FEYNMAN PATH INTEGRALS FOR POLYNOMIALLY GROWING POTENTIALS

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Dedicated to G.F. Dell'Antonio, for his 70^{th} *birthday.*

ABSTRACT. A general class of infinite dimensional oscillatory integrals with polynomially growing phase functions is studied. A representation formula of the Parseval type is proved, as well as a formula giving the integrals in terms of analytically continued absolutely convergent integrals. These results are applied to provide a rigorous Feynman path integral representation for the solution of the time dependent Schrödinger equation with a quartic anharmonic potential. The Borel summability of the asymptotic expansion of the solution in power series of the coupling constant is also proved.

Key words: Feynman path integrals, Schrödinger equation, quartic oscillator potential, asymptotic expansion.

AMS classification : 28C20, 81C35, 47D06, 35C20.

1. INTRODUCTION

More than 50 years have passed since R.P. Feynman, following a suggestion by Dirac [17], proposed a Lagrangian formulation of quantum mechanics. In Feynman's work the solution of the Schrödinger equation

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi + V\psi \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{1}$$

can be represented by a heuristic "Feynman path integral", or an "integral over histories":

$$\psi(t,x) = \quad \text{“}const \quad \int_{\{\gamma|\gamma(0)=x\}} e^{\frac{i}{\hbar}S_t(\gamma)}\psi_0(\gamma(0))D\gamma \quad \text{”} \tag{2}$$

which is intended as an integral on the space of paths arriving at time t at the point x. $S_t(\gamma)$ is the classical action of the system evaluated along the path γ and $D\gamma$ is a “flat measure” on the space of paths. The heuristic representation (2) is particularly suggestive as it shows a connection between the classical and the quantum description of the physical world: indeed it allows the study of the behavior of (2) taking into account the smallness of $\hbar$, the "semiclassical limit". In fact a heuristic stationary phase argument, in analogy with the behavior of the corresponding finite dimensional oscillatory integrals, shows that the paths contributing to the integral should be those for which the classical action S_t is stationary: these are exactly the solutions of the Hamilton equations. Such a consideration and the formula (2) are only heuristic: indeed neither the "infinite dimensional Lebesgue measure" $D\gamma$, nor the normalization constant are well defined. Nevertheless, under suitable hypothesis on the potential V and on the initial datum ψ_0, one can give to the integral (2) a rigorous mathematical meaning as a functional on a suitable class of functions. There are several definitions of the "Feynman functional", for instance by means of analytic continuation [11, 33, 25, 40, 26, 18, 29, 32, 41, 13, 39], or as an infinite dimensional distribution in the framework of Hida calculus [22, 16], or via "complex Poisson measures" [30, 1], or as a infinite dimensional oscillatory integral [23, 24, 6, 19, 3]. Indeed, when the potential V is of the following form

$$V(x) = \frac{1}{2}x \cdot \Omega^2 x + V'(x), \tag{3}$$

where Ω^2 is a positive definite symmetric $d \times d$ matrix and V' is the Fourier transform of a complex bounded variation measure on $\mathbb{R}^d$, Albeverio and Høegh-Krohn define in [6] the Feynman integral as a functional on a suitable Hilbert space of paths by means of a Parseval type formula (previous work in this direction is due to K. Ito). In [19] Elworthy and Truman define the Feynman functional by means of a sequential approach that is closer to Feynman's original work. The "infinite dimensional oscillatory integral" they propose is defined as the limit of a sequence of finite dimensional oscillatory integrals. They can also prove that, for the class of function considered in [6], the infinite dimensional oscillatory integral can be explicitly computed by means of the Parseval type formula proposed by Albeverio and Høegh-Krohn. Such an approach allows a rigorous implementation of an infinite dimensional version of the stationary phase method and was further developed in [4] and [2] in connection with the study of the asymptotic behaviour of the integral in the limit $\hbar \downarrow 0$.

Unfortunately none of the existing approaches can give a rigorous mathematical definition of the Feynman formula (2) for polynomially growing potentials. In particular the perturbation V' to the harmonic oscillator potential considered by Albeverio, Høegh-Krohn and Ito has

to belong to the class of Fourier transforms of measures, so that is bounded. An extension to Laplace transforms of measures has been given in [5, 27]. The aim of the present paper is to enlarge the class of potentials for which the Feynman functional can be defined and to include in it potentials with polynomial growth. Our approach is in the spirit of [3, 19], but has also some relations with [11, 18, 26].
In section 2 we will recall some known results about infinite dimensional oscillatory integrals. In the third and fourth sections we extend the class of functions for which a generalized infinite dimensional oscillatory integral can be computed and prove a Parseval type equality. In addition we propose an analytic continuation formula which shows a direct connection between the infinite dimensional oscillatory integral and the Wiener integral. In the fifth section we consider the Schrödinger equation for a d- dimensional quantum particle under the action of the anharmonic oscillator potential

$$V(x) = \frac{1}{2} x \cdot \Omega^2 x + \lambda C(x, x, x, x), \tag{4}$$

where C is a completely symmetric positive fourth order covariant tensor on $\mathbb{R}^d$ and $\lambda \geq 0$ is a coupling constant. If $d = 1$, (4) reduces to $V(x) = \frac{1}{2}\Omega^2 x^2 + \lambda x^4$. We give a functional integral representation for the solution of the corresponding Schrödinger equation and show that the so defined functional is analytic in the coupling constant $\lambda \in \mathbb{C}$ for $Im(\lambda) < 0$, continuous for $\lambda \in \mathbb{R}$ and coincides for $\lambda \leq 0$ with a well defined infinite dimensional oscillatory integral. We prove moreover the Borel summability of the asymptotic Dyson expansion (in powers of the coupling constant λ) for the scalar product $\langle \phi, e^{-i\frac{t}{\hbar}H}\psi_0 \rangle$, where H is the quantum mechanical Hamiltonian $H = -\frac{\hbar^2}{2} + V$ and $\phi, \psi_0 \in L^2(\mathbb{R}^d)$ are suitable vectors.

2. Oscillatory integrals and the Cameron Martin formula

In this section we recall for later use some known results, for more details we refer to [6, 19, 3]. In the following we will denote by $\mathcal{H}$ a (finite or infinite dimensional) real separable Hilbert space, whose elements are denoted by $x, y \in \mathcal{H}$ and the scalar product with $\langle x, y \rangle$. $f : \mathcal{H} \to \mathbb{C}$ will be a function on $\mathcal{H}$ and $Q : D(Q) \subseteq \mathcal{H} \to \mathcal{H}$ an invertible, densely defined and self-adjoint operator.
Let us denote by $\mathcal{M}(\mathcal{H})$ the Banach space of the complex bounded variation measures on $\mathcal{H}$, endowed with the total variation norm, that is:

$$\mu \in \mathcal{M}(\mathcal{H}), \qquad \|\mu\| = \sup \sum_i |\mu(E_i)|,$$

where the supremum is taken over all sequences $\{E_i\}$ of pairwise disjoint Borel subsets of $\mathcal{H}$, such that $\cup_i E_i = \mathcal{H}$. $\mathcal{M}(\mathcal{H})$ is a Banach algebra, where the product of two measures $\mu * \nu$ is by definition their convolution:

$$\mu * \nu(E) = \int_{\mathcal{H}} \mu(E-x)\nu(dx), \qquad \mu, \nu \in \mathcal{M}(\mathcal{H})$$

and the unit element is the vector δ_0.
Let $\mathcal{F}(\mathcal{H})$ be the space of complex functions on H which are Fourier transforms of measures belonging to $\mathcal{M}(\mathcal{H})$, that is:

$$f : \mathcal{H} \to \mathbb{C} \qquad f(x) = \int_H e^{i\langle x, \beta\rangle} \mu_f(d\beta) \equiv \hat{\mu}_f(x).$$

$\mathcal{F}(\mathcal{H})$ is a Banach algebra of functions, where the product is the pointwise one; the unit element is the function 1, i.e. $1(x) = 1 \ \forall x \in \mathcal{H}$ and the norm is given by $\|f\| = \|\mu_f\|$.

2.1. **Finite dimensional oscillatory integrals.** Let us suppose that $\mathcal{H} = \mathbb{R}^n$ and define the "Fresnel integral"

$$\int e^{\frac{i}{2\hbar}\langle x, Qx\rangle} f(x)dx$$

In the whole paper $\hbar > 0$ is a fixed parameter (we call it $\hbar$ because of its interpretation in the context of applications to quantum mechaniocs).

Definition 1. *A function $f : \mathbb{R}^n \to \mathbb{C}$ is Fresnel integrable with respect to Q if and only if for each $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\phi(0) = 1$ the limit*

$$\lim_{\epsilon \to 0} (2\pi i \hbar)^{-n/2} \int e^{\frac{i}{2\hbar}\langle x, Qx\rangle} f(x)\phi(\epsilon x)dx \tag{5}$$

exists and is independent of ϕ. In this case the limit is called the Fresnel integral of f with respect to Q and denoted by

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x, Qx\rangle} f(x)dx \tag{6}$$

The description of the full class of Fresnel integrable function is not easy, but one can find some subsets of it. Indeed the following result holds [6, 19]:

Theorem 1. *Let $f \in \mathcal{F}(\mathbb{R}^n)$, then f is Fresnel integrable and its Fresnel integral with respect to Q is given by:*

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,Qx\rangle} f(x)dx = (\det Q)^{-1/2} \int e^{\frac{-i\hbar}{2}\langle \alpha,Q^{-1}\alpha\rangle}\mu_f(d\alpha). \qquad (7)$$

2.2. **Infinite dimensional oscillatory integrals.** Let us consider an infinite dimensional real separable Hilbert space $\mathcal{H}$ and define an infinite dimensional oscillatory integral [19, 3]. Let P_n be a sequence of projectors onto n-dimensional subspaces of $\mathcal{H}$, such that $P_n \leq P_{n+1}$ and $P_n \to 1$ strongly as $n \to \infty$, (1 being the identity operator in $\mathcal{H}$).

Definition 2. *A function $f : \mathcal{H} \to \mathbb{C}$ is Fresnel integrable with respect to Q if and only if the finite dimensional approximations of the Fresnel integral of f with respect to Q*

$$(2\pi i\hbar)^{-n/2} \int_{P_n\mathcal{H}} e^{\frac{i}{2\hbar}\langle P_n x,QP_n x\rangle} f(P_n x)d(P_n x),$$

are well defined and the limit

$$\lim_{n\to\infty} (2\pi i\hbar)^{-n/2} \int_{P_n\mathcal{H}} e^{\frac{i}{2\hbar}\langle P_n x,QP_n x\rangle} f(P_n x)d(P_n x) \qquad (8)$$

exists and is independent on the sequence $\{P_n\}$.
In this case the limit is called the Fresnel integral of f with respect to Q and is denoted by

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,Qx\rangle} f(x)dx$$

One can prove One can prove [3, 19] that if $f \in \mathcal{F}(\mathcal{H})$ then $f \circ P_n \in \mathcal{F}(P_n(\mathcal{H}))$ and f is Fresnel integrable. Moreover, if $Q - I$ is trace class, the following Cameron Martin-Parseval type formula holds:

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,Qx\rangle} f(x)dx = (\det Q)^{-1/2} \int_{\mathcal{H}} e^{-\frac{i\hbar}{2}\langle \alpha,Q^{-1}\alpha\rangle}\mu_f(d\alpha) \qquad (9)$$

where $\det Q = |\det Q| e^{-\pi i \operatorname{Ind} Q}$ is the Fredholm determinant of the operator Q, $|\det Q|$ its absolute value and $\operatorname{Ind}(Q)$ is the number of negative eigenvalues of the operator Q, counted with their multiplicity.

In this setting one can give a rigorous mathematical interpretation of formula (2) in terms of an infinite dimensional oscillatory integral on a suitable Hilbert space of paths. Let us consider the Sobolev space $H^{1,2}_{(t)}([0,t],\mathbb{R}^d)$, that is the space of absolutely continuous functions $\gamma : [0,t] \to \mathbb{R}^d$, $\gamma(t) = 0$, such that $\int_0^t |\dot\gamma(s)|^2 ds < \infty$, endowed with the following scalar product

$$\langle \gamma_1, \gamma_2 \rangle = \int_0^t \dot{\gamma}_1(s) \cdot \dot{\gamma}_2(s) ds,$$

and the Schrödinger equation in $L^2(\mathbb{R}^d)$

$$i\hbar \frac{\partial}{\partial t} \psi = H\psi \tag{10}$$

with initial datum $\psi_{|t=0} = \psi_0$. Let $H = -\frac{\hbar^2}{2}\Delta + \frac{1}{2}x\Omega^2 x + V'(x)$, where $x \in \mathbb{R}^d$, $\Omega^2 \geq 0$ is a $d \times d$ matrix, $V' \in \mathcal{F}(\mathbb{R}^d)$ and $\psi_0 \in \mathcal{F}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. By considering the operator L on $H^{1,2}_{(t)}([0,t], \mathbb{R}^d)$ given by

$$\langle \gamma, L\gamma \rangle \equiv \int_0^t \gamma(s)\Omega^2\gamma(s) ds,$$

and the function $W : H_t \to \mathbb{C}$

$$W(\gamma) \equiv \int_0^t V'(\gamma(s) + x) ds + 2x\Omega^2 \int_0^t \gamma(s) ds, \qquad \gamma \in H^{1,2}_{(t)},$$

formula (2)

$$\text{“} \quad const \int_{\{\gamma | \gamma(t) = x\}} e^{\frac{i}{\hbar}\int_0^t (\frac{1}{2}\dot{\gamma}(s)^2 - \frac{1}{2}\gamma(s)\Omega^2\gamma(s) - V'(\gamma(s)))ds} \psi_0(\gamma(0)) D\gamma \quad \text{”}$$

can be interpreted as the infinite dimensional oscillatory integral on $H^{1,2}_{(t)}([0,t], \mathbb{R}^d)$

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle \gamma, (I+L)\gamma \rangle} e^{-\frac{i}{\hbar}W(\gamma)} \psi_0(\gamma(0) + x) d\gamma. \tag{11}$$

Moreover one can prove [19, 3] that (11) is a representation of the solution of (10) evaluated in $x \in \mathbb{R}^d$ at time t.

Remark: It is important to note that if $V' \in \mathcal{F}(\mathbb{R}^d)$, then V' is bounded. As a consequence the only unbounded potentials for which the Feynman functional of [6, 19, 3] can be rigourously defined are those of harmonic oscillator type. The extension to unbounded potentials which are Laplace transforms of bounded measures [5, 27] also does not cover the case of potentials which are polynomials of degree larger than 2.

3. A generalized oscillatory integral

In this section and in the following one we shall generalize formulae (7) and (9) to a larger class of phase functions.

Let us deal first of all with the finite dimensional case, i.e. $dim(\mathcal{H}) = N$. Let $A : \mathcal{H} \times \mathcal{H} \times \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a completely symmetric and positive fourth order covariant tensor on $\mathcal{H}$. After the introduction of an orthonormal basis in $\mathcal{H}$, the elements $x \in \mathcal{H}$ can be identified with $N-$ple of real numbers, i.e. $x = (x_1, \dots, x_N)$, and the action of the tensor A on the 4-ple (x,x,x,x) is represented by an homogeneus fourth order polynomial in the variables $x_1, \dots, x_N$:

$$P(x) = A(x,x,x,x) = \sum_{j,k,l,m} a_{j,k,l,m} x_j x_k x_l x_m \tag{12}$$

with $a_{j,k,l,m} \in \mathbb{R}$.
We are going to define the following generalized Fresnel integral:

$$\widetilde{\int} e^{\frac{i}{2\hbar} x\cdot(I-B)x} e^{\frac{-i\lambda}{\hbar} P(x)} f(x) dx \tag{13}$$

where I, B are $N \times N$ matrices, I being the identity, $\lambda \in \mathbb{R}$, $f \in \mathcal{F}(\mathbb{R}^N)$ and $\hbar > 0$.

Lemma 1. *Let $P : \mathbb{R}^N \to \mathbb{R}$ be given by (12). Then the Fourier transform of the distribution $\frac{e^{\frac{i}{2\hbar} x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar} P(x)}$:*

$$\tilde{F}(k) = \int_{\mathbb{R}^N} e^{ik\cdot x} \frac{e^{\frac{i}{2\hbar} x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar} P(x)} d^N x \tag{14}$$

is a bounded complex-valued entire function on $\mathbb{R}^N$ admitting, if A is strictly positive, the following representations

$$\tilde{F}(k) = \begin{cases} e^{iN\pi/8} \int_{\mathbb{R}^N} e^{ie^{i\pi/8} k\cdot x} \frac{e^{\frac{ie^{i\pi/4}}{2\hbar} x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}} e^{\frac{\lambda}{\hbar} P(x)} d^N x & \lambda < 0 \\ e^{-iN\pi/8} \int_{\mathbb{R}^N} e^{ie^{-i\pi/8} k\cdot x} \frac{e^{\frac{ie^{-i\pi/4}}{2\hbar} x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}} e^{-\frac{\lambda}{\hbar} P(x)} d^N x & \lambda > 0 \end{cases} \tag{15}$$

Moreover, for general $A \geq 0$, if $\lambda \leq 0$ and $(I-B)$ is symmetric strictly positive then $\tilde{F}(k)$ can also be represented by

$$\tilde{F}(k) = \int_{\mathbb{R}^N} e^{ie^{i\pi/4} k\cdot x} \frac{e^{-\frac{1}{2\hbar} x\cdot(I-B)x}}{(2\pi\hbar)^{N/2}} e^{\frac{i\lambda}{\hbar} P(x)} d^N x = \mathbb{E}[e^{ie^{i\pi/4} k\cdot x} e^{\frac{i\lambda}{\hbar} P(x)} e^{\frac{1}{2\hbar} x\cdot Bx}] \tag{16}$$

where $\mathbb{E}$ denotes the expectation value with respect to the centered Gaussian measure on $\mathbb{R}^N$ with covariance operator $\hbar I$.

Proof For the proof of the representation (15) and of the boundedness of $\tilde{F}$ see [7], where a more general case is handled. From the representations (15) and (16) the analyticity of $\tilde{F}(k)$, $k \in \mathbb{C}$ follows immediately.
Let us here prove representation (16) in the particular case $B = 0$ and

P of the special form $P(x)=\sum_{j=1}^N a_j x_j^4$, with $a_j\geq 0$. This is sufficient to show the main ideas of the proof, the general case is handled in the appendix.
In this case one has to study the following integral on the real line:

$$I_j(k_j)\equiv\int_{\mathbb{R}} e^{ik_jx_j}\frac{e^{\frac{i}{2\hbar}x_j^2}}{(2\pi i\hbar)^{1/2}}e^{\frac{-i\lambda}{\hbar}a_jx_j^4}dx_j, \qquad k=(k_1,\dots,k_N), k_j\in\mathbb{R}$$

and then one has

$$\tilde{F}(k)=\prod_{j=1}^N I_j(k).$$

Moreover, as $e^{\frac{i}{2\hbar}x_j^2}e^{\frac{-i\lambda}{\hbar}a_jx_j^4}$ is an even function, we have $I_j(k)=I_{j,+}(k)+I_{j,+}(-k)$, with

$$I_{j,+}(k)=\int_0^\infty e^{ik_jx_j}\frac{e^{\frac{i}{2\hbar}x_j^2}}{(2\pi i\hbar)^{1/2}}e^{\frac{-i\lambda}{\hbar}a_jx_j^4}dx_j$$

In the following we will parametrize a complex number $z\in\mathbb{C}$ by means of its modulus ρ and its phase $\theta\in[0,2\pi)$, i.e. $z=\rho e^{i\theta}$.
Since the integrand in $I_{j,+}(k)$ is oscillating, a priori it is not clear that $I_{j,+}(k)$ exists, even as an improper Riemann integral. For this reason we look at the corresponding integral in the upper halfplane of $\mathbb{C}$ with a "regularizing parameter" $0<\epsilon<\pi/4$, which we send to zero at the end. For each $R>0$ let us consider the closed path in the complex plane composed by three pieces: $\gamma_1,\gamma_2,\gamma_3$, where

$$\begin{aligned}
\gamma_1(R)&=\{z\in\mathbb{C}\quad|\quad 0\leq\rho\leq R,\quad \theta=\epsilon\}\\
\gamma_2(R)&=\{z\in\mathbb{C}\quad|\quad \rho=R,\quad \epsilon\leq\theta\leq\pi/4\}\\
\gamma_3(R)&=\{z\in\mathbb{C}\quad|\quad 0\leq\rho\leq R,\quad \theta=\pi/4\}
\end{aligned}$$

for some small $0<\epsilon<\pi/4$. From the analyticity of $z_j\mapsto e^{ik_jz_j}e^{\frac{i}{2\hbar}z_j^2}e^{\frac{-i\lambda}{\hbar}a_jz_j^4}$ and the Cauchy theorem we have:

$$\int_{\gamma_1\cup\gamma_2\cup\gamma_3} e^{ik_jz_j}e^{\frac{i}{2\hbar}z_j^2}e^{\frac{-i\lambda}{\hbar}a_jz_j^4}dz_j=0,$$

that is

$$\int_{\gamma_1} e^{ik_jz_j}e^{\frac{i}{2\hbar}z_j^2}e^{\frac{-i\lambda}{\hbar}a_jz_j^4}dz_j+iR\int_0^{\pi/4} e^{ik_jRe^{i\theta}}e^{\frac{i}{2\hbar}R^2e^{2i\theta}}e^{\frac{-i\lambda}{\hbar}a_jR^4e^{4i\theta}}e^{i\theta}d\theta+$$
$$-e^{i\pi/4}\int_0^R e^{ik_j\rho e^{i\pi/4}}e^{\frac{i}{2\hbar}\rho^2e^{i\pi/2}}e^{i\frac{\lambda}{\hbar}a_j\rho^4}d\rho=0 \quad (17)$$

Now we take the limit as $R\to+\infty$. The second integral converges to 0, as it is easy to verify by using the methods presented in appendix

A. Hence we have:

$$\lim_{R\to\infty}\int_{\gamma_1} e^{ik_jz_j}e^{\frac{i}{2\hbar}z_j^2}e^{\frac{-i\lambda}{\hbar}a_jz_j^4}dz_j = e^{i\pi/4}\int_0^\infty e^{ik_j\rho e^{i\pi/4}}e^{\frac{i}{2\hbar}\rho^2e^{i\pi/2}}e^{i\frac{\lambda}{\hbar}a_j\rho^4}d\rho.$$

The r.h.s. is independent of ϵ, hence the limit of the l.h.s. for $\epsilon\downarrow 0$ (ϵ entering in the definition of $\gamma_1(R)$) also exists and is equal to the r.h.s. So we get :

$$\begin{aligned} I_{j,+}(k) &= e^{i\pi/4}\int_0^\infty e^{ik_j\rho e^{i\pi/4}}\frac{1}{\sqrt{2\pi i\hbar}}e^{\frac{i}{2\hbar}\rho^2e^{i\pi/2}}e^{\frac{-i\lambda}{\hbar}a_j\rho^4e^{i\pi}}d\rho \\ &= \int_0^\infty e^{ik_j\rho e^{i\pi/4}}\frac{1}{\sqrt{2\pi\hbar}}e^{\frac{-\rho^2}{2\hbar}}e^{\frac{i\lambda}{\hbar}a_j\rho^4}d\rho \quad (18)\end{aligned}$$

so that

$$\begin{aligned}\tilde F(k) = \prod_{j=1}^N(I_{j,+}(k)+I_{j,+}(-k)) &= \int_{\mathbb{R}^N} e^{ie^{i\pi/4}k\cdot x}e^{i\frac{\lambda}{\hbar}P(x)}\frac{e^{-\frac{1}{2\hbar}x\cdot x}}{(2\pi\hbar)^{N/2}}d^nx = \\ &= \mathbb{E}[e^{ie^{i\pi/4}k\cdot x}e^{i\frac{\lambda}{\hbar}P(x)}] \quad (19)\end{aligned}$$

(where $\mathbb{E}$ is the expectation with respect to the gaussian measure on $\mathbb{R}^N$ of mean zero and variance $\hbar^2 I$). □

Remark 1. *A careful reading of this proof shows that the second part of the statement, that is representation* (16)*, is valid if and only if the degree of P is 4, but cannot be generalized to polynomial functions of higher (even) degree. In fact the proof is based on the analyticity of the integrand and on a deformation of the contour of integration into a region of the complex plane in which the real part of the leading term of the polynomial, that is of $Re(-i\lambda az^4)$, is negative, where $\lambda < 0$, $a > 0$. By setting $z=\rho e^{i\theta}$ one can immediately verify that this condition is satisfied if and only if $0\le\theta\le\pi/4$. By considering a polynomial of higher even degree $2M$ this condition becomes $0\le\theta\le\pi/2M$ and if $M>2$ the angle $\theta=\pi/4$ is no longer included. This angle is fundamental as the oscillatory function $\frac{e^{\frac{i}{2\hbar}z^2}}{(2\pi i\hbar)^{1/2}}$ evaluated in $z=\rho e^{i\pi/4}$ gives $e^{-i\pi/4}\frac{e^{\frac{-\rho^2}{2\hbar}}}{(2\pi\hbar)^{1/2}}$, that is the density of the normal distribution with mean zero and variance $\hbar^2$, multiplied by the factor $e^{-i\pi/4}$. These considerations also show the necessity of considering $\lambda\le 0$.*

Remark 2. *We note that to have $\lambda=0$ is equivalent to take $P=0$. In this case by a deformation of the integration contour one has* *immediately:*

$$\int_{\mathbb{R}^N} e^{ik\cdot x}\frac{e^{\frac{i}{2\hbar}x\cdot x}}{(2\pi i\hbar)^{N/2}}d^Nx =$$
$$= \int_{\mathbb{R}^N} e^{ik\cdot xe^{i\pi/4}}\frac{e^{\frac{-x\cdot x}{2\hbar}}}{(2\pi\hbar)^{N/2}}d^Nx = \mathbb{E}[e^{ik\cdot xe^{i\pi/4}}] = e^{\frac{-i\hbar}{2}k\cdot k}. \quad (20)$$

We are going to apply these results to the definition of the generalized Fresnel integral (13).

Theorem 2. *("Parseval equality") Let $f \in \mathcal{F}(\mathbb{R}^N)$, $f = \hat{\mu}_f$. Then the generalized Fresnel integral*

$$I(f) \equiv \widetilde{\int} e^{\frac{i}{2\hbar}x\cdot(I-B)x}e^{\frac{-i\lambda}{\hbar}P(x)}f(x)dx$$

is well defined and it is given by:

$$\widetilde{\int} e^{\frac{i}{2\hbar}x\cdot(I-B)x}e^{\frac{-i\lambda}{\hbar}P(x)}f(x)dx = \int \tilde{F}(k)\mu_f(dk), \quad (21)$$

where $\tilde{F}(k)$ is given by equation (15) if A in (12) is strictly positive, or by equation (16) if $A \geq 0$, $\lambda \leq 0$ and $(I-B)$ is symmetric strictly positive. The integral on the r.h.s. of (21) is absolutely convergent (hence it can be understood in Lebesgue sense).

Proof: Let us choose a test function $\psi \in \mathcal{S}(\mathbb{R}^N)$, such that $\psi(0) = 1$ and let us compute the limit

$$I(f) \equiv \lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^N}\frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(x)}\psi(\epsilon x)f(x)dx$$

By hypothesis $f(x) = \hat{\mu}_f(x) = \int e^{ikx}\mu_f(dk)$ and substituting in the previous expression we get :

$$I(f) = \lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^N}\frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(x)}\psi(\epsilon x)\int e^{ikx}\mu_f(dk)dx.$$

By Fubini theorem (which applies for any $\epsilon > 0$ since the integrand is bounded by $|\psi(\epsilon x)|$ which is dx-integrable, and μ_f is a bounded measure) the r.h.s. is

$$= \lim_{\epsilon\downarrow 0}\int\Big(\int\frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(x)}\psi(\epsilon x)e^{ikx}dx\Big)\mu_f(dk)$$
$$= \frac{1}{(2\pi)^N}\lim_{\epsilon\downarrow 0}\int\int\tilde{F}(k-\alpha\epsilon)\tilde{\psi}(\alpha)d\alpha\mu_f(dk) \quad (22)$$

(here we have used the fact that the integral with respect to x is the Fourier transform of $\frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(x)}\psi(\epsilon x)$ and the inverse Fourier transform of a product is a convolution). Now we can pass to the limit using the Lebesgue bounded convergence theorem and get the desired result:

$$\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^N}\frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(x)}\psi(\epsilon x)f(x)dx=\int_{\mathbb{R}^N}\tilde{F}(k)\mu_f(dk),$$

where we have used that $\int\tilde{\psi}(\alpha)d\alpha=(2\pi)^N\psi(0)$ and lemma 1, which assures the boundedness of $\tilde{F}(k)$. □

Corollary 1. *Let $(I-B)$ be symmetric and strictly positive, $\lambda\leq 0$ and $f\in\mathcal{F}(\mathbb{R}^N)$, $f=\hat{\mu}_f$ such that $\forall x\in\mathbb{R}^N$ the integral $\int e^{-\frac{\sqrt{2}}{2}kx}|\mu_f|(dk)$ is convergent and the positive function $g:\mathbb{R}^n\to\mathbb{R}$, defined by $g(x)=e^{\frac{1}{2\hbar}x\cdot Bx}\int e^{-\frac{\sqrt{2}}{2}kx}|\mu_f|(dk)$ is summable with respect to the centered Gaussian measure on $\mathbb{R}^N$ with covariance $\hbar I$.*
Then f extends to an analytic function on $\mathbb{C}^N$ and the corresponding generalized Fresnel integral is well defined and it is given by

$$\widetilde{\int}_{\mathbb{R}^N}\frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(x)}f(x)dx=\mathbb{E}[e^{\frac{i\lambda}{\hbar}P(x)}e^{\frac{1}{2\hbar}x\cdot Bx}f(e^{i\pi/4}x)]. \qquad (23)$$

Proof: By the assumption on the measure μ_f it follows that its Laplace transform $f^L:\mathbb{C}^N\to\mathbb{C}$, $f^L(z)=\int_{\mathbb{R}^N}e^{kz}\mu_f(dk)$, is a well defined entire function such that $f^L(ix)=f(x)$, $x\in\mathbb{R}^N$. By theorem 2 the generalized Fresnel integral can be computed by means of Parseval equality

$$\widetilde{\int}_{\mathbb{R}^N}\frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(x)}f(x)dx=\int_{\mathbb{R}^N}\tilde{F}(k)\mu_f(dk)=$$
$$=\int_{\mathbb{R}^N}\mathbb{E}[e^{ikxe^{i\pi/4}}e^{\frac{1}{2\hbar}x\cdot Bx}e^{\frac{i\lambda}{\hbar}P(x)}]\mu_f(dk)$$

By Fubini theorem, which applies given the assumptions on the measure μ_f, this is equal to

$$\mathbb{E}[e^{\frac{1}{2\hbar}x\cdot Bx}e^{\frac{i\lambda}{\hbar}P(x)}\int_{\mathbb{R}^N}e^{ikxe^{i\pi/4}}\mu_f(dk)]=\mathbb{E}[e^{\frac{1}{2\hbar}x\cdot Bx}e^{\frac{i\lambda}{\hbar}P(x)}f^L(ie^{i\pi/4}x)]=$$
$$=\mathbb{E}[e^{\frac{1}{2\hbar}x\cdot Bx}e^{\frac{i\lambda}{\hbar}P(x)}f(e^{i\pi/4}x)] \qquad (24)$$

and the conclusion follows. □

Remark 3. *The latter theorem shows that, under suitable assumptions on the function f, the generalized Fresnel integral* (13) *can be explicitly computed by means of a Gaussian integral. By mimicking the proof of lemma 1 one can be tempted to generalize equation* (23) *to a larger class of functions, that are analytic in a suitable region of $\mathbb{C}^N$, but do not belong to $\mathcal{F}(\mathbb{R}^N)$ (see* [7] *for more details). In fact this is not possible, as the definition 1 of oscillatory integral requires that the limit of the sequence of regularized integrals exists and is independent of the regularization. Let us consider the subset of the complex plane*

$$\Lambda = \{\xi \in \mathbb{C} \mid 0 < \arg(\xi) < \pi/4\} \subset \mathbb{C}, \tag{25}$$

and let $\bar{\Lambda}$ be its closure. The identity

$$\lim_{\epsilon\to 0} \int \int_{\mathbb{R}^N} \frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}P(x)} f(x)\psi(\epsilon x)dx = \mathbb{E}[e^{\frac{1}{2\hbar}x\cdot Bx} e^{\frac{i\lambda}{\hbar}P(x)} f(e^{i\pi/4}x)]$$

(with $(I-B)$ symmetric strictly positive and $\lambda \leq 0$) can only be proven by choosing a regularizing function $\psi \in \mathcal{S}$, $\psi(0) = 1$, such that the function $z \mapsto \psi(zx)$ is analytic for $z \in \Lambda$ and continuous for $z \in \bar{\Lambda}$ for each $x \in \mathbb{R}^N$. Moreover one has to assume that $|\psi(e^{i\theta}x)|$ is bounded as $|x| \to \infty$ for each $\theta \in (0, \pi/4)$.

4. Infinite dimensional generalized oscillatory integrals

Let $\mathcal{H}$ be a real separable infinite dimensional Hilbert space, with inner product $\langle\ ,\ \rangle$ and norm $|\ |$. Let ν be the finitely additive cylinder measure on $\mathcal{H}$, defined by its characteristic functional $\hat{\nu}(x) = e^{-\frac{\hbar}{2}|x|^2}$. Let $\|\ \|$ be a "measurable" norm on $\mathcal{H}$, that is $\|\ \|$ is such that for every $\epsilon > 0$ there exist a finite-dimensional projection $P_\epsilon : \mathcal{H} \to \mathcal{H}$, such that $\nu(\{x \in \mathcal{H} |\ \|P_\epsilon(x)\| > \epsilon\}) < \epsilon$. One can easily verify that $\|\ \|$ is weaker than $|\ |$. Denoted by $\mathcal{B}$ the completion of $\mathcal{H}$ in the $\|\ \|$-norm and by i the continuous inclusion of $\mathcal{H}$ in $\mathcal{B}$, one can prove that $\mu \equiv \nu \circ i^{-1}$ is a countably additive Gaussian measure on the Borel subsets of $\mathcal{B}$. The triple $(i, \mathcal{H}, \mathcal{B})$ is called an *abstract Wiener space* [21, 28]. Given $y \in \mathcal{B}^*$ one can easily verify that the restriction of y to $\mathcal{H}$ is continuous on $\mathcal{H}$, so that one can identify $\mathcal{B}^*$ as a subset of $\mathcal{H}$. Moreover $\mathcal{B}^*$ is dense in $\mathcal{H}$ and we have the dense continuous inclusions $\mathcal{B}^* \subset \mathcal{H} \subset \mathcal{B}$. Each element $y \in \mathcal{B}^*$ can be regarded as a random variable $n(y)$ on $(\mathcal{B}, \mu)$. A direct computation shows that $n(y)$ is normally distributed, with covariance $|y|^2$. More generally, given $y_1, y_2 \in \mathcal{B}^*$, one has

$$\int_B n(y_1)n(y_2)d\mu = \langle y_1, y_2\rangle.$$

The latter result allows the extension to the map $n : \mathcal{H} \to L^2(\mathcal{B}, \mu)$, because $\mathcal{B}^*$ is dense in $\mathcal{H}$. Given an orthogonal projection P in $\mathcal{H}$, with

$$P(x) = \sum_{i=1}^{n} \langle e_i, x \rangle e_i$$

for some orthonormal $e_1, \dots, e_n \in \mathcal{H}$, the stochastic extension $\tilde{P}$ of P on $\mathcal{B}$ is well defined by

$$\tilde{P}(\,\cdot\,) = \sum_{i=1}^{n} n(e_i)(\,\cdot\,)e_i.$$

Given a function $f : \mathcal{H} \to \mathcal{B}_1$, where $(\mathcal{B}_1, \|\ \|_{\mathcal{B}_1})$ is another real separable Banach space, the stochastic extension $\tilde{f}$ of f to $\mathcal{B}$ exists if the functions $f \circ \tilde{P} : \mathcal{B} \to \mathcal{B}_1$ converge to $\tilde{f}$ in probability with respect to μ as P converges strongly to the identity in $\mathcal{H}$. If $g : \mathcal{B} \to \mathcal{B}_1$ is continuous and $f := g|_{\mathcal{H}}$, then one can prove [21] that the stochastic extension of f is well defined and it is equal to g $\mu-$a.e. In this setting it is possible to extend the results of the previous section to the infinite dimensional case.

Let $A : \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \to \mathbb{R}$ be a completely symmetric positive covariant tensor operator on $\mathcal{H}$ such that the map $V : \mathcal{H} \to \mathbb{R}^+$, $x \mapsto V(x) \equiv A(x, x, x, x)$ is continuous in the $\|\ \|$ norm. As a consequence V is continuous in the $|\ |$-norm, moreover it can be extended by continuity to a random variable $\bar{V}$ on $\mathcal{B}$, with $\bar{V}|_{\mathcal{H}} = V$. By the previous considerations, the stochastic extension $\tilde{V}$ of $V : \mathcal{H} \to \mathbb{R}$ exists and coincides with $\bar{V} : \mathcal{B} \to \mathbb{R}$ $\mu-$a.e. Moreover for any increasing sequence of $n-$dimensional projectors P_n in $\mathcal{H}$, the family of bounded random variables $e^{i\frac{\lambda}{\hbar}V \circ \tilde{P}_n(\,\cdot\,)} \equiv e^{i\frac{\lambda}{\hbar}V^n(\,\cdot\,)}$ converges $\mu-$a.e. to $e^{i\frac{\lambda}{\hbar}\bar{V}(\,\cdot\,)}$. Moreover for any $h \in \mathcal{H}$ the sequence of random variables

$$\sum_{i=1}^{n} h_i n(e_i), \qquad h_i = \langle e_i, h \rangle$$

converges in $L^2(\mathcal{B}, \mu)$, and by subsequences a.e., to the random variable $n(h)$.

Let us consider a self-adjoint trace class operator $B : \mathcal{H} \to \mathcal{H}$. The quadratic form on $\mathcal{H} \times \mathcal{H}$:

$$x \in \mathcal{H} \mapsto \langle x, Bx \rangle$$

can be extended to a random variable on $\mathcal{B}$, denoted again by $\langle\,\cdot\,, B\,\cdot\,\rangle$. Indeed for each increasing sequence of finite dimensional projectors P_n converging strongly to the identity, $P_n(x) = \sum_{i=1}^{n} e_i \langle e_i, x \rangle$ ($\{e_i\}$ being a CONS in $\mathcal{H}$), the sequence of random variables

$$\omega \in \mathcal{B} \mapsto \sum_{i,j=1}^{n} \langle e_i, Be_j \rangle n(e_i)(\omega) n(e_j)(\omega)$$

is a Cauchy sequence in $L^1(\mathcal{B}, \mu)$. By passing if necessary to a subsequence, it converges to $\langle \cdot , B \cdot \rangle$ $\mu-$a.e.
Let us assume that the largest eigenvalue of B is strictly less than 1 (or, in other words, that (I-B) is strictly positive). Then one can prove that the random variable $g(\cdot) := e^{\frac{1}{2\hbar}\langle \cdot , B \cdot \rangle}$ is μ-summable. Indeed by considering a CONS $\{e_i\}$ made of eigenvectors of the operator B, b_i being the corresponding eigenvalues, the sequence of random variables

$$g_n : \mathcal{B} \to \mathbb{C}, \qquad \omega \mapsto g_n(\omega) = e^{\frac{1}{2\hbar}\sum_{i=1}^{n} b_i([n(e_i)(\omega)]^2},$$

converges to $g(\omega)$ μ-a.e..
On the other hand one has

$$\int_{\mathcal{B}} g_n(\omega) d\mu(\omega) = \prod_{i=1}^{n} \int \frac{e^{-\frac{1}{2\hbar}(1-b_i)x_i^2}}{\sqrt{2\pi\hbar}} dx_i = (\prod_{i=1}^{n}(1-b_i))^{-1/2}$$

so that $\int g_n d\mu$ converges, as $n \to \infty$, to $(\det(I-B))^{-1/2}$, where $\det(I-B)$ denotes the Fredholm determinant of $(I-B)$, which is well defined as B is trace class. Moreover $0 \le g_n \le g_{n+1}$ for each n . It follows that, as $n \to \infty$, $\int g_n d\mu \to \int g d\mu = (\det(I-B))^{-1/2}$. By an analogous reasoning one can prove that for any $y \in \mathcal{H}$, the sequence of random variables f_n:

$$\omega \mapsto f_n(\omega) = e^{\sum_{i=1}^{n} y_i n(e_i)(\omega)} e^{\frac{1}{2\hbar}\sum_{i=1}^{n} b_i([n(e_i)(\omega)]^2}$$

where $y_i = \langle y, e_i \rangle$, converges $\mu-$a.e. as n goes to ∞ to the random variable $f(\cdot) = e^{n(y)(\cdot)} e^{\frac{1}{2\hbar}\langle \cdot , B \cdot \rangle}$ and that

$$\int f_n d\mu \to \int f d\mu = (\det(I-B))^{-1/2} e^{\frac{\hbar}{2}\langle y,(I-B)^{-1}y \rangle}. \tag{26}$$

(see [28, 26]). The following result follows:

Lemma 2. *Let $B : \mathcal{H} \to \mathcal{H}$ be a self adjoint and trace class operator such that $I-B$ is strictly positive, let $k \in \mathcal{H}$ and $\lambda \le 0$. Then for any increasing sequence P_n of projectors onto n-dimensional subspaces of $\mathcal{H}$ such that $P_n \uparrow I$ strongly as $n \to \infty$, the following sequence of finite dimensional integrals:*

$$F_n(k) \equiv (2\pi i\hbar)^{-n/2} \int_{P_n\mathcal{H}} e^{i\langle P_n k, P_n x \rangle} e^{\frac{i}{2\hbar}\langle P_n x,(I-B)P_n x \rangle} e^{-i\frac{\lambda}{\hbar}V(P_n x)} d(P_n x)$$

converges, as $n \to \infty$, to the Gaussian integral on $\mathcal{B}$:

$$F(k) \equiv \mathbb{E}[e^{in(k)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\langle \omega, B\omega\rangle} e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)}] \tag{27}$$

($\mathbb{E}$ being the expectation with respect to μ on $\mathcal{B}$)

Proof: By lemma 1 one has

$$(2\pi i\hbar)^{-n/2} \int_{P_n\mathcal{H}} e^{i\langle P_nk, P_nx\rangle} e^{\frac{i}{2\hbar}\langle P_nx,(I-B)P_nx\rangle} e^{-i\frac{\lambda}{\hbar}V(P_nx)} d(P_nx) =$$
$$(2\pi\hbar)^{-n/2} \int_{P_n\mathcal{H}} e^{i\langle P_nk,P_nx\rangle e^{i\pi/4}} e^{-\frac{1}{2\hbar}\langle P_nx,P_nx\rangle} e^{\frac{1}{2\hbar}\langle P_nx,BP_nx\rangle} e^{i\frac{\lambda}{\hbar}V(P_nx)} d(P_nx) \tag{28}$$

Let us introduce an orthonormal base $\{e_i\}$ of $\mathcal{H}$ such that P_n is the projector onto the span of the first n vectors. Each element $P_nx \in P_n\mathcal{H}$ can be represented as an $n-$ple of real numbers $(x_1, \ldots, x_n)$, where $x_i = \langle x, e_i\rangle$. The latter integral can be written in the following form:

$$(2\pi\hbar)^{-n/2} \int_{\mathbb{R}^n} e^{i\sum_{i=1}^n k_ix_ie^{i\pi/4}} e^{-\frac{1}{2\hbar}\sum_{i=1}^n x_i^2} e^{\frac{1}{2\hbar}\sum_{i,j=1}^n B_{ij}x_ix_j}$$
$$e^{i\frac{\lambda}{\hbar}\sum_{ij,k,h=1}^n A_{ijkh}x_ix_jx_kx_h} dx_1 \ldots dx_n$$

where $B_{ij} = \langle e_i, Be_j\rangle$ and $A_{ijkh} = A(e_i, e_j, e_k, e_h)$.
On the other hand, this coincides with the Gaussian integral on $(\mathcal{B}, \mu)$:

$$\mathbb{E}[e^{i\sum_{i=1}^n k_in(e_i)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\sum_{i,j=1}^n \langle e_i, Be_j\rangle n(e_i)(\omega)n(e_j)(\omega)} e^{\frac{\lambda}{\hbar}V\circ\tilde{P}_n(\omega)}]$$

By Lebesgue's dominated convergence theorem (which holds because of the assumption on the strict positivity of the operator $I - B$) this converges as $n \to \infty$ to

$$\mathbb{E}[e^{in(k)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\langle \omega, B\omega\rangle} e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)}].$$

and the conclusion follows. □

The above result allows to generalize theorem 2 to the infinite dimensional case.

Theorem 3. *Let B be self-adjoint trace class, $(I-B)$ strictly positive, $\lambda \leq 0$ and $f \in \mathcal{F}(\mathcal{H})$, $f \equiv \hat{\mu}_f$, and let us suppose that the bounded variation measure μ_f satisfies the following assumption*

$$\int_{\mathcal{H}} e^{\frac{\hbar}{4}\langle k,(I-B)^{-1}k\rangle} |\mu_f|(dk) < +\infty. \tag{29}$$

Then the infinite dimensional oscillatory integral

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I-B)x\rangle} e^{-i\frac{\lambda}{\hbar}A(x,x,x,x)} f(x)dx \tag{30}$$

exists and is given by:

$$\int_{\mathcal{H}} \mathbb{E}[e^{in(k)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle} e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)}]\mu_f(dk)$$

Proof: By definition, choosing an increasing sequence of finite dimensional projectors P_n on $\mathcal{H}$, with $P_n \uparrow I$ strongly as $n \to \infty$, the oscillatory integral (30) is given by:

$$\lim_{n\to\infty}(2\pi i\hbar)^{-n/2}\int_{P_n\mathcal{H}} e^{\frac{i}{2\hbar}\langle P_nx,(I-B)P_nx\rangle} e^{-i\frac{\lambda}{\hbar}A(P_nx,P_nx,P_nx,P_nx)} f(P_nx)dP_nx. \tag{31}$$

Let $f^n : P_n\mathcal{H} \to \mathbb{C}$ be the function defined by $f^n(y) \equiv f(y)$, $y \in P_n\mathcal{H}$. One can easily verify that $f^n \in \mathcal{F}(P_n\mathcal{H})$, $f^n = \hat{\mu}^n_f$, where μ^n_f is the bounded variation measure on $P_n\mathcal{H}$ defined by $\mu^n_f(I) = \mu_f(P_n^{-1}I)$, I being a Borel subset of $P_n\mathcal{H}$, indeed:

$$f^n(y) = f(y) = \int_{\mathcal{H}} e^{i\langle y,k\rangle}\mu_f(dk) = \\ = \int_{\mathcal{H}} e^{i\langle P_ny,P_nk\rangle}\mu_f(dk) = \int_{P_n\mathcal{H}} e^{i\langle y,P_nk\rangle}\mu^n_f(dP_nk) \tag{32}$$

where $y = P_ny$. By theorem 2 the limit (31) is equal to

$$\lim_{n\to\infty}\int_{P_n\mathcal{H}} G_n(P_nk)\mu^n_f(dP_nk), \tag{33}$$

where $G_n : P_n\mathcal{H} \to \mathbb{C}$ is given by:

$$G_n(P_nk) = (2\pi\hbar)^{-n/2}\int_{P_n\mathcal{H}} e^{i\langle P_nk,P_nx\rangle e^{i\pi/4}} e^{-\frac{1}{2\hbar}\langle P_nx,(I-B)P_nx\rangle} e^{i\frac{\lambda}{\hbar}A(P_nx,P_nx,P_nx,P_nx)}dP_nx$$

This, on the other hand (see the proof of lemma 2) is equal to

$$\mathbb{E}[e^{in(P_nk)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\sum_{i,j=1}^n B_{ij}n(e_i)(\omega)n(e_j)(\omega)} e^{i\frac{\lambda}{\hbar}V^n(\omega)}],$$

where $V^n = V \circ \tilde{P}_n$. By substituting the latter expression into (33) we have

$$\lim_{n\to\infty}\int_{P_n\mathcal{H}} \mathbb{E}[e^{in(P_nk)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\sum_{i,j=1}^n B_{ij}n(e_i)(\omega)n(e_j)(\omega)} e^{i\frac{\lambda}{\hbar}V^n(\omega)}]\mu^n_f(dP_nk) = \\ = \lim_{n\to\infty}\int_{\mathcal{H}} \mathbb{E}[e^{in(P_nk)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\sum_{i,j=1}^n B_{ij}n(e_i)(\omega)n(e_j)(\omega)} e^{i\frac{\lambda}{\hbar}V^n(\omega)}]\mu_f(dk) = \\ = \lim_{n\to\infty}\int_{\mathcal{H}} F_n(k)\mu_f(dk) \tag{34}$$

By lemma 2 and the dominated convergence theorem, applicable to the integral with respect to μ_f, due to assumption (29), we then get

$$\int_{\mathcal{H}} F(k)\mu_f(dk) = \int_{\mathcal{H}} \mathbb{E}[e^{in(k)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle} e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)}]\mu_f(dk)$$

and the conclusion follows. □

Corollary 1 can be generalized to the infinite dimensional case. Indeed due to the assumption (29) the function f on the real Hilbert space $\mathcal{H}$ can be extended to those vectors $y \in \mathcal{H}^{\mathbb{C}}$ in the complex Hilbert space $\mathcal{H}^{\mathbb{C}}$ of the form $y = zx$, $x \in \mathcal{H}$, $z \in \mathbb{C}$ as the integral

$$\int_{\mathcal{H}} e^{iz\langle x,k\rangle}\mu_f(dk)$$

is absolutely convergent. Moreover the latter can be uniquely extended to a random variable on $\mathcal{B}$, denoted again by f, by

$$f^z(\omega) \equiv f(z\omega) \equiv \int_{\mathcal{H}} e^{izn(k)(\omega)}\mu_f(dk), \quad \omega \in \mathcal{B}. \tag{35}$$

Moreover the random variable $e^{\frac{1}{2\hbar}\langle\ \cdot, B\ \cdot\rangle} f^z(\ \cdot)$ belongs to $L^1(\mathcal{B},\mu)$ if $Im(z)^2 \leq 1/2$.

Theorem 4. *Let $B : \mathcal{H} \to \mathcal{H}$ be self-adjoint trace class, $I - B$ strictly positive, $\lambda \leq 0$ and $f \in \mathcal{F}(\mathcal{H})$ be the Fourier transform of a bounded variation measure μ_f satisfying assumption* (29)*. Then the infinite dimensional oscillatory integral* (30) *is well defined and it is given by:*

$$\widetilde{\int}_{\mathcal{H}} e^{\frac{i}{2\hbar}\langle x,(I-B)x\rangle} e^{-i\frac{\lambda}{\hbar}A(x,x,x,x)} f(x)dx = \mathbb{E}[e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)} e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle} f(e^{i\pi/4}\omega)] \tag{36}$$

Proof: By theorem 3 the infinite dimensional oscillatory integral (30) can be computed by means of the Parseval-type formula:

$$\widetilde{\int}_{\mathcal{H}} e^{\frac{i}{2\hbar}\langle x,(I-B)x\rangle} e^{-i\frac{\lambda}{\hbar}A(x,x,x,x)} f(x)dx =$$
$$= \int_{\mathcal{H}} \mathbb{E}[e^{in(k)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle} e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)}]\mu_f(dk) \tag{37}$$

By Fubini theorem, which can be applied under the assumption (29), the integral on the r.h.s. of (37) is equal to

$$\mathbb{E}[e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)} e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle} \int_{\mathcal{H}} e^{in(k)(\omega)e^{i\pi/4}}\mu_f(dk)] =$$
$$= \mathbb{E}[e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)} e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle} f^{e^{i\pi/4}}(\omega)] = \mathbb{E}[e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)} e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle} f(e^{i\pi/4}\omega)]$$

The integral on the r.h.s. is absolutely convergent as $|e^{i\frac{\lambda}{\hbar}\bar{V}}|=1$ and $e^{\frac{1}{2\hbar}\langle\cdot,B\cdot\rangle}f^{e^{i\pi/4}}\in L^1(\mathcal{B},\mu)$ as $Im(e^{i\pi/4})=1/\sqrt{2}$. □

Remark 4. *In the simpler case $\lambda=0$, under the above assumptions on the function f and the operator B, the infinite dimensional oscillatory integral (given by* (36) *with $V=0$) can also be explicitly computed by means of the absolutely convergent integrals:*

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I-B)x\rangle}f(x)dx=\frac{1}{\sqrt{\det(I-B)}}\int_{\mathcal{H}} e^{-\frac{i\hbar}{2}\langle k,(I-B)^{-1}k}\mu_f(dk) \quad (38)$$

In fact, by means of different methods (see section 2), equation (38) *can be proven even without the assumption on the positivity of the operator $(I-B)$ (it suffices that $(I-B)$ be invertible).*

Remark 5. *So far we have proven, under suitable assumptions on the function $f:\mathcal{H}\to\mathbb{C}$ and the operator B, that if $\lambda\leq 0$ the infinite dimensional generalized Fresnel integral* (30)

$$I^F(\lambda)\equiv\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I-B)x\rangle}e^{-i\frac{\lambda}{\hbar}A(x,x,x,x)}f(x)dx$$

on the Hilbert space $\mathcal{H}$ is exactly equal to a Gaussian integral on $\mathcal{B}$:

$$I^G(\lambda)\equiv\int_{\mathcal{H}}\mathbb{E}[e^{in(k)(\omega)e^{i\pi/4}}e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle}e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)}]\mu_f(dk)$$

(theorem 3), and to

$$I^A(\lambda)\equiv\mathbb{E}[e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)}e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle}f(e^{i\pi/4}\omega)]$$

(theorem 4). One can easily verify that I^G and I^A are analytic functions of the complex variable λ in the region of the complex λ plane $\{Im(\lambda)>0\}$, while they are continuous in $\{Im(\lambda)=0\}$ and coincide with I^F in $\{Im(\lambda)=0,Re(\lambda)\leq 0\}$.

5. Application to the Schrödinger equation

Let us consider the Schrödinger equation

$$i\hbar\frac{d}{dt}\psi=H\psi \quad (39)$$

on $L^2(\mathbb{R}^d)$ for an anharmonic oscillator hamiltonian H of the following form:

$$H=-\frac{\hbar^2}{2}\Delta+\frac{1}{2}x\Omega^2x+\lambda C(x,x,x,x), \quad (40)$$

where C is a completely symmetric positive fourth order covariant tensor on $\mathbb{R}^d$, Ω is a positive symmetric $d \times d$ matrix, $\lambda \geq 0$ a positive constant. It is well known, see [36], that H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$. By means of the results of the previous section we are going to give mathematical meaning to the "Feynman path integral" representation of the solution of equation (39):

$$\psi(t,x) = \text{“}\int_{\gamma(0)=x} e^{\frac{i}{\hbar}\int_0^t \frac{\dot\gamma(s)^2}{2}ds - \frac{i}{\hbar}\int_0^t [\frac{1}{2}\gamma(s)\Omega^2\gamma(s)+\lambda C(\gamma(s),\gamma(s),\gamma(s),\gamma(s))]ds}\psi_0(\gamma(t))D\gamma\text{”},$$

as the analytic continuation (in the parameter λ) of an infinite dimensional generalized oscillatory integral on a suitable Hilbert space.
Let us consider the Cameron-Martin space H_t, that is the Hilbert space of absolutely continuous paths $\gamma : [0,t] \to \mathbb{R}^d$, with $\gamma(0) = 0$ and inner product $\langle \gamma_1, \gamma_2 \rangle = \int_0^t \dot\gamma_1(s)\dot\gamma_2(s)ds$. The cylindrical gaussian measure on H_t with covariance operator the identity extends to a σ-additive measure on the Wiener space $C_t = \{\omega \in C([0,t];\mathbb{R}^d) \mid \gamma(0) = 0\}$: the Wiener measure W. (i, H_t, C_t) is an abstract Wiener space.
Let us consider moreover the Hilbert space $\mathcal{H} = \mathbb{R}^d \times H_t$, and the Banach space $\mathcal{B} = \mathbb{R}^d \times C_t$ endowed with the product measure $N(dx) \times W(d\omega)$, N being the gaussian measure on $\mathbb{R}^d$ with covariance equal to the $d \times d$ identity matrix. $(i, \mathcal{H}, \mathcal{B})$ is an abstract Wiener space.
Let us consider two vectors $\phi, \psi_0 \in L^2(\mathbb{R}^d) \cap \mathcal{F}(\mathbb{R}^d)$.We are going to define the following infinite dimensional oscillatory integral on $\mathcal{H}$:

$$\text{”}\int_{\mathbb{R}^d\times H_t} \bar\phi(x) e^{\frac{i}{2\hbar}\int_0^t \dot\gamma(s)^2 ds} e^{-\frac{i}{2\hbar}\int_0^t [(\gamma(s)+x)\Omega^2(\gamma(s)+x)ds} e^{\frac{i\lambda}{\hbar}C(\gamma(s)+x,\gamma(s)+x,\gamma(s)+x,\gamma(s)+x)ds}\psi_0(\gamma(t)+x)dx D\gamma\text{”} \quad (41)$$

Let us consider the operator $B : \mathcal{H} \to \mathcal{H}$ given by:

$$(x,\gamma) \longrightarrow (y,\eta) = B(x,\gamma),$$

$$y = t\Omega^2 x + \Omega^2 \int_0^t \gamma(s)ds, \qquad \eta(s) = \Omega^2 x(ts - \frac{s^2}{2}) - \int_0^s \int_t^u \Omega^2\gamma(r)drdu \quad (42)$$

and the fourth order tensor operator A given by:

$$A((x_1,\gamma_1),(x_2,\gamma_2),(x_3,\gamma_3),(x_4,\gamma_4)) = \\ = \int_0^t C(\gamma_1(s)+x_1, \gamma_2(s)+x_2, \gamma_3(s)+x_3, \gamma_4(s)+x_4)ds. \quad (43)$$

Let us consider moreover the function $f : \mathcal{H} \to \mathbb{C}$ given by

$$f(x,\gamma) = (2\pi i\hbar)^{d/2} e^{-\frac{i}{2\hbar}|x|^2}\bar\phi(x)\psi_0(\gamma(t)+x) \quad (44)$$

with this notation expression (41) can be written in the following form:

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}(|x|^2+|\gamma|^2)} e^{-\frac{i}{2\hbar}\langle (x,\gamma),B(x,\gamma)\rangle} e^{-\frac{i\lambda}{\hbar}A((x,\gamma),(x,\gamma),(x,\gamma),(x,\gamma))} f(x,\gamma)dxd\gamma \quad (45)$$

Under suitable assumptions on Ω, λ the theory of the latter section applies, as we shall see below. In the following we will denote by Ω_i, $i = 1, \dots, d$, the eigenvalues of the matrix Ω.

Theorem 5. *Let us assume that $\lambda \leq 0$, and that for each $i = 1, \dots, d$ the following inequalities are satisfied*

$$\Omega_i t < \frac{\pi}{2}, \qquad 1 - \Omega_i \tan(\Omega_i t) > 0. \quad (46)$$

Let $\phi, \psi_0 \in L^2(\mathbb{R}^d) \cap \mathcal{F}(\mathbb{R}^d)$. Let μ_0 be the complex bounded variation measure on $\mathbb{R}^d$ such that $\hat{\mu}_0 = \psi_0$. Let μ_ϕ be the complex bounded variation measure on $\mathbb{R}^d$ such that $\hat{\mu}_\phi(x) = (2\pi i\hbar)^{d/2} e^{-\frac{i}{2\hbar}|x|^2}\bar{\phi}(x)$. Assume in addition that the measures μ_0, μ_ϕ satisfy the following assumption:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\frac{\hbar}{4}x\Omega^{-1}\tan(\Omega t)x} e^{(y+\cos(\Omega t)^{-1}x)(1-\Omega\tan(\Omega t))^{-1}(y+\cos(\Omega t)^{-1}x)} |\mu_0|(dx)|\mu_\phi|(dy) < \infty \quad (47)$$

Then the function $f : \mathcal{H} \to \mathbb{C}$, given by (44) is the Fourier transform of a bounded variation measure μ_f on $\mathcal{H}$ satisfying

$$\int_{\mathcal{H}} e^{\frac{\hbar}{4}\langle (y,\eta),(I-B)^{-1}(y,\eta)\rangle} |\mu_f|(dyd\eta) < \infty \quad (48)$$

(B being given by (42)) and the infinite dimensional oscillatory integral (45) is well defined and is given by:

$$\int_{\mathbb{R}^d\times H_t}\Big(\int_{\mathbb{R}^d\times C_t} e^{ie^{i\pi/4}(x\cdot y+\sqrt{\hbar}n(\gamma)(\omega))} e^{\frac{1}{2\hbar}\int_0^t(\sqrt{\hbar}\omega(s)+x)\Omega^2(\sqrt{\hbar}\omega(s)+x)ds}$$
$$e^{i\frac{\lambda}{\hbar}\int_0^t C(\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x)ds} W(d\omega)\frac{e^{-\frac{|x|^2}{2\hbar}}}{(2\pi\hbar)^{d/2}}dx\Big)\mu_f(dyd\gamma). \quad (49)$$

This is also equal to

$$(i)^{d/2}\int_{\mathbb{R}^d\times C_t} e^{i\frac{\lambda}{\hbar}\int_0^t C(\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x)ds}$$
$$e^{\frac{1}{2\hbar}\int_0^t(\sqrt{\hbar}\omega(s)+x)\Omega^2(\sqrt{\hbar}\omega(s)+x)ds}\bar{\phi}(e^{i\pi/4}x)\psi_0(e^{i\pi/4}\sqrt{\hbar}\omega(t)+e^{i\pi/4}x)W(d\omega)dx. \quad (50)$$

Proof: By the assumptions on ϕ, one can easily verify that the function $(2\pi i\hbar)^{d/2}e^{-\frac{i}{2\hbar}|x|^2}\bar{\phi}(x)$ is the Fourier transform of the bounded variation measure on $\mathbb{R}^d \times \mathcal{H}$, which is the product measure $\mu_\phi(dx) \times \delta_0(d\gamma)$, where $\delta_0(d\gamma)$ is the measure on H_t concentrated on the vector $0 \in H_t$. Analogously the function $(x, \gamma) \mapsto \psi_0(\gamma(t) + x)$ is the Fourier transform of the bounded variation measure μ_ψ on $\mathbb{R}^d \times \mathcal{H}$ given by :

$$\int_{\mathbb{R}^d \times H_t} f(x,\gamma)\mu_\psi(dxd\gamma) = \int_{\mathbb{R}^d \times H_t} f(x, x\gamma)\delta_{G_t}(d\gamma)\mu_0(dx),$$

where G_t is the vector in $\mathcal{H}$ given by $G_t(s) = s$. As $\mathcal{F}(\mathbb{R}^d \times \mathcal{H})$ is a Banach algebra, the product $f(x,\gamma) := (2\pi i\hbar)^{d/2}e^{-\frac{i}{2\hbar}|x|^2}\bar{\phi}(x)\psi(\gamma(t) + x)$ still belongs to $\mathcal{F}(\mathbb{R}^d \times \mathcal{H})$, in fact it is the Fourier transform of the convolution $\mu_f \equiv (\mu_\phi \times \delta_0) * \mu_\psi$. A direct computation shows that μ_f satisfies assumptions (29) of theorem 3, that is (48), if and only if μ_0 and μ_ϕ satisfy (47).
By simple calculations one can verify that the operator B given by (42) is bounded symmetric and trace class. Moreover if assumptions (46) are satisfied, $I - B$ is positive definite (see appendix B for more details).
A direct computation shows that the function $V : \mathcal{H} \to \mathbb{R}$, $V(x,\gamma) = A((x,\gamma),(x,\gamma),(x,\gamma),(x,\gamma))$ is continuous in the norm of the Banach space B and extends to a function $\bar{V}$ on it.
By applying theorem 3 and theorem 4 the conclusion follows. □

Remark 6. *The class of states $\phi, \psi_0 \in L^2(\mathbb{R}^d) \cap \mathcal{F}(\mathbb{R}^d)$ satisfying assumption (47) is sufficiently rich. Indeed both ϕ and ψ_0 can be chosen in two dense subsets of the Hilbert space $L^2(\mathbb{R}^d)$. More precisely one can take for instance $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$ of the form $\psi_0(x) = P(x)e^{-\alpha\frac{x\Omega^{-1}x}{2\hbar}}$, and $\phi \in \mathcal{S}(\mathbb{R}^d)$ of the form $\phi(x) = Q(x)e^{-x(\frac{\beta\Omega}{2\hbar}+i\gamma)x}$, with $\alpha, \beta, \gamma > 0$ and with P, Q arbitrary polynomials. Moreover α and β have to satisfy the following conditions, for all $i = 1, \ldots, d$:*

$$\begin{cases} \frac{1}{\beta\Omega_i} - \frac{1}{2(1-\Omega_i\tan(\Omega_i t)} > 0 \\ \frac{1}{\alpha} - \frac{1}{2}\left(\frac{\tan(\Omega_i t)+\Omega_i}{1-\Omega_i\tan(\Omega_i t)}\right) > 0) \\ \left(\frac{1-\Omega_i\tan(\Omega_i t)}{\alpha\Omega_i} - \frac{(\tan(\Omega_i t)+\Omega_i)}{2\Omega_i}\right)\left(\frac{1-\Omega_i\tan(\Omega_i t)}{\beta\omega_i} - \frac{1}{2}\right) > \left(\frac{1}{2\cos(\Omega_i t)}\right)^2 \end{cases} \tag{51}$$

Let us denote respectively by D_1 and D_2 the set of vectors ϕ and ψ_0 of the above form. It is easy to see that both D_1 and D_2 are dense in $L^2(\mathbb{R}^d)$.

The oscillatory integral (45) can heuristically be written in the following form:

$$(\phi, \psi(t)) = \text{“}\int_{\mathbb{R}^d} \bar\phi(x) \quad \int_{\{\gamma|\gamma(t)=x\}} e^{\frac{i}{\hbar}S_t(\gamma)}\psi_0(\gamma(0))D\gamma dx\text{”}$$

and interpreted as a rigorous realization of the Feynman path integral representing the inner product between the vector $\phi \in L^2(\mathbb{R}^d)$ and the solution of the Schrödinger equation (39) with initial datum ψ_0. However the infinite dimensional oscillatory integral (45) is well defined only if $\lambda \leq 0$. By the considerations in remark 5 the absolutely convergent integrals (49) and (50) are analytic functions of the complex variable λ if $Im(\lambda) > 0$, continuous in $Im(\lambda) = 0$ and coinciding with (45) if $\lambda \leq 0$. We shall prove that when $\lambda \geq 0$ the Gaussian integrals (49) and (50) represent the inner product $\langle \phi, \psi(t)\rangle$, where $\psi(t)$ is the solution of the Schrödinger equation. We will prove moreover the Borel summability of the formal Dyson expansion for $\langle \phi, \psi(t)\rangle$.

Lemma 3. *Let $\lambda = 0$, $\psi_0, \phi \in \mathcal{S}(\mathbb{R}^d)$. Let μ_0, resp μ_ϕ, be such that $\hat\mu_0 = \psi_0$, resp. $\hat\mu_\phi(x) = (2\pi i\hbar)^{d/2}e^{-\frac{i}{2\hbar}|x|^2}\bar\phi(x)$. Assume moreover that μ_0, μ_ϕ satisfy condition (47). Then the scalar product between ϕ and the solution ψ_t of the Schrödinger equation with initial datum ψ_0 is given by:*

$$\langle \phi, \psi_t\rangle = \int_{\mathbb{R}^d\times H_t} \Big(\int_{\mathbb{R}^d\times C_t} e^{ie^{i\pi/4}\sqrt{\hbar}(x\cdot y+n(\gamma)(\omega))} \\ e^{\frac{1}{2}\langle (x,\omega), B(x,\omega)\rangle} W(d\omega)\frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{d/2}}dx\Big)\mu_f(dyd\gamma) \quad (52)$$

where μ_f is the complex bounded variation measure on $\mathbb{R}^d \times H_t$ whose Fourier transform is the function $f : \mathcal{H} \to \mathbb{C}$, given by $f(x,\gamma) := (2\pi i\hbar)^{d/2}e^{-\frac{i}{2\hbar}|x|^2}\bar\phi(x)\psi(\gamma(t)+x)$ and B is the continuous extension on $\mathbb{R}^d \times C_t$ of the operator (42).

Proof: In order to avoid the use of a complicate notation we assume $d = 1$. The proof holds in a completely similar way in the case $d > 1$. As $\psi_0 \in S(\mathbb{R}^d)$, the solution of the Schrödinger equation with $\lambda = 0$, i.e. with the free Hamiltonian, and initial datum ψ_0 is given by

$$\psi(t,x) = (2\pi i\hbar)^{-1/2}\sqrt{\frac{\Omega}{\sin\Omega t}}\int_{\mathbb{R}} e^{\frac{i\Omega}{2\hbar\sin\Omega t}(\cos\Omega t(x^2+y^2)-2xy)}\psi_0(y)dy, \quad (53)$$

$t>0, x\in\mathbb{R}$, so that

$$\langle\phi,\psi_t\rangle = (2\pi i\hbar)^{-1/2}\sqrt{\frac{\Omega}{\sin\Omega t}}\int_{\mathbb{R}}\bar{\phi}(x)\int_{\mathbb{R}} e^{\frac{i\Omega}{2\hbar\sin\Omega t}(\cos\Omega t(x^2+y^2)-2xy)}\psi_0(y)dydx \tag{54}$$

Let $(2\pi i\hbar)^{1/2}e^{\frac{-i|x|^2}{2\hbar}}\bar{\phi}(x) = \int_{\mathbb{R}} e^{ik\cdot x}\mu_\phi(dk)$ and $\psi_0(y) = \int_{\mathbb{R}} e^{il\cdot y}\mu_0(dl)$, so that (54) becomes:

$$\frac{1}{\sqrt{\cos\Omega t}}\int_{\mathbb{R}}\frac{e^{-x^2/2}}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{ie^{i\pi/4}\sqrt{\hbar}xk}e^{-\frac{i\hbar\tan\Omega t l^2}{2\Omega}}e^{\frac{\Omega\tan\Omega t x^2}{2}}e^{\frac{i\sqrt{\hbar}e^{i\pi/4}xl}{\cos\Omega t}}\mu_\phi(dk)\mu_0(dl)dx.$$

A direct computation (see appendix B) shows that the latter expression is exactly equal to the integral (52), that is to

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}\Big(\int_{\mathbb{R}^d\times C_t} e^{ie^{i\pi/4}\sqrt{\hbar}(x\cdot k+x\cdot l+n(G_t^l)(\omega))} e^{\frac{1}{2}\langle(x,\omega),B(x,\omega)\rangle}W(d\omega)\frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{d/2}}dx\Big)\mu_f(dyd\gamma) \tag{55}$$

(where $G_t^l(s)=ls$ and n has been defined in section 4) and the conclusion follows. □

Remark 7. *By Fubini's theorem expression* (52) *is also equal to*

$$(i)^{d/2}\int_{\mathbb{R}^d\times C_t} e^{\frac{1}{2\hbar}\int_0^t(\sqrt{\hbar}\omega(s)+x)\Omega^2(\sqrt{\hbar}\omega(s)+x)ds} \bar{\phi}(e^{i\pi/4}x)\psi_0(e^{i\pi/4}\sqrt{\hbar}\omega(t)+e^{i\pi/4}x)W(d\omega)dx \tag{56}$$

Lemma 4. *Let $\lambda=0$ and $\psi_0\in\mathcal{S}(\mathbb{R}^d)$, such that for each $x\in\mathbb{R}^d$*

$$\int_{\mathbb{R}^d} e^{kx}e^{\frac{\hbar}{4}\langle k,\Omega^{-1}\tan\Omega t k\rangle}|\tilde{\psi}_0(k)|dk<\infty. \tag{57}$$

Then the solution ψ_t of the Schrödinger equation (39) *is an analytic function in the variable $z\in\mathbb{C}^d$ and its value in $z=e^{i\pi/4}x$, $x\in\mathbb{R}^d$ is given by:*

$$\psi_t(e^{i\pi/4}x)=\int_{C_t}\psi_0(e^{i\pi/4}x+e^{i\pi/4}\sqrt{\hbar}\omega(t))e^{\frac{1}{2\hbar}\langle(x,\sqrt{\hbar}\omega),B(x,\sqrt{\hbar}\omega)\rangle}W(d\omega)$$

Proof: In order to avoid the use of a complicate notation we assume $d=1$. The proof holds in a completely similar way in the case $d>1$.

Since $\psi_0 \in \mathcal{S}(\mathbb{R})$, $\lambda = 0$, one has (53). By Parseval equality this is also equal to

$$\psi_t(x) = \sqrt{\frac{1}{\cos \Omega t}} e^{-\frac{i\Omega \tan(\Omega t) x^2}{2\hbar}} \int_{\mathbb{R}} e^{-\frac{i\hbar \tan(\Omega t) k^2}{2\Omega}} e^{\frac{ikx}{\cos \Omega t}} \tilde{\psi}_0(k) dk$$

The analyticity of $\psi_t(z)$, $z \in \mathbb{C}$, follows by Morera and Fubini theorems. Moreover $\psi_t(e^{i\pi/4}x)$ is given by

$$\psi_t(e^{i\pi/4}x) = \sqrt{\frac{1}{\cos \Omega t}} e^{\frac{\Omega \tan(\Omega t) x^2}{2\hbar}} \int_{\mathbb{R}} e^{-\frac{i\hbar \tan(\Omega t) k^2}{2\Omega}} e^{\frac{ie^{i\pi/4} kx}{\cos \Omega t}} \tilde{\psi}_0(k) dk \qquad (58)$$

On the other hand, by Fubini's theorem (which holds thanks to the assumption (57)), one has:

$$\int_{C_t} \psi_0(e^{i\pi/4}x + e^{i\pi/4}\sqrt{\hbar}\omega(t)) e^{\frac{\Omega^2}{2\hbar}\int_0^t (\sqrt{\hbar}\omega(s)+x)^2 ds} W(d\omega) =$$
$$= \int_{\mathbb{R}} \tilde{\psi}_0(k) e^{ikxe^{i\pi/4}} e^{\frac{\Omega^2 t x^2}{2\hbar}} \int_{C_t} e^{\frac{\Omega^2}{2}\int_0^t \omega^2(s) ds} e^{\frac{\Omega^2 x}{\sqrt{\hbar}}\int_0^t \omega(s) ds} e^{ik\sqrt{\hbar} e^{i\pi/4}\omega(t)} W(d\omega) dk \qquad (59)$$

By a direct computation (see appendix B) the latter expression is equal to (58) and the conclusion follows. □

Theorem 6. *Let $\phi, \psi_0 \in \mathcal{S}(\mathbb{R}^d)$ satisfy assumption* (47). *Then the power series expansions (in powers of λ) of the expression* (50) *coincides with the Dyson expansion for the scalar product between ϕ and the solution of the Schrödinger equation* (39).

Proof: In order to avoid a complicated notation we assume $d = 1$, but the proof is valid also in the case $d > 1$.
First of all one can easily verify that expression (50) is an analytic function of the variable $\lambda \in \mathbb{C}$ in the upper halfplane $Im(\lambda) > 0$ and continuous in $\lambda \in \mathbb{R}$. By expanding it in power series of λ around $\lambda = 0$ we have for any $N \in \mathbb{N}$:

$$(i)^{d/2} \sum_{n=0}^{N-1} \frac{1}{n!}\Big(\frac{i\lambda}{\hbar}\Big)^n \int_0^t ds_1 \cdots \int_0^t ds_n \int_{\mathbb{R}\times C_t} \prod_{i=1}^n (\sqrt{\hbar}\omega(s_i) + x)^4$$
$$e^{\frac{\Omega^2}{2\hbar}\int_0^t (\sqrt{\hbar}\omega(s)+x)^2 ds} \bar{\phi}(e^{i\pi/4}x) \psi_0(e^{i\pi/4}\sqrt{\hbar}\omega(t) + e^{i\pi/4}x) W(d\omega) dx + R_N, \qquad (60)$$

with R_N a remainder term. Because of the symmetry of the integrand, (60) is equal to

$$(i)^{d/2}\sum_{n=0}^{N-1}\Big(\frac{i\lambda}{\hbar}\Big)^n\int\cdots\int_{\Delta_n}ds_1\cdots ds_n\int_{\mathbb{R}\times C_t}\prod_{i=1}^n(\sqrt{\hbar}\omega(s_i)+x)^4$$
$$e^{\frac{\Omega^2}{2\hbar}\int_0^t(\sqrt{\hbar}\omega(s)+x)^2ds}\bar{\phi}(e^{i\pi/4}x)\psi_0(e^{i\pi/4}\sqrt{\hbar}\omega(t)+e^{i\pi/4}x)W(d\omega)dx+R_N \tag{61}$$

where $\Delta_n=\{(s_1,\dots,s_n)\in[0,t]^n:0\le s_1\le\cdots\le s_n\le t\}$. The integral over $\mathbb{R}\times C_t$ can be evaluated by partitioning the interval $[0,t]$ into $n+1$ subintervals $[s_0\equiv 0,s_1]$, $[s_1,s_2]$, …, $[s_{n-1},s_n]$, $[s_n,s_{n+1}\equiv t]$. Let us denote by $\omega_i:[s_i,s_{i+1}]\to\mathbb{R}$ the Wiener process on the interval $[s_i,s_{i+1}]$, $\omega_i(s_i)=0$, by C_i the space of continuous paths on $[s_i,s_{i+1}]$ and by $\mathbb{E}_{[s_i,s_{i+1}]}$ the expectation with respect to the Wiener measure on it. With these notations expression (61) becomes

$$(i)^{d/2}\sum_{n=0}^{N-1}\Big(\frac{-i\lambda}{\hbar}\Big)^n\int\cdots\int_{\Delta_n}ds_1\cdots ds_n\int_{\mathbb{R}}dx\bar{\phi}(e^{i\pi/4}x)\mathbb{E}_{[0,s_1]}[(\sqrt{\hbar}e^{i\pi/4}\omega_0(s_1)+$$
$$+xe^{i\pi/4})^4e^{\frac{\Omega^2}{2\hbar}\int_0^{s_1}(\sqrt{\hbar}\omega_0(s)+x)^2ds}\mathbb{E}_{[s_1,s_2]}[(\sqrt{\hbar}e^{i\pi/4}\omega_1(s_2)+\sqrt{\hbar}e^{i\pi/4}\omega_0(s_1)+xe^{i\pi/4})^4$$
$$e^{\frac{\Omega^2}{2\hbar}\int_{s_1}^{s_2}(\sqrt{\hbar}\omega_1(s)++\sqrt{\hbar}\omega_0(s_1)+x)^2ds}\dots\mathbb{E}_{[s_n,t]}[e^{\frac{\Omega^2}{2\hbar}\int_{s_n}^t(\sqrt{\hbar}\omega_n(s)+\sqrt{\hbar}\sum_{i=0}^{n-1}\omega_i(s_{i+1})+x)^2ds}$$
$$\psi_0(e^{i\pi/4}\sqrt{\hbar}\sum_{i=0}^n\omega_i(s_{i+1})+e^{i\pi/4}x)]\dots]]+R_N$$

By lemma 3 and lemma 4 the latter expression is equal to

$$\sum_{n=0}^{N-1}\Big(\frac{-i\lambda}{\hbar}\Big)^n\int\cdots\int_{\Delta_n}ds_1\cdots ds_n\langle\phi,e^{-i\frac{s_1}{\hbar}H_0}Ve^{-i\frac{(s_2-s_1)}{\hbar}H_0}V\cdots$$
$$\cdots e^{-i\frac{(s_n-s_{n-1})}{\hbar}H_0}Ve^{-i\frac{(t-s_n)}{\hbar}H_0}\psi_0\rangle+R_N$$

and, by the change of variables $s_i\to t-s_{n+1-i}$, to

$$\sum_{n=0}^{N-1}\Big(\frac{-i\lambda}{\hbar}\Big)^n\int\cdots\int_{\Delta_n}ds_1\cdots ds_n\langle\phi,e^{-i\frac{(t-s_n)}{\hbar}H_0}Ve^{-i\frac{(s_n-s_{n-1})}{\hbar}H_0}V\cdots$$
$$\cdots e^{-i\frac{(s_2-s_1)}{\hbar}H_0}Ve^{-i\frac{s_1}{\hbar}H_0}\psi_0\rangle+R_N$$

where $H_0\equiv-\frac{\hbar^2}{2}\Delta+\frac{x\Omega^2x}{2}$ is the harmonic oscillator hamiltonian and $V(x)\equiv x^4$. The latter expression is Dyson's expansion for the scalar product between ϕ and the solution ψ_t of the Schrödinger equation (39) with Hamiltonian $H=H_0+\lambda V$ and the conclusion follows. □

Theorem 7. *Let $\lambda \geq 0$, and let $\phi, \psi_0 \in \mathcal{S}(\mathbb{R}^d)$ satisfy assumption* (47). *Then the scalar product between ϕ and the solution of the Schrödinger equation* (39) *with initial datum ψ_0 is given by the absolutely convergent integrals* (49) *and* (50).

Proof Let us consider the anharmonic oscillator Hamiltonian H given by (40). H is a positive selfadjoint operator and generates an analytic semigroup $T^z(t) = e^{-\frac{ztH}{\hbar}}$, $t \geq 0$, $z \in \mathbb{C}$, $Re(z) \geq 0$ (see for instance [36]). Given $t \geq 0$ and $\phi, \psi_0 \in L^2(\mathbb{R}^d)$, the function $F : \bar{D} \to \mathbb{C}$, where $D = \{z \in \mathbb{C}, Re(z) > 0\}$ and $\bar{D}$ is the closure of D,

$$F(z) \equiv \langle \phi, T^z(t)\psi_0 \rangle \tag{62}$$

is analytic in D and continuous in $\bar{D}$. If $z = i$, $F(z)$ is the scalar product between ϕ and the solution $\psi(t)$ of the Schrödinger equation (39) with initial datum ψ_0, while if $z \in \mathbb{R}^+$, $F(z)$ is the scalar product between ϕ and the solution of the heat equation

$$\frac{\partial}{\partial t}\psi = -\frac{z}{\hbar}H\psi \tag{63}$$

In this case $F(z)$ can be computed by means of the Feynman-Kac formula (see for instance [37]):

$$\begin{aligned} F(z) = &\int_{\mathbb{R}^d} \bar{\phi}(x) \int_{C_t} e^{-\frac{z}{2\hbar}\int_0^t (\sqrt{\hbar z}\omega(s)+x)\Omega^2(\sqrt{\hbar z}\omega(s)+x)ds} \\ &e^{-\frac{z\lambda}{\hbar}\int_0^t C(\sqrt{\hbar z}\omega(s)+x,\sqrt{\hbar z}\omega(s)+x,\sqrt{\hbar z}\omega(s)+x,\sqrt{\hbar z}\omega(s)+x)ds}\psi_0(\sqrt{\hbar z}\omega(t)+x)W(d\omega)dx \\ = &\, z^{d/2}\int_{\mathbb{R}^d} \bar{\phi}(\sqrt{z}x) \int_{C_t} e^{-\frac{z^2}{2\hbar}\int_0^t (\sqrt{\hbar}\omega(s)+x)\Omega^2(\sqrt{\hbar}\omega(s)+x)ds} \\ &e^{-\frac{z^3\lambda}{\hbar}\int_0^t C(\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x)ds}\psi_0(\sqrt{\hbar z}\omega(t)+\sqrt{z}x)W(d\omega)dx \end{aligned} \tag{64}$$

By the assumptions on the vectors ϕ, ψ_0, the r.h.s. of (64) makes sense for $z \in \bar{D}$. Moreover, by the analyticity of the semigroup $T^z(t)$, it represents for $z = i$ the scalar product $\langle \phi, e^{-\frac{it}{\hbar}H}\psi_0 \rangle$, that is:

$$\begin{aligned} &i^{d/2}\int_{\mathbb{R}^d} \bar{\phi}(e^{i\pi/4}x) \int_{C_t} e^{\frac{1}{2\hbar}\int_0^t (\sqrt{\hbar}\omega(s)+x)\Omega^2(\sqrt{\hbar}\omega(s)+x)ds} \\ &e^{\frac{i\lambda}{\hbar}\int_0^t C(\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x)ds}\psi_0(\sqrt{\hbar}e^{i\pi/4}\omega(t)+e^{i\pi/4}x)W(d\omega)dx \end{aligned} \tag{65}$$

This coincides with expression (50) and the conclusion follows. □

Theorem 8. *Let $\lambda \geq 0$, and let $\phi, \psi_0 \in \mathcal{S}(\mathbb{R}^d)$ satisfy assumption*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{(y+\cos[\Omega(t+t\delta)]^{-1}x)(1-(1+\delta)\Omega\tan[\Omega(t+t\delta)])^{-1}(y+\cos[\Omega(t+t\delta)]^{-1}x)} e^{\frac{\hbar}{4(1+\delta)}x\Omega^{-1}\tan[\Omega(t+t\delta)]x}|\mu_0|(dx)|\mu_\phi|(dy)<\infty \quad (66)$$

for some $\delta > 0$. Then the Dyson expansion for the scalar product between ϕ and the solution of the Schrödinger equation (39) *with initial datum ψ_0 is Borel summable.*

Proof By theorems 6 and 7 it is sufficient to show the Borel summability of the power series expansions (in powers of λ) of expression (50).
In order to avoid a complicated notation we assume $d = 1$, but the proof is valid also in the case $d \geq 1$.
As already remarked before lemma 3, the expression (50) is an analytic function of the variable $\lambda \in \mathbb{C}$ in the upper halfplane $Im(\lambda) > 0$ and continuous in $\lambda \in \mathbb{R}$. Moreover the rest R_N of its asymptotic expansion (60) is equal to:

$$R_N = \int_{\mathbb{R}^d\times H_t}\Big(\int_{\mathbb{R}\times C_t}\sum_{n=N}^{\infty}\frac{1}{n!}\Big(\frac{i\lambda}{\hbar}\Big)^n\int_0^t ds_1\cdots\int_0^t ds_n\prod_{i=1}^n(\sqrt{\hbar}\omega(s_i)+x)^4$$
$$e^{\frac{\Omega^2}{2\hbar}\int_0^t(\sqrt{\hbar}\omega(s)+x)^2ds}e^{ie^{i\pi/4}(x\cdot y+\sqrt{\hbar}n(\gamma)(\omega))}W(d\omega)\frac{e^{-\frac{|x|^2}{2\hbar}}}{(2\pi\hbar)^{d/2}}dx\Big)\mu_f(dyd\gamma)$$

R_N satisfies the following uniform estimate in $Im(\lambda) \geq 0$:

$$|R_N| = |\int_{\mathbb{R}^d\times H_t}\Big(\int_{\mathbb{R}\times C_t}\frac{1}{N-1!}\Big(\frac{i\lambda}{\hbar}\Big)^N\int_0^t ds_1\cdots\int_0^t ds_N\prod_{i=1}^N(\sqrt{\hbar}\omega(s_i)+x)^4$$
$$\int_0^1 du(1-u)^{N-1}e^{i\frac{u\lambda}{\hbar}\int_0^t(\sqrt{\hbar}\omega(s)+x)^4ds}e^{\frac{\Omega^2}{2\hbar}\int_0^t(\sqrt{\hbar}\omega(s)+x)^2ds}$$
$$e^{ie^{i\pi/4}(x\cdot y+\sqrt{\hbar}n(\gamma)(\omega))}W(d\omega)\frac{e^{-\frac{|x|^2}{2\hbar}}}{(2\pi\hbar)^{d/2}}dx\Big)\mu_f(dyd\gamma)|$$
$$\leq |\lambda|^N\hbar^N\frac{1}{N!}\int_0^t ds_1\cdots\int_0^t ds_N\int_{\mathbb{R}^d\times H_t}\int_{\mathbb{R}\times C_t}e^{\frac{\Omega^2}{2}\int_0^t(\omega(s)+x)^2ds}$$
$$\prod_{i=1}^N(\omega(s_i)+x)^4e^{-\frac{\sqrt{2}}{2}\sqrt{\hbar}(x\cdot y+n(\gamma)(\omega))}W(d\omega)\frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{d/2}}dx|\mu_f|(dyd\gamma) \quad (67)$$

By denoting G_i the vector in $C_t^* \subset H_t$ equal to $G_i(s) = 1_{[0,s_i]}s$, $(|G_i|_{H_t} = s_i)$, the gaussian integral

$$\int_{\mathbb{R}\times C_t} e^{\frac{\Omega^2}{2}\int_0^t(\omega(s)+x)^2ds}\prod_{i=1}^N(\omega(s_i)+x)^4 e^{-\frac{\sqrt{2}}{2}\sqrt{\hbar}(x\cdot y+n(\gamma)(\omega))}W(d\omega)\frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{d/2}}dx =$$

$$= \int_{\mathbb{R}\times C_t} e^{\frac{1}{2}\langle(x,\omega),B(x,\omega)\rangle}\prod_{i=1}^N(n(G_i)(\omega)+x)^4 e^{-\frac{\sqrt{2}}{2}\sqrt{\hbar}(x\cdot y+n(\gamma)(\omega))}W(d\omega)\frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{d/2}}dx$$

is equal to

$$H_{4N}\Big(i\frac{\sqrt{\hbar}}{2}(I-B)^{-1/2}(G_1,1),\dots,i\frac{\sqrt{\hbar}}{2}(I-B)^{-1/2}(G_N,1)\Big|(\sqrt{\frac{\hbar}{2}}\gamma,\sqrt{\frac{\hbar}{2}}y)\Big)\frac{e^{\frac{\hbar}{4}\langle(\gamma,y),(I-B)^{-1}(\gamma,y)\rangle}}{\sqrt{\det(I-B)}} \quad (68)$$

where

$$D_{x_1}\dots D_{x_n}e^{-x^2} = (-1)^n H_n(x_1,\dots,x_n|x)e^{-x^2}$$

By the assumption (66) on ψ_0,ϕ involving a $\delta>0$, we have

$$\int_{H_t\times R} e^{(1+\delta)\frac{\hbar}{4}\langle(\gamma,y),(I-B)^{-1}(\gamma,y)\rangle}|\mu_f|(d\gamma dy) < \infty$$

By using this and the estimate on Hermite polynomials H_n derived in [35] (formula (2.9)) we see that expression (68) is bounded by

$$ac^N\prod_{i=1}^N(s_i+1)^4\Big(\frac{1+\delta}{\delta}\Big)^{2N}2N!,$$

where $a,c>0$ are suitable constants. By inserting such an estimate into (67) and by using the identity $2N! = 2^{2N}N!(N-1/2)!/\sqrt{\pi}$, we have:

$$|R_N| \leq AC^N|\lambda|^N N!$$

This and the analyticity of (50) in $Im(\lambda)>0$, by Nevanlinna theorem [34], assure the Borel summability of asymptotic expansion (60). □

6. Concluding remarks

There are relations between our approach in the definition of the Feynman integral and those in [11, 26, 18, 33]. Indeed formula (50) often appears in the literature for a more restricted class of potentials and initial conditions. We would like however to underline that here we achieved to prove (50) and related formulae for potentials of polynomial growth. This involves our extension of the definition of infinite dimensional oscillatory integrals (in the spirit of [6, 19, 3]) to a class of phase functions much larger than the usual "quadratic + Fourier transform of measure". In [11, 26, 18, 33] the authors define the Feynman functional by means of a gaussian integral depending on a parameter (which in some cases can be identified with the mass m), prove the analyticity of such a functional in a suitable region of the complex plane and show that when it approaches the imaginary axis the corresponding functional gives a representation of the solution of the Schrödinger equation for a restricted class of potentials. In work of the euclidean approach to quantum field theory, the representation of solution of the perturbed heat equation via a Feynman-Kac formula and integrals with respect to gaussian (Wiener resp. Orstein-Uhlenbeck) measures are used to provide via an "analytic continuation in time" solutions of the Schrödinger equation. In [10] this approach provides a semiclassical expansion for the Schrödinger equation. In our case, under suitable assumptions on the initial datum ψ_0, we prove that the infinite dimensional oscillatory integral we define *coincides* with a gaussian integral. In the case of the quartic potential $V = \lambda x^4$ we prove that the gaussian integral representing the solution of the Schrödinger equation is an analytic function of the complex variable λ in the upper halfplane which coincides for $\lambda \leq 0$ with a well defined infinite dimensional oscillatory integral. We plan to use our representation for discussing rigorously asymptotic expansions in fractional powers of $\hbar$ (semiclassical expansions).

Appendix A. Proof of lemma 1

Let us study the Fourier transform of the complex-valued distribution $\frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(x)}$, $x \in \mathbb{R}^N$, where $(I-B)$ is symmetric and strictly positive, $\lambda \leq 0$ and P is given by (12):

$$\tilde{F}(k) = \int_{\mathbb{R}^N} e^{ik\cdot x}\frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(x)}d^N x$$

Without loss of generality we can assume that the quadratic form $x \cdot (I - B)x$ is equal to $x \cdot x$, as it can always be reduced to this form by a change of coordinates.

Let us compute the $N-$dimensional integral defining $\tilde{F}(k)$ by introducing the polar coordinates in $\mathbb{R}^N$:

$$\int_{\mathbb{R}^N} e^{ik\cdot x}\frac{e^{\frac{i}{2\hbar}x\cdot x}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(x)}d^Nx = \\ = \int_{S_{N-1}}\Big(\int_0^{+\infty} e^{i|k|rf(\phi_1,\dots,\phi_{N-1})}\frac{e^{\frac{i}{2\hbar}r^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(r)}r^{N-1}dr\Big)d\Omega_{N-1} \tag{69}$$

where instead of N cartesian coordinates we use $N-1$ angular coordinates $(\phi_1,\dots,\phi_{N-1})$ and the variable $r=|x|$. S_{N-1} denotes the $(N-1)$-dimensional spherical surface, $d\Omega_{N-1}$ is the Haar measure on it, $f(\phi_1,\dots,\phi_{N-1})=(k\cdot x)/|k|r$, $P(r)$ is a fourth order polynomial in the variable r with coefficients depending on the $N-1$ angular variables $(\phi_1,\dots,\phi_{N-1})$, namely:

$$P(r)=r^4A\Big(\frac{x}{|x|},\frac{x}{|x|},\frac{x}{|x|},\frac{x}{|x|}\Big)=r^4a(\phi_1,\dots,\phi_{N-1}) \tag{70}$$

where $a(\phi_1,\dots,\phi_{N-1})>0$ for all $(\phi_1,\dots,\phi_{N-1})\in S_{N-1}$. Let us focus on the integral

$$\int_0^{+\infty} e^{i|k|rf(\phi_1,\dots,\phi_{N-1})}\frac{e^{\frac{i}{2\hbar}r^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(r)}r^{N-1}dr.$$

This can be interpreted as the Fourier transform of the distribution on the real line

$$F(r)=\theta(r)r^{N-1}\frac{e^{\frac{i}{2\hbar}r^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(r)},$$

with $\theta(r)=1$ if $r\geq 0$ $\theta(r)=0$ otherwise, $\lambda<0$ and $P(r)=ar^4$, $a>0$:

$$\int_0^{+\infty} e^{ikr}\frac{e^{\frac{i}{2\hbar}r^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(r)}r^{N-1}dr. \tag{71}$$

Let us consider the complex plane and set $z=re^{i\theta}$. We have

$$\int_0^{+\infty} e^{ikr}\frac{e^{\frac{i}{2\hbar}r^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(r)}r^{N-1}dr = \\ =\lim_{\epsilon\downarrow 0}\int_{z=\rho e^{i\epsilon}} e^{ikz}\frac{e^{\frac{i}{2\hbar}z^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(z)}z^{N-1}dz \\ =\lim_{\epsilon\downarrow 0}\lim_{R\to+\infty}\int_0^R e^{ik\rho e^{i\epsilon}}\frac{e^{\frac{i}{2\hbar}\rho^2e^{2i\epsilon}}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(\rho e^{i\epsilon})}\rho^{N-1}e^{Ni\epsilon}d\rho \tag{72}$$

Given:

$$\begin{array}{llll} \gamma_1(R) = \{z \in \mathbb{C} & | & 0 \le \rho \le R, \quad \theta = \epsilon\} \\ \gamma_2(R) = \{z \in \mathbb{C} & | & \rho = R, \quad \epsilon \le \theta \le \pi/4 - \epsilon\} \\ \gamma_3(R) = \{z \in \mathbb{C} & | & 0 \le \rho \le R, \quad \theta = \pi/4 - \epsilon\} \end{array}$$

with $\epsilon > 0$ small, from the analyticity of the integrand and the Cauchy theorem we have

$$\int_{\gamma_1(R)\cup\gamma_2(R)\cup\gamma_3(R)} e^{ikz}\frac{e^{\frac{i}{2\hbar}z^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(z)}z^{N-1}dz = 0.$$

In particular:

$$\begin{aligned} &\Big|\int_{\gamma_2(R)} e^{ikz}\frac{e^{\frac{i}{2\hbar}z^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(z)}z^{N-1}dz\Big| = \\ &= R^N\Big|\int_\epsilon^{\pi/4-\epsilon} e^{ikRe^{i\theta}}\frac{e^{\frac{ie^{i2\theta}}{2\hbar}R^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(Re^{i\theta})}e^{iN\theta}d\theta\Big| \\ &\le R^N\int_\epsilon^{\pi/4-\epsilon} e^{-kR\sin(\theta)}\frac{e^{\frac{-\sin(2\theta)}{2\hbar}R^2}}{(2\pi\hbar)^{N/2}}e^{\frac{\lambda}{\hbar}(aR^4\sin(4\theta))}d\theta \\ &\le R^N\int_\epsilon^{\pi/8} e^{-k'R\theta}\frac{e^{\frac{-2\theta}{\pi\hbar}R^2}}{(2\pi\hbar)^{N/2}}e^{\frac{\lambda}{\hbar}(aR^4\frac{8}{\pi})\theta}d\theta + \\ &+ R^N e^{\frac{\lambda}{\hbar}2aR^4}\int_{\pi/8}^{\pi/4-\epsilon} e^{-k'R\theta}\frac{e^{\frac{-2\theta}{\pi\hbar}R^2}}{(2\pi\hbar)^{N/2}}e^{\frac{\lambda}{\hbar}(-aR^4\frac{8}{\pi})\theta}d\theta \\ &= \frac{R^N}{(2\pi\hbar)^{N/2}}\Big\{\Big(\frac{e^{(\frac{8a\lambda}{\pi\hbar}R^4-\frac{2}{\pi\hbar}R^2-k'R)\pi/8}-e^{(\frac{8a\lambda}{\pi\hbar}R^4-\frac{2}{\pi\hbar}R^2-k'R)\epsilon}}{(\frac{8a\lambda}{\pi\hbar}R^4-\frac{2}{\pi\hbar}R^2-k'R)}\Big) + \\ &+ \Big(\frac{e^{\frac{8\epsilon a\lambda}{\pi\hbar}R^4}e^{(-\frac{2}{\pi\hbar}R^2-k'R)(\pi/4-\epsilon)}-e^{\frac{a\lambda}{\hbar}R^4}e^{(-\frac{2}{\pi\hbar})R^2-k'R)\pi/8}}{(\frac{-8a\lambda}{\pi\hbar}R^4-\frac{2}{\pi\hbar}R^2+-k'R)}\Big)\Big\} \end{aligned} \quad (73)$$

where $k' \in \mathbb{R}$ is a suitable constant. We have used the fact that if $\alpha \in [0, \pi/2]$ then $\frac{2}{\pi}\alpha \le \sin(\alpha) \le \alpha$, while if $\alpha \in [\pi/2, \pi]$ then $\sin(\alpha) \ge 2 - \frac{2}{\pi}\alpha$. From the last line one can deduce that

$$\Big|\int_{\gamma_2(R)} e^{ikz}\frac{e^{\frac{i}{2\hbar}z^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(z)}z^{N-1}dz\Big| \to 0, \qquad R\to\infty,$$

so that

$$\int_{z=\rho e^{i\epsilon}} e^{ikz}\frac{e^{\frac{i}{2\hbar}z^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(z)}z^{N-1}dz = \int_{z=\rho e^{i(\pi/4-\epsilon)}} e^{ikz}\frac{e^{\frac{i}{2\hbar}z^2}}{(2\pi i\hbar)^{N/2}}e^{\frac{-i\lambda}{\hbar}P(z)}z^{N-1}dz$$

By taking the limit as $\epsilon \downarrow 0$ of both sides one gets:

$$\int_0^{+\infty} e^{ikr} \frac{e^{\frac{i}{2\hbar}r^2}}{(2\pi i\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}P(r)} r^{N-1} dr = \int_0^{+\infty} e^{ik\rho e^{i\pi/4}} \frac{e^{\frac{-\rho^2}{2\hbar}\rho^2}}{(2\pi\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}P(\rho e^{i\pi/4})} \rho^{N-1} d\rho \tag{74}$$

By substituting into (69) we get the final result:

$$\int_{\mathbb{R}^N} e^{ik\cdot x} \frac{e^{\frac{i}{2\hbar}x\cdot x}}{(2\pi i\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}P(x)} d^N x$$
$$= \int_{\mathbb{R}^N} e^{ie^{i\pi/4}k\cdot x} \frac{e^{\frac{-x\cdot x}{2\hbar}}}{(2\pi\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}P(xe^{i\pi/4})} d^N x$$
$$= \mathbb{E}[e^{ie^{i\pi/4}k\cdot x} e^{\frac{-i\lambda}{\hbar}P(xe^{i\pi/4})}] \tag{75}$$

Appendix B. Some explicit calculations

B.1. **The positivity of the operator $I - B$.** Let us study the spectrum of the self-adjoint operator B on $\mathcal{H}$ given by (42). In order to avoid the use of too many indexes we will assume $d = 1$, but our reasonings remain valid also in the case $d > 1$. A positive real number c_l and a vector $(x_l, \gamma_l) \in \mathcal{H}$ are respectively an eigenvalue and an eigenvector of B if and only if:

$$\begin{cases} t\Omega^2 x_l + \Omega^2 \int_0^t \gamma_l(s)ds = c_l x_l \\ \Omega^2 x_l(ts - \frac{s^2}{2}) - \int_0^s \int_t \Omega^2 \gamma_l(r)drdu = c_l \gamma_l(s) \end{cases}$$

More precisely the vector $(x_l, \gamma_l) \in \mathcal{H}$ solves the following system:

$$\begin{cases} t\Omega^2 x_l + \Omega^2 \int_0^t \gamma_l(s)ds = c_l x_l \\ c_l \ddot{\gamma}_l(s) + \Omega^2 \gamma_l(s) = -\Omega^2 x_l \\ \gamma_l(0) = 0 \\ \dot{\gamma}_l(t) = 0 \end{cases}$$

By a direct calculation one can verify that the latter system indeed admits a (unique) solution if and only if C_l satisfies the following equation

$$\frac{\Omega}{\sqrt{c_l}} \tan \frac{\Omega t}{\sqrt{c_l}} = 1$$

A graphical representation of the position of the solutions shows that the operator B is trace class. Moreover if the conditions (46) are fulfilled the maximum eigenvalue of B is strictly less than 1, so that $(I-B)$ is positive definite.

B.2. **Estimate of** $\int_{\mathbb{R}^d\times C_t} e^{\sqrt{\hbar}xy+\sqrt{\hbar}n(\eta)(\omega)}e^{\frac{1}{2}\langle (x,\omega),B(x,\omega)\rangle}N(dx)W(d\omega)$**.** Let us consider the following function $F:\mathcal{H}\to\mathbb{C}$ given by

$$F(y,\eta)=\int_{\mathbb{R}^d\times C_t} e^{\sqrt{\hbar}xy+\sqrt{\hbar}n(\eta)(\omega)}e^{\frac{1}{2}\langle (x,\omega),B(x,\omega)\rangle}N(dx)W(d\omega).$$

Let us assume Ω, t satisfy assumption (46). By a direct computation and by Fubini theorem, F is equal to

$$\begin{aligned}F(y,\eta)&=(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{\sqrt{\hbar}xy}e^{-x\frac{(I-t\Omega^2)}{2}x}\int_{C_t} e^{\sqrt{\hbar}n(\eta)(\omega)}e^{x\int_0^t\Omega^2\omega(s)ds}\\&\qquad e^{\frac{1}{2}\int_0^t\omega(s)\Omega^2\omega(s)ds}W(d\omega)dx\\&=(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{\sqrt{\hbar}xy}e^{-x\frac{(I-t\Omega^2)}{2}x}\int_{C_t} e^{\sqrt{\hbar}n(\eta)(\omega)}e^{n(v_x)(\omega)}e^{\frac{1}{2}\langle\omega,L\omega}W(d\omega)dx,\end{aligned}\tag{76}$$

where $L:H_t\to H_t$ is the operator given by

$$L\gamma(s)=-\int_0^s\int_t^{s'}\Omega^2\gamma(s'')ds''ds'$$

and $v_x\in H_t$ is the vector given by $v_x(s)=\Omega^2x(ts-\frac{s^2}{2})$. One can easily verify that L is symmetric and trace class. Indeed by denoting by α^2,γ respectively the eigenvalues and the eigenvectors of the operator L, we have

$$-\Omega^2\ddot{\gamma}(s)=\alpha^2\gamma(s),\qquad \gamma(0)=0,\dot{\gamma}(t)=0$$

Without loss of generality we can assume Ω^2 is diagonal with eigenvalues Ω_i^2, $i=1,\dots d$. The components γ_i, $i=1,\dots d$, of the eigenvector γ corresponding to the eigenvalue α^2 are equal to

$$\gamma_i(s)=A_i\sin\frac{\Omega_i s}{\alpha}.$$

By imposing the condition $\dot{\gamma}(t)=0$, we have $\Omega_it/\alpha=\pi/2+n_i\pi$, $n_i\in\mathbb{Z}$. The possible α^2 are of the form $\alpha^2=\Omega_i^2t^2/(n_i+\frac{1}{2})^2\pi^2$. It follows that the operator $I-L$ is positive definite if and only if $\Omega_it<\pi/2$ for all $i=1,\dots d$. Moreover the Fredholm determinant of L can easily be computed by means of the equality $\cos x=\prod(1-\frac{x^2}{\pi^2(n+1/2)^2})$ and it is equal to $\det\cos\Omega t$.

By the considerations in section 4 the function $G:\mathbb{R}^d\to\mathbb{R}$ given by

$$G(x)=\int_{C_t} e^{\sqrt{\hbar}n(\eta)(\omega)+n(v_x)(\omega)}e^{\frac{1}{2}\langle\omega,L\omega\rangle}W(d\omega)\tag{77}$$

is equal to

$$\frac{1}{\sqrt{\det\cos\Omega t}}e^{\frac{1}{2}\langle\sqrt{\hbar}\eta+v_x,(I-L)^{-1}(\sqrt{\hbar}\eta+v_x)\rangle}$$

where $(I-L)^{-1}$ is given by

$$(I-L)^{-1}\gamma(s)=\Omega^{-1}[\int_0^s \sin[\Omega(s-s')]\ddot{\gamma}(s')ds'+\sin(\Omega s)\dot{\gamma}(0)]+$$
$$+\sin(\Omega s)[\cos(\Omega t)]^{-1}\int_0^t \sin[\Omega(t-s')]\dot{\gamma}(s')ds' \quad (78)$$

Moreover by direct computation we see that

$$G(x)=\frac{1}{\sqrt{\det\cos\Omega t}}e^{\frac{1}{2}\langle\sqrt{\hbar}\eta,(I-L)^{-1}\sqrt{\hbar}\eta\rangle}e^{\frac{1}{2}x(-t\Omega^2+\Omega\tan\Omega t)x}e^{\langle v_x,(I-L)^{-1}\sqrt{\hbar}\eta\rangle} \quad (79)$$

By inserting this into (76), we have

$$F(y,\eta)=\frac{(2\pi)^{-d/2}}{\sqrt{\det\cos\Omega t}}e^{\frac{1}{2}\langle\sqrt{\hbar}\eta,(I-L)^{-1}\sqrt{\hbar}\eta\rangle}\int_{\mathbb{R}^d}e^{\sqrt{\hbar}xy}e^{-\frac{1}{2}x(I-\Omega\tan\Omega t)x}e^{\langle v_x,(I-L)^{-1}\sqrt{\hbar}\eta\rangle}dx$$

In particular by taking $\eta=\beta G_t$, $\beta\in\mathbb{C}$, $G_t\in H_t$, $G_t(s)=zs$, $z\in\mathbb{R}^d$ we get

$$F(y,\eta)=\frac{e^{\frac{\hbar\beta^2}{2}z(\Omega^{-1}tan\Omega t)z}}{\sqrt{\det(\cos\Omega t-\Omega\sin\Omega t)}}$$
$$e^{\frac{\hbar}{2}(y+\beta\cos\Omega t^{-1}(1-\cos\Omega t)z)(1-\Omega\tan\Omega t)^{-1}(y+\beta\cos\Omega t^{-1}(1-\cos\Omega t)z)}. \quad (80)$$

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