14, 56100 Trento, Italy
E-mail address: goldoni@science.unitn.it