CYCLE COVER PROPERTY AND CPP=SCC PROPERTY ARE NOT EQUIVALENT

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Abstract

Let G be an undirected graph. The Chinese Postman Problem (CPP) asks for a shortest postman tour in G, i.e. a closed walk using each edge at least once. The Shortest Cycle Cover Problem (SCC) asks for a family C of circuits of G such that each edge is in some circuit of C and the total length of all circuits in C is as small as possible. Clearly, an optimal solution of CPP can not be greater than a solution of SCC. A graph G has the CPP = SCC property when the solutions to the two problems have the same value.

Graph G is said to have the cycle cover property if for every Eulerian 1,2-weighting w : E(G) → {1,2} there exists a family C of circuits of G such that every edge e is in precisely w_e circuits of C. The cycle cover property implies the CPP = SCC property.

We give a counterexample to a conjecture of Zhang [8, 9, 2, 10] stating the equivalence of the cycle cover property and the CPP = SCC property for 3-connected graphs. This is also a counterexample to the stronger conjecture of Lai and Zhang, stating that every 3-connected graph with the CPP = SCC property has a nowhere-zero 4-flow. We actually obtain infinitely many cyclically 4-connected counterexamples to both conjectures.

Key words: cycle cover, faithful cover, Petersen graph, 4-flow, counterexample.

1 Introduction

Let G = (V,E) be an undirected graph, possibly with parallel edges. A postman tour (Euler tour) in G is a closed walk using each edge at least (exactly) once. The Chinese Postman Problem (CPP) asks for a shortest postman tour in G. We denote by V_o(G) the set of nodes with odd degree in G. Mei Gu Guan [4] observed that CPP is equivalent to the problem of finding a minimum V_o(G)-join in G, i.e. a subgraph J of G with V_o(J) = V_o(G), since the graph obtained by G duplicating the edges in J will be Eulerian, hence will admit an Euler tour. The first to efficiently solve CPP were Edmonds and Johnson [3]. (See [1] for a simpler method inspired by results of Sebő [7]).

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A cycle is a closed walk $C$ where repetition of nodes is forbidden. Denote by $|C|$ the length of $C$, i.e. the number of nodes in $C$. The Shortest Cycle Cover Problem (SCC) asks for a family $\mathcal{C}$ of cycles of $G$ with $\sum_{C \in \mathcal{C}} |C|$ as small as possible and such that each edge of $G$ is in some cycle of $\mathcal{C}$. An optimal solution of $CPP$ can not be greater than a solution of SCC, since, when $G$ is connected, it is always possible to read out a postman tour of $G$ from a cycle cover of $G$. A graph $G$ has the $CPP = SCC$ property when the solutions to the two problems have the same value. A well known graph without the $CPP = SCC$ property is the Petersen graph $\mathcal{P}$, shown in Figure 1 on the left. Indeed, the 1-factors of $\mathcal{P}$ are the minimum $V_0(\mathcal{P})$-joins in $\mathcal{P}$ and, since they are all isomorphic, we essentially have to consider only the 1-factor shown in Figure 1 in the middle. To do so, just check that the edge weighting shown in Figure 1 on the right is bad in the sense that no family $\mathcal{C}$ of cycles exists in $\mathcal{P}$ such that every edge is taken precisely the indicated number of times.

**LEFT:** The Petersen graph.   **MIDDLE:** A 1-factor.   **RIGHT:** A bad weighting.

![Figure 1: The Petersen graph does not have the $CPP = SCC$ property.](image)

A weight function $w : E(G) \mapsto \{1, 2\}$ is called Eulerian if $\sum_{e \in \delta(S)} w_e$ is even for every cut $\delta(S)$ of $G$. Denote by $\mathcal{W}_G$ the set of all Eulerian weight functions for $G$. A $w \in \mathcal{W}_G$ is said to be bad when there exists no family $\mathcal{C}$ of cycles of $G$ such that each edge $e$ of $G$ is in precisely $w_e$ cycles of $\mathcal{C}$. When no $w \in \mathcal{W}_G$ is bad then $G$ is said to have the cycle cover property. Note that the cycle cover property implies the $CPP = SCC$ property.

In Section 2, we give a counterexample to the following conjecture of Zhang [8, 9, 2, 10].

**Conjecture 1** The cycle cover property and the $CPP = SCC$ property are equivalent for 3-connected graphs.

This will also be a counterexample to the stronger conjecture of Lai and Zhang stating that every 3-connected graph with the $CPP = SCC$ property has a nowhere-zero 4-flow. In Section 3, we derive infinitely many cyclically 4-connected counterexamples to both conjectures. Since the cycle cover property implies the $CPP = SCC$ property, the following conjecture of Jackson [6] would eventually come into play when one is willing to consider graphs with higher connectivity.

**Conjecture 2** The Petersen graph is the only cyclically 5-connected cubic graph without the cycle cover property.
2 A first counterexample

In Figure 2, a first counterexample to Conjecture 1 is given.

**LEFT:** A counterexample. **MIDDLE:** $CPP = SCC$. **RIGHT:** A bad $w \in \mathcal{W}_G$.

![Graph with $CPP = SCC$ property](image)

Figure 2: A graph $G$ with the $CPP = SCC$ property but without the cycle cover property.

Graph $G$, given in Figure 2 on the left, is indeed 3-connected. Let $\mathcal{C}$ be the family of cycles shown in Figure 2 in the middle. Every edge of $G$ belongs to either 1 or 2 of the cycles in $\mathcal{C}$. Moreover, the edges of $G$ belonging to 2 cycles in $\mathcal{C}$ give a 1-factor of $G$ and hence a minimum $V_0(G)$-join of $G$. Hence $G$ has the $CPP = SCC$ property. Consider now the weighting $w$ indicated in Figure 2 on the right. Note that $w \in \mathcal{W}_G$. We will show that $w$ is bad, hence $G$ does not have the cycle cover property. Assume on the contrary that there exists a family of cycles $\mathcal{C}$ such that every edge $e$ is in precisely $w_e$ cycles of $\mathcal{C}$. Let $e, f, g$ be the three edges of $G$ indicated in Figure 2 on the left. Let $C_1$ and $C_2$ be the two cycles of $\mathcal{C}$ containing $f$. We can assume w.l.o.g. that $e$ belongs to $C_1$ and $g$ belongs to $C_2$. Let $G_A$ and $G_B$ be the two connected components of $G \setminus \{e, f, g\}$. Now $\mathcal{C} \setminus \{C_1, C_2\}$ can be partitioned into $\mathcal{C}_A$ and $\mathcal{C}_B$, where $\mathcal{C}_A$ is the set of those cycles in $\mathcal{C}$ which are cycles of $G_A$ and $\mathcal{C}_B$ is the set of those cycles in $\mathcal{C}$ which are cycles of $G_B$. Consider the Petersen graph $\mathcal{P}$ obtained from $G$ by identifying all nodes in $V(G_B)$ into a single node. Here $\mathcal{C}_A \cup \{C_1 \setminus E(G_B), C_2 \setminus E(G_B)\}$ would be a cycle cover of $\mathcal{P}$ contradicting the fact that the edge weighting shown in Figure 1 on the right is bad for $\mathcal{P}$.

3 Infinitely many cyclically 4-connected counterexamples

Although the original conjectures were about 3-connected graphs, it is now pertinent to investigate what happens for higher connectivity values. In this section, we show that infinitely many cyclically 4-connected counterexamples exist. To do so, we consider an operation that merges two cubic graphs, endowed by Eulerian 1,2-weightings, into a single cubic graph, endowed by a corresponding Eulerian 1,2-weighting. This operation is called dot product, since it is a natural extension of the celebrated operation introduced by Isaacs in [5] to generate new snarks by combining old ones.
We are given two pairs \((G_1, w_1)\) and \((G_2, w_2)\), with \(G_i\) cubic and \(w_i \in W_{G_i}\), for \(i = 1, 2\). Let \(hk\) and \(xy\) be two edges of \(G_1\) and assume \(w_1(hk) = w_1(xy) = 1\). Let \(uv, uu_A, uu_B, vv_A, vv_B\), be edges of \(G_2\) and assume \(w_2(uv) = 2\), whereas \(w_2(uu_A) = w_2(uu_B) = w_2(vv_A) = w_2(vv_B) = 1\). Then the dot product \((G_1, w_1) \cdot (G_2, w_2)\) is the pair \((G, w)\) obtained from \((G_1, w_1)\) and \((G_2, w_2)\) by removing nodes \(u\) and \(v\) and removing edges \(hk, xy, uw, uu_A, uu_B, vv_A, \) and \(vv_B\) and adding edges \(u_Ax, u_By, v_Ay\) and \(v_Bk\) with \(w(u_Ax) = w(u_By) = w(v_Ay) = w(v_Bk) = 1\). Every other edge \(e\) of \(G\) either belongs to \(G_1\) or to \(G_2\) and we set \(w(e) = w_1(e)\) or \(w(e) = w_2(e)\), accordingly. The operation is shown in Figure 3 and had been introduced by Jackson in [6] for the special case when the edges of weight 2 form a 1-factor. In [6], the following lemma had also been given.

**Lemma 3** If \(w_1 \in W_{G_1}\) and \(w_2 \in W_{G_2}\) are bad, and \((G, w) = (G_1, w_1) \cdot (G_2, w_2)\), then \(w\) is bad for \(G\).

**Proof:** Assume \(w\) is not bad for \(G\). Let \(C\) be a family of cycles of \(G\) such that each edge \(e\) of \(G\) is in precisely \(w_e\) cycles of \(C\). Let \(C\) be the unique cycle in \(C\) containing edge \(u_Ax\). If \(C\) contains also edge \(u_By\) then we have a contradiction with the fact that \(w_1\) was bad for \(G_1\). Otherwise we have a contradiction with the fact that \(w_2\) was bad for \(G_2\). \(\square\)

Let \(G\) be a cubic graph with the \(CPP = SCC\) property but without the cycle cover property. If \(G\) is 3-connected, then \(G\) is bridgeless and hence, by Petersen’s theorem, \(G\) has a 1-factor. Therefore, when \(\mathcal{C}\) is a shortest cycle cover of \(G\), and since \(G\) has the \(CPP = SCC\) property, then the edges of \(G\) which are contained in two cycles of \(\mathcal{C}\) form a 1-factor of \(G\), denoted by \(F_G(\mathcal{C})\). Let \(hk\) and \(xy\) be any two edges of \(G\). Graph \(G\) is called an \(hk, xy\)-counterexample if there exists a shortest cycle cover \(\mathcal{C}\) of \(G\) with \(hk, xy \notin F_G(\mathcal{C})\) and a bad \(\bar{w}_G \in W_G\) with \(\bar{w}(hk) = \bar{w}(xy) = 1\). Note that the graph \(G\) given in Figure 2 is an \(hk, xy\)-counterexample. Denote by \(\bar{w}_P\) the bad weighting of \(P\) given in Figure 1 on the right. When \(G\) is an \(hk, xy\)-counterexample, then in the dot product \((H, w_H) = (G, \bar{w}_G) \cdot (P, \bar{w}_P)\), graph \(H\) has the \(CPP = SCC\) property, as shown in Figure 4. Moreover, by Lemma 3, \(w_H\) is a bad weighting for \(H\). Hence, \(H\) too is a cubic graph with the \(CPP = SCC\) property but without the cycle cover property. Moreover many choices for \(hk\) and \(xy\) are possible in \(H\) so that \(H\) is actually an \(hk, xy\)-counterexample. (One such choice is indicated in Figure 4). This means that the above operation can be repeated indefinitely many times, and in several ways.
For the graph $\mathcal{G}$, the choice of $hk$ and $xy$ indicated in Figure 2 was particularly fortunate: under this choice, the graph $\mathcal{H} = \mathcal{G} \cdot \mathcal{P}$, also displayed in Figure 4, is cyclically 4-connected. Finally, the property of being cyclically 4-connected is maintained when further dot product operations are performed.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{A cyclically 4-connected graph with the $CPP = SCC$ but without the cycle cover property.}
\end{figure}

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