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Rigorous construction and Hadamard property of the Unruh state in the full Schwarzschild spacetime

Claudio Dappiaggi[1,3,a], **Valter Moretti**[2,3,b], **Nicola Pinamonti**[1,3,c]

[1] II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, D-22761 Hamburg, Germany.

[2] Dipartimento di Matematica, Università di Trento and Istituto Nazionale di Fisica Nucleare – Gruppo Collegato di Trento, via Sommarive 14 I-38050 Povo (TN), Italy.

[3] Istituto Nazionale di Alta Matematica "F.Severi"– GNFM

E-mail: [a]claudio.dappiaggi@desy.de, [b]moretti@science.unitn.it, [c]nicola.pinamonti@desy.de

Abstract. The discovery of the radiation properties of black holes prompted the search for a natural candidate quantum ground state for a massless scalar field theory on the Schwarzschild spacetime. Among the several available proposals in the literature, an important physical role is played by the so-called Unruh state which is supposed to be appropriate to capture the physics of a real spherically symmetric black hole formed by collapsing matter. One of the aims of this paper, referring to a massless Klein-Gordon field, is to rigorously construct that state globally, i.e. on the algebra of Weyl observables localized in the union of the static external region, the future event horizon and the non-static black hole region. The Unruh state is constructed following the traditional recipe that it is the vacuum state with respect to the affine parameter U of the geodesic forming the whole past horizon whereas it is the vacum state with respect to the Schwarzschild Killing time t on the past light infinity, interpreting these data within our algebraic formalism. Eventually, making use of the microlocal-analysis approach, we prove that the Unruh state built up following our procedure fulfills the so-called Hadamard condition everywhere it is defined and, hence, it is perturbatively stable, realizing the natural candidate with which one could study purely quantum phenomena such as the role of the back reaction of Hawking's radiation.
The achieved results are obtained by means of a bulk-to-boundary reconstruction technique which exploits the Killing (horizon) structure and the conformal asymptotic structure of the underlying background, employing Hörmander's theorem on propagation of singularities, some recent results about passive state extended to our case, and a careful analysis of the remaining part of the wavefront set of the state. A crucial technical role is played by the recent results due to Dafermos and Rodnianski on the peeling behaveour of the solutions of Klein-Gordon equations in Schwarzschild spacetime.

Contents

1 Introduction

In the wake of Hawking's discovery of the radiating properties of black holes [Haw74], several investigations on the assumptions leading to such result were prompted. In between them, that of Unruh [Un76] caught the attention of the scientific community, since he first emphasized the need to identify a physically sensible candidate quantum state which could be called the vacuum for a quantum massless scalar field theory on the Schwarzschild spacetime. Especially when viewing it as the spacetime of a real black hole obtained by spherically symmetric collapsed matter. This is in counterposition with the spacetime of the, less physical, eternal black hole, described by the whole Kruskal manifold. In view of Birkhoff's theorem (outside the collapsing matter) the metric of the black hole is identical of that of Schwarzschild, both for the internal and the external region, including the future event horizon. Adopting the standard notation (e.g. see [Wa94]), this spacetime can be identified with the union of regions I and III in Kruskal manifold including the future horizon, but omitting the the remaining two regions with their boundaries [Wa84, Wa94]. To the date, in the literature, three candidate background states are available, going under the name of *Boulware* (for the external region), *Hartle-Hawking* (for the complete Kruskal manifold) and *Unruh* state (for the union of the external and black hole region, including the future event horizon). The goal of this paper is to focus on the latter, mostly due to its remarkable physical properties. As a matter of fact, earlier works (see for example [Ca80, Ba84, Ba01]) showed that such a state could be employed to compute the expectation value of the regularized stress-energy tensor for a massless scalar field in the physical region of Schwarzschild spacetime pointed out above. The outcome turns out to be a regular expression on the future event horizon and it corresponds at future null infinity to an outgoing flux of radiation compatible with that of a blackbody at the black hole temperature. As preannounced, this result, together with Birkhoff's theorem, lead to the conjecture that the very same Unruh state, say ω_U, is the natural candidate to be used in the description of the gravitational collapse of a spherically symmetric star. However, to this avail one is also led to assume that ω_U fulfils the so-called *Hadamard*

property [KW91, Wa94], a natural prerequisite for states on curved background to be indicated as natural ground ones, which assures the existence of a well behaved averaged stress energy tensor [Wa94]. From a heuristic point of view, this condition is tantamount to require that the ultraviolet behaviour mimics that of Minkowski vacuum, yielding to a physically clear prescription to remove the singularites of the averaged stress-energy tensor in order to compute the back reaction of the quantum matter on the gravitational background through Einstein equations. Furthermore, in the specific scenario we are considering, the relevance of the Hadamard condition is further borne out by the analysis in [FH90]. There, assuming the existence of suitable algebraic states of Hadamard form, it is shown that the Hawking radiation brought by the state at large times is precisely related to a scaling limit of the two-point function of the state on the 2-sphere where the star radius crosses the Schwarzschild radius.

It is therefore manifest the utmost importance to verify whether ω_U satisfies or not the Hadamard property as it might appear to be reasonable, at least in the static region of Schwarzschild spacetime (outside the event future horizon and the internal region) also in view both of the former analysis in [Ca80] and of the general results achieved in [SV00] applied to some of the results presented in [DK86-87]. Indeed this check is one of the main purposes to write this paper. However our aim is broader, as we shall make a novel use of the Killing and conformal structure of Schwarzschild spacetime in order to construct rigorously and unambiguously the Unruh state, contemporary in the static region, inside the internal region and on the future event horizon. To this avail, we shall exploit techniques which in the recent past have been successfully applied to manifold with Killing horizons, asymptotically flat spacetimes and cosmological backgrounds [MP05, DMP06, Mo06, Da08a, Da08b, Mo08, DMP09a, DMP09b]. Though for different physical goals, a mathematically similar technology was employed in [Ho00] including a proof of the Hadamard property of the relevant states.

The general approach of [DMP06, Mo06, Da08a, Da08b, Mo08, DMP09a, DMP09b] can be summarized as follows. First one defines a state on the algebra of observables of a certain codimension 1 null submanifolds of the background, one is interested in. Then, a corresponding distinguished state is induced on the algebra of observables of the bulk through a certain pull-back procedure. As showed in the mentioned references, the induced state enjoys several important physical properties, related with the symmetries of the spacetime or with uniqueness and energy positivity. In particular, at first glance, the Hadamard property seems to be satisfied automatically as a consequence of the very construction and known results of microlocal analysis about composition of wave front sets. Actually this feature has to be verified case by case since it depends on subtle geometrical details. As a matter of fact, the proofs of the Hadamard property presented in the mentioned literature differ to each other in relation with the effective possibility of exploiting some result of microlocal analysis depending on the details of the geometry of the spacetime. Here we shall use an even different procedure as outlined below.

In our scenario the role of the relevant null hypersurfaces of the spacetime will be played by the union of the complete Killing past horizon and null past infinity. The state on the algebra referred to this null manifold will be then defined following just the original recipe by Unruh: a vacuum state on the horizon, defined with respect to the affine parameter of null geodesics forming the horizon and a vacuum state with respect to the Schwarzshild Killing vector ∂_t at past null infinity. The two-point function of the part of the state on the horizon, when restricting to the subalgebra smeared by compactly supported functions, takes a distinguished shape already noticed in [Sw82, DK86-87, KW91]. The most difficult effort at this step is to prove that this state extends to the full algebra of the horizon, since the full algebra is constrained by recent achievements by Dafermos and Rodnianski mentioned below. A similar problem will be tackled concerning the state defined on the algebra at the null infinity. However both problems will be solved and the full proceure will be implemented defining the Unruh state, ω_U, in the spacetime.

The net advantage our approach will be the possibility to present a global definition for the spacetime

including the future horizon, the external and the internal region contemporarily. On the other hand, our approach will allow us to avoid most of the technical cumbersomeness, encountered in the earlier approaches. The most remarkable being [DK86-87] (see also [Ka85a]), where the Unruh state was defined via S-matrix out of the solutions of the corresponding field equation of motion in asymptotic Minkowski spacetimes, but the definition was established only for the static region and the Hadamard condition was not checked.
Differently, our boundary-to-bulk construction, as preannounced, will allow us to make a full use of the powerful techniques of microlocal analysis, thus leading to a verification of the Hadamard condition using the global microlocal characterisation discovered by Radzikowski [Ra96a, Ra96b] and fruitfully exploited in all the subsequent literature. Differently from the proofs of the Hadamard property presented in [Mo08] and [DMP09b] here we shall adopt a more indirect procedure (also to avoid complicated issues related to null geodesics reaching i^+ from the interior of the Schwarzschild region). The Hadamard property will be first established in the static region making use of an extension of the formalism and the results presented in [SV00] for *passive states*. The black hole region together with the future horizon will be finally encompassed by a profitable use of celebrated Hörmander's propagation of singularity theorem joined with a direct computation of the relevant remaining part of wavefront set of the involved distributions, all in view of well-established results of microlocal analysis.
¿From a mathematical point of view, it is certainly worth acknowledging that the results we present in this paper are obtainable thanks to several remarkable achievements presented in a recent series of papers due to Dafermos and Rodnianski [DR05, DR07, DR08, DR09], who discussed in great details the behaviour of a solution φ of the Klein-Gordon equation in Schwarzschild spacetime improving a classical result of Kay and Wald [KW87]. Particularly they proved peeling estimates for φ both on the horizons and at null infinity, thus proving the long-standing conjecture known as Price law [DR05]. As a byproduct, the very same results will be here used to guarantee that the behaviour of the wave front set of ω_U is of an Hadamard form.

In detail, the paper will be divided as follows.
In section 2.1, we recall the geometric properties of Schwarzschild spherically symmetric solution of Einstein's equations. Particularly, we shall introduce, characterise and discuss all the different regions of the background which will play a distinguished role in the paper.
Subsequently, in section 2.2 and 2.3, we will define the relevant Weyl C^*-algebras of observables respectively in the bulk and in the codimension 1 submanifolds, we are interested in, namely the past horizon and null infinity.
Eventually, in section 2.4, we shall relate bulk and boundary data by means of an certain isometric $*$-homomorphism which will be proved to exist.
Section 3 will be instead devoted to a detailed discussion on the relation between bulk and boundary states. In particular we will focus on the state defined by Kay and Wald for a (smaller) algebra associated with the past horizon $\mathcal{H}$ [KW91], showing that that state che be extended to the (larger) algebra relevant for our purposes.
The core of our results will be in section 4 where we shall first define the Unruh state and, then, we will prove that it fulfils the Hadamard property employing techniques proper of microlocal analysis. Eventually we draw some conclusions.
Appendix A contains further geometric details on the conformal structure of Schwarzschild spacetime, while Appendix C encompasses the proofs of most propositions. At the same time Appendix B is noteworthy because it summarises several different definitions of the KMS condition and their relation is briefly sketched.

1.1. *Notation, mathematical conventions.*
Throughout, $A \subset B$ (or $A \supset B$) includes the case $A = B$, moreover $\mathbb{R}_+ \doteq [0, +\infty)$, $\mathbb{R}_+^* \doteq (0, +\infty)$, $\mathbb{R}_- \doteq (-\infty, 0]$, $\mathbb{R}_-^* \doteq (-\infty, 0)$ and $\mathbb{N} \doteq \{1, 2, \ldots\}$. For smooth manifolds $\mathscr{M}, \mathscr{N}$, $C^\infty(\mathscr{M}; \mathscr{N})$ is the space of smooth functions $f : \mathscr{M} \to \mathscr{N}$. $C_0^\infty(\mathscr{M}; \mathscr{N}) \subset C^\infty(\mathscr{M}; \mathscr{N})$ is the subspace of compactly-supported functions. If $\chi : \mathscr{M} \to \mathscr{N}$ is a diffeomorphism, χ^* is the natural extension to tensor bundles (counter-, co-variant and mixed) from $\mathscr{M}$ to $\mathscr{N}$ (Appendix C in [Wa84]). A **spacetime** $(\mathscr{M}, g)$ is a Hausdorff, second-countable, smooth, four-dimensional connected manifold $\mathscr{M}$, whose smooth metric has signature $- + ++$. We shall also assume that a spacetime is *oriented* and *time oriented*. The symbol $\Box_g$ denotes the standard **D'Alembert operator** associated with the unique metric, torsion free, affine connection $\nabla_{(g)}$ constructed out of the metric g. $\Box_g$ is locally individuated by $g_{ab}\nabla^a_{(g)}\nabla^b_{(g)}$. We adopt definitions and results about causal structures as in [Wa84, O'N83], but we take recent results [BS03-05, BS06] into account, too. If $(\mathscr{M}, g)$ and $(\mathscr{M}', g')$ are spacetimes and $S \subset \mathscr{M} \cap \mathscr{M}'$, then $J^\pm(S; \mathscr{M})$ $(I^\pm(S; \mathscr{M}))$ and $J^\pm(S; \mathscr{M}')$ $(I^\pm(S; \mathscr{M}'))$ indicate the **causal** (resp. **chronological**) **sets** generated by S in the spacetime $\mathscr{M}$ or $\mathscr{M}'$, respectively. An (anti)symmetric bilinear map over a real vector space $\sigma : V \times V \to \mathbb{R}$ is **nondegenerate** when $\sigma(u, v) = 0$ for all $v \in V$ entails $u = 0$.

2 Quantum Field theories - bulk to boundary relations

2.1. *Schwarzschild-Kruskal spacetime.*
In this paper we will be interested in the analysis of a Klein-Gordon scalar massless field theory on Schwarzschild spacetime and, therefore, we shall first recall the main geometric properties of the background we shall work with. Within this respect, we shall follow section 6.4 of [Wa84] and we will focus on the *physical region* $\mathscr{M}$ of the full Kruskal manifold $\mathscr{K}$ (represented in figure 2 in the appendix), associated with a black hole of mass $m > 0$.
$\mathscr{M}$ is made of the union of three pairwisely disjoint parts, $\mathscr{W}, \mathscr{B}, \mathcal{H}_{ev}$ we go to describe. According to figure 1 (and figure 2 in the appendix), we individuate $\mathscr{W}$ as the (open) **Schwarzschild wedge**, the (open) **black hole** region is denoted by $\mathscr{B}$ and their common boundary, the **event horizon** is indicated by $\mathcal{H}_{ev}$.
The underlying metric is easily described if we make use of the standard **Schwarzschild coordinates** t, r, θ, ϕ, where $t \in \mathbb{R}$, $r \in (r_S, +\infty)$, $(\theta, \phi) \in \mathbb{S}^2$ in $\mathscr{W}$, whereas $t \in \mathbb{R}$, $r \in (0, r_S)$, $(\theta, \phi) \in \mathbb{S}^2$ in $\mathscr{B}$. Within this respect the metric in both $\mathscr{W}$ and $\mathscr{B}$ assumes the standard Schwarzschild form:

$$-\left(1 - \frac{2m}{r}\right) dt \otimes dt + \left(1 - \frac{2m}{r}\right)^{-1} dr \otimes dr + r^2 h_{\mathbb{S}^2}(\theta, \phi) \,, \tag{1}$$

where $h_{\mathbb{S}^2}$ is the standard metric on the unit 2-sphere. Here, per direct inspection, one can recognize that the locus $r = 0$ corresponds to proper metrical singularity of this spacetime, whereas $r = r_S = 2m$ individuates the apparent singularity on the event horizon.
It is also convenient to work with the **Schwarzschild light** or **Eddington-Finkelstein coordinates** [KW91, Wa94] u, v, θ, ϕ which cover $\mathscr{W}$ and $\mathscr{B}$ separately, such that $(u, v) \in \mathbb{R}^2$, $(\theta, \phi) \in \mathbb{S}^2$ and

$$u \doteq t - r^* \text{ in } \mathscr{W}, \quad u \doteq -t - r^* \text{ in } \mathscr{B}, \tag{2}$$

$$v \doteq t + r^* \text{ in } \mathscr{W}, \quad v \doteq t - r^* \text{ in } \mathscr{B}, \tag{3}$$

$$r^* \doteq r + 2m \ln \left| \tfrac{r}{2m} - 1 \right| \in \mathbb{R} \,. \tag{4}$$

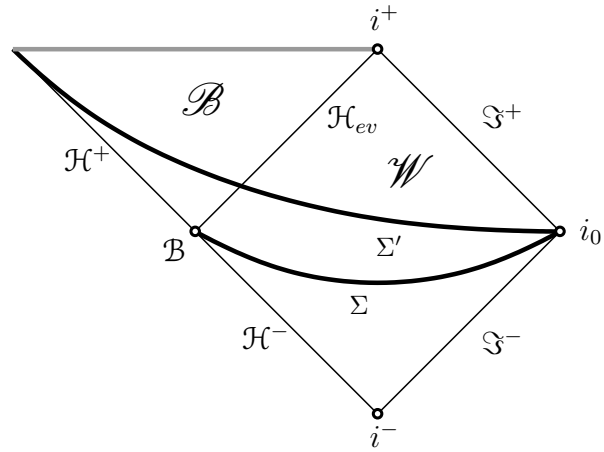

Figure 1: The overall picture represents $\mathscr{M}$. The regions $\mathscr{W}$ and $\mathscr{B}$ respectively correspond to regions I and III in fig 2. The thick horizontal line denotes the metric singularity at $r = 0$, Σ is a spacelike Cauchy surface for $\mathscr{M}$ while Σ' is a spacelike Cauchy surface for $\mathscr{W}$.

A third convenient set of **global null coordinates** U, V, θ, ϕ can be introduced on the whole Kruskal spacetime [Wa84], such that, in particular:

$$U = -e^{-u/(4m)}\,, \quad V = e^{v/(4m)} \quad \text{in } \mathscr{W}, \tag{5}$$

$$U = e^{u/(4m)}\,, \quad V = e^{v/(4m)} \quad \text{in } \mathscr{B}\,. \tag{6}$$

In this frame,

$$\mathscr{W} \equiv \{(U, V, \theta, \phi) \in \mathbb{R}^2 \times \mathbb{S}^2 \mid U < 0, V > 0\}\,, \tag{7}$$

$$\mathscr{B} \equiv \{(U, V, \theta, \phi) \in \mathbb{R}^2 \times \mathbb{S}^2 \mid UV < 1\,, U, V > 0\}\,, \tag{8}$$

$$\mathscr{M} \doteq \mathscr{W} \cup \mathscr{B} \cup \mathcal{H}_{ev} \equiv \{(U, V, \theta, \phi) \in \mathbb{R}^2 \times \mathbb{S}^2 \mid UV < 1\,, V > 0\}\,. \tag{9}$$

Each of the three mentioned spacetimes is globally hyperbolic. The event horizon of $\mathscr{W}$, $\mathcal{H}_{ev}$ is one of the two horizons we shall consider. The other is the complete **past horizon** of $\mathscr{M}$, $\mathcal{H}$ which is part of the boundary of $\mathscr{M}$ in the Kruskal manifold. These horizons are respectively individuated by:

$$\mathcal{H}_{ev} \equiv \{(U, V, \theta, \phi) \in \mathbb{R}^2 \times \mathbb{S}^2 \mid U = 0, V > 0\}\,, \quad \mathcal{H} \equiv \{(U, V, \theta, \phi) \in \mathbb{R}^2 \times \mathbb{S}^2 \mid V = 0, U \in \mathbb{R}\}\,. \tag{10}$$

For future convenience, we decompose $\mathcal{H}$ into the *disjoint* union $\mathcal{H} = \mathcal{H}^- \cup \mathcal{B} \cup \mathcal{H}^+$ where $\mathcal{H}^\pm$ are defined according to $U > 0$ or $U < 0$ and $\mathcal{B}$ is the **bifurcation surface** at $U = 0$. *i.e.* the spacelike 2-sphere with radius r_S where $\mathcal{H}$ meets (the closure of) $\mathcal{H}_{ev}$.
The metric on $\mathscr{M}$ (and in the whole Kruskal manifold) takes the form:

$$g = -\frac{16m^3}{r} e^{-\frac{r}{2m}} (dU \otimes dV + dV \otimes dU) + r^2 h_{\mathbb{S}^2}(\theta, \phi) \quad \text{with} \quad UV = -\left(\frac{r(UV)}{2m} - 1\right) e^{r(UV)/2m}\,, \tag{11}$$

where it is manifest that the apparent Schwarzschild-coordinate singularity on $\mathcal{H}_{ev}$ has disappeared. The smooth function $r = r(UV)$ is obtained by solving the second equation in (11). It coincides with the radial Schwarzschild coordinate in both $\mathscr{W}$ and $\mathscr{B}$, hence taking the constant value r_s on $\mathcal{H}_{ev} \cup \mathcal{B}$ and yielding the metric singularity, located at $r = 0$, corresponding to $UV = 1$.
Let us pass focusing on the Killing vectors structure. Per direct inspection of either (1) or (11), one realizes that there exist a space of Killing vectors generated both by all the complete Killing fields associated

with the spherical symmetry – the vector fields ∂_ϕ for every choice of the polar axis z – and by a further smooth Killing field X. It coincides with ∂_t in both $\mathscr{W}$ and $\mathscr{B}$, although it is timelike and complete in the therefore static region $\mathscr{W}$, while it is spacelike in $\mathscr{B}$. Moreover X becomes light-like and tangent to $\mathcal{H}$ and $\mathcal{H}_{ev}$ (and to the whole completion of $\mathcal{H}_{ev}$ in the Kruskal manifold) and vanishes exactly on $\mathscr{B}$, giving rise to the structure of a *bifurcate Killing horizon* [KW91]. It is finally useful remarking that the coordinates u and v are respectively well defined on $\mathcal{H}_{ev}$ and $\mathcal{H}^\pm$ where it turns out that:

$$X = \mp\partial_u \text{ on } \mathcal{H}^\pm, \quad X = \partial_v \text{ on } \mathcal{H}_{ev}. \tag{12}$$

To conclude this short digression on the geometry of Kruskal-Schwarzschild spacetime, we notice that, by means of a conformal completion procedure, outlined in Appendix A, one can coherently introduce the notion of future and past **null infinity** $\Im^\pm$. Along the same lines (see again figure 1 and figure 2 in the appendix), we also shall refer to the formal *points at infinity* $i^\pm$, i^0, often referred to as **future**, **past** and **spatial infinity** respectively.

2.2. *The Algebra of field observables of the spacetime.* We are interested in the quantisation of the free massless scalar field φ [KW91, Wa94] on the globally hyperbolic spacetime $(\mathscr{N}, g)$. The real field φ is supposed to be smooth and to satisfy the massless **Klein-Gordon** equation in $(\mathscr{N}, g)$:

$$P_g\varphi = 0, \qquad P_g \doteq -\Box_g + \frac{1}{6}R_g\ . \tag{13}$$

Since we would like to use some conformal techniques, we have made explicit the conformal coupling with the metric, even if it has no net effect for the case $\mathscr{N} = \mathscr{M}$, because the curvature R_g vanishes therein. However, this allows us make a profitable use of the discussion in Appendix A when $\mathscr{N} = \mathscr{M}$ and $\widetilde{\mathscr{M}} \supset \mathscr{M}$ the conformal extension (see also figure 2 in the appendix) of the previously introduced physical part of Kruskal spacetime $\widetilde{\mathscr{M}}$, equipped with the metric $\widetilde{g}$ which coincides with g/r^2 in $\mathscr{M}$. In that case, if the smooth real function $\widetilde{\varphi}$ solves the Klein-Gordon equation in $\widetilde{\mathscr{M}}$ (where now $R_{\widetilde{g}} \neq 0$):

$$P_{\widetilde{g}}\widetilde{\varphi} = 0, \qquad P_{\widetilde{g}} \doteq -\Box_{\widetilde{g}} + \frac{1}{6}R_{\widetilde{g}}\ , \tag{14}$$

$\varphi \doteq \frac{1}{r}\widetilde{\varphi}\restriction_{\mathscr{N}}$ solves (13) in $\mathscr{M}$.
Generally speaking, we shall focus our attention to the class $\mathsf{S}(\mathscr{N})$ of real smooth solutions of (13) having compact support when restricted on a (and thus every) spacelike smooth Cauchy surface of a globally hyperbolic spacetime $(\mathscr{N}, g)$, mainly in the case $\mathscr{N} = \mathscr{M}$. This real vector space becomes a symplectic space $(\mathsf{S}(\mathscr{N}), \sigma_{\mathscr{N}})$ when equipped with the well-known non-degenerate symplectic form [KW91, Wa94, BGP96], for $\varphi_1, \varphi_2 \in \mathsf{S}(\mathscr{N})$,

$$\sigma_{\mathscr{N}}(\varphi_1, \varphi_2) \doteq \int_{\Sigma_n} (\varphi_2\nabla_n\varphi_1 - \varphi_1\nabla_n\varphi_2)d\mu_g(\Sigma_{\mathscr{N}})\ , \tag{15}$$

$\Sigma_{\mathscr{N}}$ being any spacelike smooth Cauchy surface of $\mathscr{N}$ with measure $\mu_g(\Sigma_{\mathscr{N}})$ induced by g and future-directed normal unit vector n. $\sigma_{\mathscr{N}}$ turns out to be independent form Σ.
Furthermore, if $\mathscr{N}' \subset \mathscr{N}$ and $(\mathscr{N}', g\restriction_{\mathscr{N}'})$ is globally hyperbolic, the following inclusion of symplectic subspaces holds

$$(\mathsf{S}(\mathscr{N}'), \sigma_{\mathscr{N}'}) \subset (\mathsf{S}(\mathscr{N}), \sigma_{\mathscr{N}})\ .$$

In can be proved by (15) and out of independence of σ from the used smooth spacelike Cauchy surface, using the fact that every compact portion of a spacelike Cauchy surface of $\mathscr{N}'$ can be viewed as a portion

of a smooth spacelike Cauchy surface of $\mathscr{N}$ by [BS06] and that any spacelike Cauchy surface is acausal (it being achronal and it holding Lemma 42 from Chap. 14 in [O'N83]).

The quantisation procedure within the algebraic approach, along the guidelines given in [KW91, Wa94, BGP96] goes on as follows: the elementary observables associated with the field φ are the (self-adjoint) elements of the **Weyl** (C^*-) **algebra** $\mathcal{W}(\mathsf{S}(\mathscr{N}))$ [Ha92, BR022, KW91, Wa94, BGP96] whose generators will be denoted by $W_{\mathscr{N}}(\varphi)$, $\varphi \in \mathsf{S}(\mathscr{N})$, as discussed in the Appendix B.

In order to interpret the elements in $\mathcal{W}(\mathsf{S}(\mathscr{N}))$ as *local observables* smeared with functions of $C_0^\infty(\mathscr{N};\mathbb{R})$, we introduce some further technology. In general, globally hyperbolicity of the underlying spacetime, as for $(\mathscr{N}, g)$, entails the existence of the **causal propagator**, $E_{P_g} : A_{P_g} - R_{P_g} : C_0^\infty(\mathscr{N};\mathbb{R}) \to \mathsf{S}(\mathscr{N})$ associated to P_g and defined as the difference of the advanced and retarded fundamental solution, which exist in globally hyperbolic spacetimes and are uniquely determined by their causal properties [Wa94, BGP96]. $E_{P_g} : C_0^\infty(\mathscr{N};\mathbb{R}) \to \mathsf{S}(\mathscr{N})$ is linear, surjective with $KerE_{P_g} = P_g(C_0^\infty(\mathscr{N};\mathbb{R}))$ and it is continuous with respect to the natural topologies of $C_0^\infty(\mathscr{N};\mathbb{R})$ and $C^\infty(\mathscr{N};\mathbb{R})$. Finally, it is possible to prove that [Wa94, BGP96], given $\psi \in \mathsf{S}(\mathscr{N})$ and any open neighborhood $\mathscr{N}'$ of any fixed smooth spacelike Cauchy surface of $\mathscr{N}$, there is $f_\psi \in C_0^\infty(\mathscr{N}';\mathbb{R})$ with $E_{P_g} f_\psi = \psi$. Consequently, $supp\psi \subset J^+(suppf_\psi;\mathscr{N}) \cup J^-(suppf_\psi;\mathscr{N})$.

The observable algebra $\mathcal{W}_{\mathscr{N}}[\mathscr{O}]$ **localised** in the open relatively compact set $\mathscr{O} \subset \mathscr{N}$ is then defined as the sub C^*-algebra of $\mathcal{W}(\mathsf{S}(\mathscr{N}))$ generated by all $W_{\mathscr{N}}(E_{P_g} f)$ with supp $f \subset \mathscr{O}$.

The standard Hilbert space picture, where the generators $W_{\mathscr{N}}(E_{P_g} f)$ are interpreted as exponentials of standard field operators, $e^{i\Phi(f)}$, can be introduced in the *GNS representation*, $(\mathfrak{H}_\omega, \Pi_\omega, \Psi_\omega)$, of any fixed algebraic state $\omega : \mathcal{W}(\mathsf{S}(\mathscr{N})) \to \mathbb{C}$ [Ha92, Wa94], such that the unitary one-parameter group $\mathbb{R} \ni t \mapsto \Pi_\omega\left(W_{\mathscr{N}}(tE_{\mathscr{N}} f)\right)$ is strongly continuous. The **field operators** $\Phi_\omega(f)$ obtained as the self-adjoint generators of those unitary one-parameter groups, $\Pi_\omega\left(W_{\mathscr{N}}(E_{\mathscr{N}} tf)\right) = \exp\{it\Phi_\omega(f)\}$, enjoy all the standard properties of usual quantization procedure of Klein-Gordon scalar field based on CCR [KW91, Wa94]. A different but equivalent definition is presented in the Appendix B. A physically important point, deserving particular attention, is the choice of physically meaningful states. We shall come back to this issue later.

2.3. *Algebras on $\mathcal{H}$ and $\Im^\pm$.* Let us consider the case $\mathscr{N} = \mathscr{M}$, the latter being the physical part of Kruskal spacetime beforehand introduced. The null 3-surfaces $\mathcal{H}$, $\Im^\pm$, as well as, with a certain difference, $\mathcal{H}_{ev}$, $\mathcal{H}^\pm$, can be equipped with a Weyl algebra of observable along the guidelines given in [MP05, DMP06, Mo06, Mo08, DMP09a]. These play a central role in defining physically interesting states for $\mathcal{W}(\mathsf{S}(\mathscr{M}))$ in the bulk. To keep the paper sufficiently self-contained, we briefly sketch the construction. Let $\mathcal{N}$ be any 3-submanifold of a spacetime – either $(\mathscr{M}, g)$ or its conformal completion $(\widetilde{\mathscr{M}}, \widetilde{g})$, in our case – whose metric, when restricted to $\mathcal{N}$, takes the **complete Bondi form**:

$$c_{\mathcal{N}}\left(-d\Omega \otimes d\ell - d\ell \otimes d\Omega + h_{\mathbb{S}^2}(\theta,\phi)\right) \tag{16}$$

where $c_{\mathcal{N}}$ is a non vanishing constant, while $(\ell, \Omega, \theta, \phi)$ define a coordinate patch in a neighbourhood of $\mathcal{N}$ seen as the locus $\Omega = 0$ though such that $d\Omega\restriction_{\mathcal{N}} \neq 0$. Out of this last condition we select $\ell \in \mathbb{R}$ as a complete parameter along the integral lines of $(d\Omega)^a$ and, in view of the given hypotheses, $\mathcal{N}$ turns out to be a null embedded codimension 1-submanifold diffeomorphic to $\mathbb{R} \times \mathbb{S}^2$.[1] It is possible to construct a symplectic space $(\mathsf{S}(\mathcal{N}), \sigma_{\mathcal{N}})$, where $\mathsf{S}(\mathcal{N})$ is a real linear space of smooth real-valued functions on $\mathcal{N}$

[1] In [DMP06, Mo06, Mo08, DMP09a] it was, more strongly, assumed and used the **geodetically complete Bondi form** of the metric, i.e. the integral lines of $(d\Omega)^a$ forming $\mathcal{N}$ are complete null geodesics with $\ell \in \mathbb{R}$ as an affine parameter. It happens if and only if, in the considered coordinates, $\partial_\Omega g_{\ell\ell}\restriction_{\mathcal{N}} = 0$ for all $\ell \in \mathbb{R}$ and $(\phi, \theta) \in \mathbb{S}^2$. This stronger requirement holds here for $\mathcal{H}$ and $\Im^\pm$.

which includes $C_0^\infty(\mathcal{N};\mathbb{R})$ and such that the right-hand side of

$$\sigma_{\mathcal{N}}(\psi,\psi') \doteq c_{\mathcal{N}} \int_{\mathcal{N}} \left(\psi' \frac{\partial \psi}{\partial \ell} - \psi \frac{\partial \psi'}{\partial \ell} \right) d\ell \wedge d\mathbb{S}^2 \,, \qquad \psi,\psi' \in \mathsf{S}(\mathcal{N}) \tag{17}$$

can be interpreted in the sense of $L^1(\mathbb{R}\times\mathbb{S}^2; d\ell\wedge d\mathbb{S}^2)$, where $d\mathbb{S}^2$ is the standard volume form on $\mathbb{S}^2$. Similarly to what it has been done in the bulk, since the only structure of symplectic space is necessary, one may define the Weyl algebra $\mathcal{W}(\mathsf{S}(\mathcal{N}))$, since the assumption that $C_0^\infty(\mathcal{N};\mathbb{R}) \subset \mathsf{S}(\mathcal{N})$ entails that $\sigma_{\mathcal{N}}$ is non-degenerate, hence $\mathcal{W}(\mathsf{S}(\mathcal{N}))$ is well-defined.
An interpretation of $\sigma_{\mathcal{N}}$ can be given thinking of ψ,ψ' as boundary values of fields $\varphi,\varphi' \in \mathsf{S}(\mathscr{M})$. The right hand side of (17) can then be seen as the integral over $\mathcal{N}$ of the 3-form $\eta[\varphi,\varphi']$ associated with $\varphi,\varphi' \in \mathsf{S}(\mathscr{K})$

$$\eta[\varphi,\varphi'] \doteq \frac{1}{6}\left(\varphi\nabla^a\varphi' - \varphi'\nabla^a\varphi\right)\sqrt{-g}\epsilon_{abcd}dx^b\wedge dx^b\wedge dx^c \,, \tag{18}$$

where ϵ_{abcd} is totally antisymmetric with $\epsilon_{1234}=1$ and where $\psi \doteq \varphi\restriction_{\mathcal{N}}$, $\psi' \doteq \varphi'\restriction_{\mathcal{N}}$. Furthermore, in order to give a sense to the integration of $\eta[\varphi,\varphi']$ over $\mathcal{N}$, we assume that $\mathcal{N}$ is positively oriented with respect to its future-directed normal vector. The crucial observation is now that, integrating $\eta[\varphi,\varphi']$ over a spacelike Cauchy surface $\Sigma \subset \mathscr{M}$, one gets exactly the standard symplectic form $\sigma_{\mathscr{M}}(\varphi,\varphi')$ in (15) (or that appropriate for the globally hyperbolic spacetime containing $\mathcal{N}$). In view of the validity of Klein-Gordon equation for φ and φ', the form $\eta[\varphi,\varphi']$ satisfies $d\eta[\varphi,\varphi']=0$. Therefore one aspects that, as a consequence of Stokes-Poincaré theorem it can happen that $\sigma_{\mathscr{M}}(\varphi,\varphi') = \sigma_{\mathcal{N}}(\varphi\restriction_{\mathcal{N}},\varphi'\restriction_{\mathcal{N}})$. If this result is valid, it implies the existence of an identification of $\mathcal{W}(\mathsf{S}(\mathscr{M}))$ (or some relevant sub algebra) and $\mathcal{W}(\mathsf{S}(\mathcal{N}))$. This is nothing but the idea we want to implement shortly with some generalisations.

In the present case we shall consider the following manifolds $\mathcal{N}$ equipped with the Bondi metric and thus the associated symplectic spaces $(\mathsf{S}(\mathcal{N}),\sigma_{\mathcal{N}})$:

(a) $\mathcal{H}$ with $\ell \doteq U$ where $c_{\mathcal{N}} = r_S^2$, r_S being the Schwarzschild radius,

(b) $\Im_R^\pm$ with $\ell \doteq u$ or, respectively, $\ell \doteq v$ where $c_{\mathcal{N}}=1$.

In the cases (b), the metric restricted to $\mathcal{N}$ with Bondi form is the conformally rescaled and extended Kruskal metric $\widetilde{g}$, with $\widetilde{g}\restriction_{\mathscr{M}} = g/r^2$, defined in the conformal completion $\widetilde{\mathscr{M}}$ of $\mathscr{M}$, as discussed in the Appendix A.
It is worth stressing that ℓ in Eq. (17) can be replaced, without affecting the left-hand side of (17), by any other coordinate $\ell' = f(\ell)$, where $f:\mathbb{R}\to(a,b)\subset\mathbb{R}$ is any smooth diffeomorphism. This allows us to consider the further case of symplectic spaces $(\mathsf{S}(\mathcal{N}),\sigma_{\mathcal{N}})$ where $\mathcal{N}$ is:

(c) $\mathcal{H}^\pm$ with $\ell \doteq u$ and $c_{\mathcal{N}} = r_S^2$,

independently from the fact that, in the considered coordinates, the metric g over $\mathcal{H}^\pm$ does not take the Bondi form.

2.4. *Injective isometric $*$-homomorphism between the Weyl algebras.* To conclude this section, as promised in the introduction, we establish the existence of some injective (isometric) $*$-homomorphisms mapping the Weyl algebras in the bulk into Weyl subalgebras defined on appropriate subsets of the piecewise smooth null 3-surfaces $\Im^- \cup \mathcal{H}$. To this end we have to specify the definition of $\mathsf{S}(\mathcal{H})$, $\mathsf{S}(\mathcal{H}^\pm)$ and $\mathsf{S}(\Im^\pm)$. From now on, referring to the definition of the preferred coordinate ℓ as pointed out in the above-mentioned list and with the identification of $\mathcal{H}$, $\mathcal{H}_{ev}$, $\mathcal{H}^\pm$ and $\Im^\pm$ with $\mathbb{R}\times\mathbb{S}^2$ as appropriate:

$$\mathsf{S}(\mathcal{H}) \doteq \left\{ \psi \in C^\infty(\mathbb{R}\times\mathbb{S}^2;\mathbb{R}) \;\middle|\; \exists\exists M_\psi > 1,\, C_\psi, C'_\psi \geq 0 \text{ with } |\psi(\ell,\theta,\phi)| < \frac{C_\psi}{\ln|\ell|} \;, \right.$$
$$\left. |\partial_\ell\psi(\ell,\theta,\phi)| < \frac{C'_\psi}{|\ell|\ln|\ell|} \quad \text{if } |\ell| > M_\psi \,, (\theta,\phi)\in\mathbb{S}^2 \right\} \,, \tag{19}$$

where $\ell = U$ on $\mathcal{H}$, and

$$\mathsf{S}(\Im^{\pm}) \doteq \Big\{\psi \in C^\infty(\mathbb{R}\times\mathbb{S}^2;\mathbb{R}) \;\Big|\; \psi(\ell)=0 \text{ in a neighbourhood of } i^0 \text{ and } \exists\exists C_\psi, C'_\psi \geq 0 \text{ with} \\ |\psi(\ell,\theta,\phi)| < \frac{C_\psi}{\sqrt{1+|\ell|}} \;\;, |\partial_\ell\psi(\ell,\theta,\phi)| < \frac{C'_\psi}{1+|\ell|}\,, \quad (\ell,\theta,\phi)\in\mathbb{R}\times\mathbb{S}^2\Big\}\,, \tag{20}$$

where $\ell = u$ on $\Im^+$ or $\ell = v$ on $\Im^-$, and, finally,

$$\mathsf{S}(\mathcal{H}_{ev})\,,\mathsf{S}(\mathcal{H}^{\pm}) \doteq \Big\{\psi \in C^\infty(\mathbb{R}\times\mathbb{S}^2;\mathbb{R}) \;\Big|\; \psi(\ell)=0 \text{ in a neighbourhood of } \mathcal{B} \text{ and } \exists\exists C_\psi, C'_\psi \geq 0 \text{ with} \\ |\psi(\ell,\theta,\phi)| < \frac{C_\psi}{1+|\ell|} \;\;, |\partial_\ell\psi(\ell,\theta,\phi)| < \frac{C'_\psi}{1+|\ell|}\,, \quad (\ell,\theta,\phi)\in\mathbb{R}\times\mathbb{S}^2\Big\}\,, \tag{21}$$

where $\ell = v$ on $\mathcal{H}_{ev}$ and $\ell = u$ on $\mathcal{H}^{\pm}$.
It is a trivial task to verify that the sets defined above are real vector spaces, they include $C_0^\infty(\mathbb{R}\times\mathbb{S}^2;\mathbb{R})$ and, if ψ belongs to one of them, $\psi\partial_\ell\psi \in L^1(\mathbb{R}\times\mathbb{S}^2, d\ell\wedge d\mathbb{S}^2)$ as requested. The reason of the definitions above relies upon the fact that the restrictions of wavefunctions of $\mathsf{S}(\mathscr{M})$ to the relevant boundaries of $\mathscr{M}$ satisfy the fall-off conditions in the definitions (19), (20), (21) approaching $i^{\pm}$ and this fact will play a crucial role shortly.
To go on, notice that, given two real symplectic spaces (with nondegenerate symplectic forms) (S_1,σ_1) and (S_2,σ_2), we can define the direct sum of them, as the real symplectic space $(\mathsf{S}_1\oplus\mathsf{S}_1, \sigma_1\oplus\sigma_2)$, where the nondegenerate symplectic form $\sigma_1\oplus\sigma_2 : (\mathsf{S}_1\oplus\mathsf{S}_2)\times(\mathsf{S}_1\oplus\mathsf{S}_2)\to\mathbb{R}$ is

$$\sigma_1\oplus\sigma_2((f,g),(f',g')) \doteq \sigma_1(f,f') + \sigma_2(g,g')\,, \quad \text{for all } f,f'\in\mathsf{S}_1 \text{ and } g,g'\in\mathsf{S}_2. \tag{22}$$

Passing to the Weyl algebras $\mathcal{W}(\mathsf{S}_1)$, $\mathcal{W}(\mathsf{S}_2)$, $\mathcal{W}(\mathsf{S}_1\oplus\mathsf{S}_2)$, it is natural to identity the C^*-algebra $\mathcal{W}(\mathsf{S}_1\oplus\mathsf{S}_2)$ with $\mathcal{W}(\mathsf{S}_1)\otimes\mathcal{W}(\mathsf{S}_2)$ providing, in this way, the algebraic tensor product of the two C^*-algebras with a natural C^*-norm (there is no canonical C^*-norm for the tensor product of two generic C^*-algebras). This identification is such that $W_{\mathsf{S}_1\oplus\mathsf{S}_2}((f_1,f_2))$ is identified to $W_{\mathsf{S}_1}(f_1)\otimes W_{\mathsf{S}_2}(f_2)$ for all $f_1\in\mathsf{S}_1$ and $f_2\in\mathsf{S}_2$.

We are now in place to state and to prove the main theorems of this section, making profitable use of the results achieved in [DR09]. Most notably, we are going to show that $\mathcal{W}(\mathsf{S}(\mathscr{M}))$ is isomorphic to a sub C^*-algebra of $\mathcal{W}(\mathsf{S}(\mathcal{H}))\otimes\mathcal{W}(\mathsf{S}(\Im^-))$. As a starting point, let us notice that, if φ and φ' are solutions of the KG equation with compact support on any spacelike Cauchy surface Σ of $\mathscr{M}$, the value of $\sigma_{\mathscr{M}}(\varphi,\varphi')$ is independent on the used Σ and, therefore, we can deform it preserving the value of $\sigma_{\mathscr{M}}(\varphi,\varphi')$. A tricky issue as well as a remarkable result arise if one performs a limit deformation where the final surface tends to $\mathcal{H}\cup\Im^-$ since one gets:

$$\sigma_{\mathscr{M}}(\varphi,\varphi') = \sigma_{\Im_{\mathcal{H}}}(\varphi_{\mathcal{H}},\varphi'_{\mathcal{H}}) + \sigma_{\Im^-}(\varphi_{\Im^-},\varphi'_{\Im^-})\;, \tag{23}$$

where the arguments of the symplectic forms in the right-hand side (which turn out to belong to the appropriate spaces (19), (20)) are obtained either as restrictions to $\mathcal{H}$ or as limit values (suitably rescaled) towards $\Im^-$ of φ and φ'. As the map $\varphi\mapsto(\varphi_{\mathcal{H}},\varphi_{\Im^-})$ is linear and the sum of symplectic forms appearing above in the right-hand side is the symplectic form σ on $\mathsf{S}\doteq\mathsf{S}(\mathcal{H})\oplus\mathsf{S}(\Im^-)$, this entails that we have built up a symplectomorphism from $\mathsf{S}(\mathscr{M})$ to S, $\varphi\mapsto(\varphi_{\mathcal{H}},\varphi_{\Im^-})$ which must be injective, that is, $(\varphi_{\Im^-},\varphi_{\mathcal{H}})=0$ entails $\varphi=0$. This is because, in the said hypotheses:

$$\sigma_{\mathscr{M}}(\varphi,\varphi') = \sigma_{\Im_{\mathcal{H}}}(\varphi_{\mathcal{H}},\varphi'_{\mathcal{H}}) + \sigma_{\Im^-}(\varphi_{\Im^-},\varphi'_{\Im^-}) = 0\,, \quad \forall\varphi'\in\mathsf{S}(\mathscr{M})$$

and $\sigma_{\mathscr{M}}$ is non-degenerate. In view of known theorems [BR022], this entails the existence of an isometric $*$-homomorphism $\imath : \mathcal{W}(\mathsf{S}(\mathscr{M}))\to\mathcal{W}(\mathsf{S}(\mathcal{H}))\otimes\mathcal{W}(\mathsf{S}(\Im^-))$. Our goal now is to formally state and to prove

the result displayed in (23).

Theorem 2.1. *For every $\varphi \in \mathsf{S}(\mathscr{M})$, define*

$$\varphi_{\Im^-} \doteq \lim_{\to \Im^-} r\varphi \,, \quad \textit{and} \quad \varphi_{\mathcal{H}} \doteq \varphi \restriction_{\mathcal{H}} \,.$$

Then the following facts hold.

(a) *The linear map*

$$\Gamma : \mathsf{S}(\mathscr{M}) \ni \varphi \mapsto (\varphi_{\Im^-}, \varphi_{\mathcal{H}}) \,,$$

is an injective symplectomorphism of $\mathsf{S}(\mathscr{M})$ into $\mathsf{S}(\Im^-) \oplus \mathsf{S}(\mathcal{H})$ equipped with the symplectic form, such that, for $\varphi, \varphi' \in \mathsf{S}(\mathscr{M})$:

$$\sigma_{\mathsf{S}(\Im^-)\oplus\mathsf{S}(\mathcal{H})}(\varphi, \varphi') \doteq \sigma_{\Im^-}\left(\varphi_{\Im^-}, \varphi'_{\Im^-}\right) + \sigma_{\Im_{\mathcal{H}}}\left(\varphi_{\mathcal{H}}, \varphi'_{\mathcal{H}}\right). \tag{24}$$

(b) *There exist a corresponding injective isometric $*$-homomorphism*

$$\imath : \mathcal{W}(\mathsf{S}(\mathscr{M})) \to \mathcal{W}(\mathsf{S}(\Im^-)) \otimes \mathcal{W}(\mathsf{S}(\mathcal{H})) \,,$$

which is unambiguously individuated by

$$\imath\left(W_{\mathscr{M}}(\varphi)\right) = W_{\Im^-}\left(\varphi_{\Im^-}\right) \otimes W_{\mathcal{H}}\left(\varphi_{\mathcal{H}}\right).$$

Proof. Let us start from point (a). If $\varphi \in \mathsf{S}(\mathscr{M})$, we can think of it as a restriction to $\mathscr{M}$ of a solution φ' of Klein-Gordon equation in the whole Kruskal manifold (to this end notice that the Cauchy data of φ on a spacelike Cauchy surface of $\mathscr{M}$ can be seen as Cauchy data on a spacelike Cauchy surface of the whole Kruskal manifold by direct application of the results in [BS06] and on the fact that the spacelike Cauchy surface are acausal as they being achronal and spacelike [O'N83]). Therefore $\varphi_{\mathcal{H}} \doteq \varphi' \restriction_{\mathcal{H}}$ is well-defined and smooth. Similarly, the functions $\varphi_{\Im^-} \doteq \lim_{\to \Im^-} r\varphi$ are well defined, smooth and vanish in a neighbourhood of the relevant i^0 in view of the following lemma whose proof is sketched in the Appendix C.

Lemma 2.1. *If $\varphi \in \mathsf{S}(\mathscr{M})$, $r\varphi$ uniquely extends to a smooth function $\widetilde{\varphi}$ defined in $\mathscr{M}$ joined with open neighborhoods of $\Im^+$ and $\Im^-$ included in the conformal extension $\widetilde{\mathscr{M}}$ of $\mathscr{M}$ discussed in the Appendix A. Furthermore, there are constants $v^{(\varphi)}, u^{(\varphi)} \in (-\infty, \infty)$ such that $\widetilde{\varphi}$ vanishes in $\mathscr{W}$ if $u < u^{(\varphi)}, v > v^{(\varphi)}$ and thus – by continuity – it vanishes in the corresponding limit regions on $\Im^+ \cup \Im^-$.*

Since the map Γ is linear by construction, it remains to prove that (i) $\varphi_{\mathcal{H}} \in \mathsf{S}(\mathcal{H})$ and $\varphi_{\Im^\pm} \in \mathsf{S}(\Im^\pm)$ as defined in (19) and (20), and (ii) that Γ preserves the symplectic forms, *i.e.*,

$$\sigma_{\mathscr{M}}(\varphi_1, \varphi_2) = \sigma_{\mathsf{S}(\mathcal{H})\oplus\mathsf{S}(\Im^-)}\left(\Gamma\varphi_1, \Gamma\varphi_2\right) \,. \tag{25}$$

Notice that, since $\sigma_{\mathscr{M}}$ is nondegenerate, the identity above implies that the linear map Γ is injective. Let us tackle point (i): since the behaviour of $\varphi_{\Im^-}$ about i^0 is harmless, what he have to establish is only that $\varphi_{\mathcal{H}}$ and $\varphi_{\Im^-}$ vanish, approaching the relevant i^-, as fast as requested in the definitions (19) and (20). Such a result is a consequence of the following proposition whose proof, based on the results achieved by Dafermos and Rodnianski in [DR09], is in the Appendix C.

Proposition 2.1. *Let Σ a smooth spacelike Cauchy surface of $\mathscr{M}$ with normal future-oriented versor n. Fix $\hat{R} > r_S$. The following facts hold.*
(a) *If $\varphi \in \mathsf{S}(\mathscr{M})$ and $\widetilde{\varphi}$ extends $r\varphi$ across $\Im^{\pm}$ as stated in Lemma 2.1, there are constants $C_1, C_2 \geq 0$ depending on φ and C_3, C_4 depending on φ and $\hat{R}$, such that the following pointwise bounds hold in $\mathscr{W} \cup \mathcal{H}_{ev}$ and $\mathscr{W} \cup \mathcal{H}^-$:*

$$|\varphi| \leq \frac{C_1}{\max\{2, v\}}\,, \qquad |X(\varphi)| \leq \frac{C_2}{\max\{2, v\}}\,, \tag{26}$$

and, respectively,

$$|\varphi| \leq \frac{C_1}{\max\{2, -u\}}\,, \qquad |X(\varphi)| \leq \frac{C_2}{\max\{2, -u\}}\,. \tag{27}$$

Similarly, assuming also $r \geq \hat{R}$ and $t > 0$ (including the points on $\Im^+$),

$$|\widetilde{\varphi}| \leq \frac{C_3}{\sqrt{1+|u|}}\,, \qquad |X(\widetilde{\varphi})| \leq \frac{C_4}{1+|u|}\,, \tag{28}$$

or, assuming $r \geq \hat{R}$ but $t < 0$ (including the points on $\Im^-$),

$$|\widetilde{\varphi}| \leq \frac{C_3}{\sqrt{1+|v|}}\,, \qquad |X(\widetilde{\varphi})| \leq \frac{C_4}{1+|v|}. \tag{29}$$

X is the smooth Killing vector field on the conformally extended Kruskal spacetime with $X = \partial_t$ in $\mathscr{W}$, $X = \partial_v$ on $\mathcal{H}_{ev}$, $X = \partial_u$ on $\mathcal{H}^-$, $X = \partial_u$ on $\Im^+$ and $X = \partial_v$ on $\Im^-$.
(b) *If the Cauchy data $(\varphi\restriction_\Sigma, \nabla_n\varphi\restriction_\Sigma)$ on Σ of φ tend to 0 in the sense of the test function (product) topology on $C_0^\infty(\Sigma;\mathbb{R})$, then the associated constants C_i tend to 0, for $i = 1, 3$.*
If the Cauchy data $(\varphi'\restriction_\Sigma, \nabla_n\varphi'\restriction_\Sigma)$ on Σ of $\varphi' \doteq X(\varphi)$ tend to 0 in the sense of the test function (product) topology on $C_0^\infty(\Sigma;\mathbb{R})$, then the associated constants C_i tend to 0, for $i = 2, 4$

Since φ and $\widetilde{\varphi}$ are smooth, $X(\varphi) = \partial_u\varphi$ on $\mathcal{H}^-$ and $X(\widetilde{\varphi}) = \partial_v\widetilde{\varphi}$ on $\Im^-$, it comes out of a direct inspection that $\varphi_{\mathcal{H}} \in \mathsf{S}(\mathcal{H})$ and $\varphi_{\Im^-} \in \mathsf{S}(\Im^-)$ since the definitions (19) and (20) are fulfilled, for $\ell = U = e^{u/(4m)}$ and $\ell = v$ respectively; furthermore, in view of the last statement of the above proposition, it holds $\varphi_{\Im^+} \in \mathsf{S}(\Im^+)$.
In order to conclude, let us finally prove item (ii), that is (25), making use once more of Proposition 2.1. Consider $\varphi, \varphi' \in \mathsf{S}(\mathscr{M})$ and a spacelike Cauchy surface $\Sigma_{\mathscr{M}}$ of $\mathscr{M}$ so that,

$$\sigma_{\mathscr{M}}(\varphi, \varphi') = \int_{\Sigma_{\mathscr{M}}} (\varphi'\nabla_n\varphi - \varphi\nabla_n\varphi')\; d\mu_g(\Sigma_{\mathscr{M}})$$

where n is the unit normal to the surface $\Sigma_{\mathscr{M}}$ and $\mu_g(\Sigma_{\mathscr{M}})$ is the metric induced measure on $\Sigma_{\mathscr{M}}$ (we shall write $d\mu_g$ in place of $d\mu_g(\Sigma_{\mathscr{M}})$ and Σ in palce of $\Sigma_{\mathscr{M}}$ in the following). Since both φ and φ' vanish for sufficiently large U, we can use the surface Σ, defined as the locus $t = 0$ in $\mathscr{W}$, and, out of the Poincaré theorem (employing the 3-form η as discussed in Sec. 2.3), we can write

$$\sigma_{\mathscr{M}}(\varphi, \varphi') = \int_{\Sigma\cap\mathscr{W}} \varphi' X(\varphi) - \varphi X(\varphi') d\mu_g + r_S^2 \int_{\mathcal{H}^+} (\varphi'\partial_U\varphi - \varphi\partial_U\varphi')\, dU \wedge d\mathbb{S}^2, \tag{30}$$

where we have used the fact that $\mathcal{B}\cap\Sigma$ has measure zero. We shall prove that, restricting the integration to $\mathscr{W}$,

$$\int_{\Sigma\cap\mathscr{W}} \varphi' X(\varphi) - \varphi X(\varphi') d\mu_g = r_S^2 \int_{\mathcal{H}^-} \left(\varphi'\partial_U\varphi - \varphi\partial_U\varphi'\right) dU\wedge d\mathbb{S}^2 + \int_{\Im^-} \left(\widetilde{\varphi}'\partial_v\widetilde{\varphi} - \widetilde{\varphi}\partial_v\widetilde{\varphi}'\right) du\wedge d\mathbb{S}^2 \,. \tag{31}$$

Since, with the same procedure, one gets an analogous statement (with the integration in dV extended over $\mathbb{R}^-$ and the other performed on $\Im^-$) for the portion of the initial integration taken in $\mathscr{W}$, this will conclude the proof.
To prove the identity (31) we first notice that:

$$\int_{\Sigma\cap\mathscr{W}} \varphi' X(\varphi) - \varphi X(\varphi') d\mu_g = \int_{[r_S,+\infty)\times\mathbb{S}^2} \left.\frac{r^2\left(\varphi' X(\varphi) - \varphi X(\varphi')\right)}{1-2m/r}\right|_{(t=0,r,\theta,\phi)} dr\wedge d\mathbb{S}^2(\theta,\phi)\,.$$

Next we break the integral in the right-hand side into two pieces passing to the coordinate r^*:

$$\begin{aligned}&\int_{[r_S,+\infty)\times\mathbb{S}^2} \left.\frac{r^2\left(\varphi' X(\varphi) - \varphi X(\varphi')\right)}{1-2m/r}\right|_{(t=0,r,\theta,\phi)} dr\wedge d\mathbb{S}^2\\ &= \int_{(-\infty,\hat{R}^*)\times\mathbb{S}^2} r^2\left.\left(\varphi' X(\varphi) - \varphi X(\varphi')\right)\right|_{(t=0,r^*,\theta,\phi)} dr^*\wedge d\mathbb{S}^2\\ &+ \int_{[\hat{R}^*,+\infty)\times\mathbb{S}^2} \left.\left(r\varphi' X(r\varphi) - r\varphi X(r\varphi')\right)\right|_{(t=0,r^*,\theta,\phi)} dr^*\wedge d\mathbb{S}^2\,.\end{aligned} \tag{32}$$

We started assuming Σ as the surface with $t=0$ in (30), however, the value of t is immaterial, since we can work, with a different surface Σ_t obtained by evolving Σ along the flux of the Killing vector X. We remind that $X=\partial_t$ in $\mathscr{W}$ and $X=0$ exactly on $\mathcal{B}$, which is a fixed submanifold of the flux as a consequence. We know that the value of the symplectic form $\sigma_{\mathscr{M}}(\varphi,\varphi')$ does not change varying t, by construction. Since $\mathcal{B}$ is fixed under the flux of X, by direct application of Stokes-Poincaré theorem, one sees that this invariance holds for the integration restricted to $\mathscr{W}$, too. In other words, for every $t>0$, it holds:

$$\begin{aligned}\int_{\Sigma\cap\mathscr{W}} \varphi' X(\varphi) - \varphi X(\varphi') d\mu_g = \int_{\Sigma_t\cap\mathscr{W}} \varphi' X(\varphi) - \varphi X(\varphi') d\mu_g &= \int_{\mathbb{R}\times\mathbb{S}^2} r^2\left.\left(\varphi' X(\varphi) - \varphi X(\varphi')\right)\right|_{(t,v-t,\theta,\phi)} dv\wedge d\mathbb{S}^2\\ &+ \int_{\mathbb{R}\times\mathbb{S}^2} \left.\left(\widetilde{\varphi}' X(\widetilde{\varphi}) - \widetilde{\varphi} X(\widetilde{\varphi}')\right)\right|_{(t,t-u,\theta,\phi)} du\wedge d\mathbb{S}^2\,,\end{aligned}$$

where we have also changed the variables of integration, passing from r^* to the variable $v=t+r^*$ or to the variable $u=t-r^*$. Thus, provided that both the limit exist, we have:

$$\begin{aligned}\int_{\Sigma\cap\mathscr{W}} \varphi' X(\varphi) - \varphi X(\varphi') d\mu_g &= \lim_{t\to-\infty} \int_{\mathbb{R}\times\mathbb{S}^2} r^2\left.\left(\varphi' X(\varphi) - \varphi X(\varphi')\right)\right|_{(t,t-u,\theta,\phi)} du\wedge d\mathbb{S}^2\\ &+ \lim_{t\to-\infty} \int_{\mathbb{R}\times\mathbb{S}^2} \left.\left(\widetilde{\varphi}' X(\widetilde{\varphi}) - \widetilde{\varphi} X(\widetilde{\varphi}')\right)\right|_{(t,v-t,\theta,\phi)} dv\wedge d\mathbb{S}^2\,.\end{aligned} \tag{33}$$

Formally, the former limit should give rise to an integral over $\mathcal{H}^-$, whereas the latter should give rise to an analogous integral over $\Im^-$. Let us examine the two limits of integrals in the right-hand side of

(33) separately. We start from the latter. First of all we notice that, in view of (a) in Lemma 2.1, the integration in v can be performed in $(-\infty, v_0]$ for some constant $v_0 \in \mathbb{R}$, without affecting the integral for every $t < 0$. Therefore

$$\lim_{t\to-\infty} \int_{\mathbb{R}\times\mathbb{S}^2} \Big(\widetilde{\varphi}' X(\widetilde{\varphi}) - \widetilde{\varphi} X(\widetilde{\varphi}')\Big)\Big|_{(t,v-t,\theta,\phi)} dv\wedge d\mathbb{S}^2 = \lim_{t\to-\infty} \int_{(-\infty,v_0]\times\mathbb{S}^2} \Big(\widetilde{\varphi}' X(\widetilde{\varphi}) - \widetilde{\varphi} X(\widetilde{\varphi}')\Big)\Big|_{(t,v-t,\theta,\phi)} dv\wedge d\mathbb{S}^2.$$

In view of the uniform bounds, associated with the constants C_3 and C_4, given by v-integrable functions (in $(\infty, v_0]$!) as stated in Proposition 2.1, we can now apply Lebesgue's dominated convergence theorem to the limit in the right-hand side, obtaining that

$$\lim_{t\to-\infty} \int_{\mathbb{R}\times\mathbb{S}^2} \Big(\widetilde{\varphi}' X(\widetilde{\varphi}) - \widetilde{\varphi} X(\widetilde{\varphi}')\Big)\Big|_{(t,v-t,\theta,\phi)} dv \wedge d\mathbb{S}^2 = \int_{\Im^-} \Big(\widetilde{\varphi}' \partial_v \widetilde{\varphi} - \widetilde{\varphi} \partial_v \widetilde{\varphi}'\Big)\, dv \wedge d\mathbb{S}^2 \,. \tag{34}$$

Let us now pass to the former integral in the right-hand side of (33). To this end, fix $u_0 \in \mathbb{R}$ and decompose:

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{S}^2} r^2\, (\varphi' X(\varphi) - \varphi X(\varphi'))|_{(t,t-u,\theta,\phi)} du \wedge d\mathbb{S}^2 &= \int_{\Sigma_t^{(u_0)}} \varphi' X(\varphi) - \varphi X(\varphi') d\mu_g \\ &+ \int_{(-\infty,u_0]\times\mathbb{S}^2} r^2\, (\varphi' X(\varphi) - \varphi X(\varphi'))|_{(t,t-u,\theta,\phi)} du \wedge d\mathbb{S}^2 \,. \end{aligned} \tag{35}$$

We have used the initial expression for the first integral, which is performed over the compact subregion $\Sigma_t^{(u_0)}$ of $\Sigma_t \cap \mathscr{W}$ containing the points with global null coordinate U included in $[-\exp\{-u_0/(4m)\}, 0]$. That integral is, in fact, an integral the smooth 3-form $\eta \doteq \eta[\varphi, \varphi']$ defined in (18). In view of Klein-Gordon equation, $d\eta = 0$. Thus, by means of an appropriate use of the Stokes-Poincaré theorem, this integral can be re-written as an integral of η over the compact subregion of $\mathcal{H}^+$ containing the points with coordinate U which belongs to $[U_0, 0]$, where $U_0 \doteq -e^{-u_0/(4m)}$, plus an integral of the same 3-form over the compact null 3-surface $S_t^{(u_0)}$ formed of the points in $\mathscr{M}$ with $U = U_0$ lying between Σ_t and $\mathcal{H}^-$, i.e.:

$$\int_{\Sigma_t^{(u_0)}} \varphi' X(\varphi) - \varphi X(\varphi') d\mu_g = \int_{\mathcal{H}^- \cap \{U_0 \le U \le 0\}} \eta + \int_{S_t^{(u_0)}} \eta \,.$$

Adopting coordinates U, V, θ, ϕ, the direct evaluation of the first integral in the right-hand side produces:

$$\int_{\Sigma_t^{(u_0)}} \varphi' X(\varphi) - \varphi X(\varphi') d\mu_g = r_S^2 \int_{\mathcal{H}^- \cap \{U_0 \le U \le 0\}} (\varphi' \partial_U \varphi - \varphi \partial_U \varphi')\, dU \wedge d\mathbb{S}^2 + \int_{S_t^{(u_0)}} \eta \,. \tag{36}$$

We have obtained that

$$\begin{aligned}\lim_{t\to-\infty} \int_{\mathbb{R}\times\mathbb{S}^2} r^2\, (\varphi' X(\varphi) - \varphi X(\varphi'))|_{(t,t-u,\theta,\phi)} du \wedge d\mathbb{S}^2 &= r_S^2 \int_{\mathcal{H}_R^+ \cap \{U_0 \le U \le 0\}} (\varphi' \partial_U \varphi - \varphi \partial_U \varphi')\, dU \wedge d\mathbb{S}^2 \\ &+ \lim_{t\to-\infty} \int_{S_t^{(u_0)}} \eta + \lim_{t\to-\infty} \int_{(-\infty,u_0]\times\mathbb{S}^2} r^2\, (\varphi' X(\varphi) - \varphi X(\varphi'))|_{(t,t-u,\theta,\phi)} du \wedge d\mathbb{S}^2 \,. \end{aligned} \tag{37}$$

Taking the limit as $t \to -\infty$, one has $\int_{S_t^{(u_0)}} \eta \to 0$, because it is the integral of a smooth form over a vanishing surface (as $t \to -\infty$), whereas

$$\begin{aligned}\lim_{t\to-\infty} \int_{(-\infty,u_0]\times\mathbb{S}^2} r^2\, (\varphi' X(\varphi) - \varphi X(\varphi'))|_{(t,t-u,\theta,\phi)} du \wedge d\mathbb{S}^2 &= \int_{\mathcal{H}^- \cap \{u_0 \ge u\}} r_S^2\, (\varphi' \partial_u \varphi - \varphi \partial_u \varphi')\, du \wedge d\mathbb{S}^2 \\ &= r_S^2 \int_{\mathcal{H}^- \cap \{U_0 \ge U\}} (\varphi' \partial_U \varphi - \varphi \partial_U \varphi')\, dU \wedge d\mathbb{S}^2 \end{aligned} \tag{38}$$

where we stress that the final integrals are evaluated over $\mathcal{H}^-$ and we have used again Lebesgue's dominated convergence theorem thanks to the estimates associated with the constants C_1 and C_2 in Proposition 2.1. Inserting the achieved results in the right-hand side of (37), we find that:

$$\lim_{t\to-\infty}\int_{\mathbb{R}\times\mathbb{S}^2} r^2\left(\varphi' X(\varphi)-\varphi X(\varphi')\right)\big|_{(t,t-u,\theta,\phi)} du\ \wedge d\mathbb{S}^2 = r_S^2\int_{\mathcal{H}^-}\left(\varphi'\partial_U\varphi-\varphi\partial_U\varphi'\right)dU\wedge d\mathbb{S}^2_{\mathbb{S}^2}\ .$$

That identity inserted in (33) together with (34), yields (31) concluding the proof of (a).
The item (b) can be proved as follows. In the following $\mathsf{S}\doteq\mathsf{S}(\mathcal{H})\oplus\mathsf{S}(\Im^-)$ and σ is the natural symplectic form on that space. Consider the closure of the sub $*$-algebra generated by all the generators $W_{\mathsf{S}}(\Gamma\varphi)\in\mathcal{W}(\mathsf{S})$ for all $\varphi\in\mathsf{S}(\mathscr{M})$. This is a C^*-algebra which, in turn, defines a realization of $\mathcal{W}(\mathsf{S}(\mathscr{M}))$ because Γ is a isomorphism of the symplectic space $(\mathsf{S}(\mathscr{M}),\sigma_{\mathscr{M}})$ onto the symplectic space $(\Gamma(\mathsf{S}(\mathscr{M})),\sigma\restriction_{\Gamma(\mathsf{S}(\mathscr{M}))\times\Gamma(\mathsf{S}(\mathscr{M}))})$. As a consequence of Theorem 5.2.8 in [BR022], there is a $*$-isomorphism (which is isometric as a consequence) between $\mathcal{W}(\mathsf{S}(\mathscr{M}))$ and the other, just found, realization of the same Weyl algebra, unambiguously individuated by the requirement $\imath_{\mathscr{M}}(W_{\mathscr{M}})(\varphi)=W_{\mathsf{S}}(\Gamma\varphi)$. This $*$-isomorphism individuates an injective (because it is isometric) $*$-homomorphism of $\mathcal{W}(\mathsf{S}(\mathscr{M}))$ into $\mathcal{W}(\mathsf{S},\sigma)\equiv\mathcal{W}(\mathsf{S}(\mathcal{H}))\otimes\mathcal{W}(\mathsf{S}(\Im^-))$. $\square$

As a byproduct and a straightforward generalization, the proof of the above theorem also establishes the following:

Theorem 2.2. *With the same definitions as in Theorem 2.1 and defining, for $\varphi\in\mathsf{S}(\mathscr{W})$, $\varphi_{\mathcal{H}^-}\doteq\lim_{\to\mathcal{H}^-}\varphi$ and $\varphi_{\mathcal{H}_{ev}}\doteq\lim_{\to\mathcal{H}_{ev}}\varphi$, the linear maps*

$$\Gamma_-:\mathsf{S}(\mathscr{W})\ni\varphi\mapsto(\varphi_{\mathcal{H}^-},\varphi_{\Im^-})\in\mathsf{S}(\mathcal{H}^-)\oplus\mathsf{S}(\Im^-)\ ,\quad \Gamma_+:\mathsf{S}(\mathscr{W})\ni\varphi\mapsto(\varphi_{\mathcal{H}_{ev}},\varphi_{\Im^+})\in\mathsf{S}(\mathcal{H}_{ev})\oplus\mathsf{S}(\Im^+)$$

are well-defined injective symplectomorphisms. As a consequence, there exist a corresponding injective isometric $*$-homomorphisms:

$$\imath^-:\mathcal{W}(\mathsf{S}(\mathscr{W}))\to\mathcal{W}(\mathsf{S}(\mathcal{H}^-))\otimes\mathcal{W}(\mathsf{S}(\Im^-))\ ,\quad \imath^+:\mathcal{W}(\mathsf{S}(\mathscr{W}))\to\mathcal{W}(\mathsf{S}(\mathcal{H}_{ev}))\otimes\mathcal{W}(\mathsf{S}(\Im^+))\ ,$$

which are respectively unambiguously individuated by the requirements for $\varphi\in\mathsf{S}(\mathscr{W})$

$$\imath^-\left(W_{\mathscr{W}}(\varphi)\right)=W_{\mathcal{H}^-}\left(\varphi_{\mathcal{H}^-}\right)\otimes W_{\Im^-}\left(\varphi_{\Im^-}\right)\ ,\qquad \imath^+\left(W_{\mathscr{W}}(\varphi)\right)=W_{\mathcal{H}_{ev}}\left(\varphi_{\mathcal{H}_{ev}}\right)\otimes W_{\Im^+}\left(\varphi_{\Im^+}\right).$$

Before concluding the present section we would like to stress that a similar result as the one presented in Theorem 2.1 and in Theorem 2.2 can be obtained for the algebra of observables defined on the whole Kruskal extension $\mathscr{K}$ of the Schwarzschild spacetime. In the case, an injective isometric $*$-homomorphisms $\imath_{\mathscr{K}}:\mathcal{W}(\mathsf{S}(\mathscr{K}))\to\mathcal{W}(\mathsf{S}(\Im^+_L))\otimes\mathcal{W}(\mathsf{S}(\mathcal{H}))\otimes\mathcal{W}(\mathsf{S}(\Im^-))$ can be constructed out of the projection $\Gamma_{\mathscr{K}}:\mathsf{S}(\mathscr{K})\ni\varphi\mapsto(\varphi_{\Im^+_L},\varphi_{\mathcal{H}},\varphi_{\Im^-})\in\mathsf{S}(\Im^+_L)\oplus\mathsf{S}(\mathcal{H})\oplus\mathsf{S}(\Im^-)$ from the requirement $\imath_{\mathscr{K}}\left(W_{\mathscr{K}}(\varphi)\right)=W_{\Im^+_L}\left(\varphi_{\Im^+_L}\right)\otimes W_{\mathcal{H}}\left(\varphi_{\mathcal{H}}\right)\otimes W_{\Im^-}\left(\varphi_{\Im^-}\right)$ where $\Im^+_L$ is for the future null infinity of the left Schwarzschild wedge in the Kruskal spacetime $\mathscr{K}$).

3 Interplay of bulk states and boundary states.

3.1. *Bulk states in induced form boundary states by means of the pullback of $\imath$ and $\imath^-$.* In this section we construct the mathematical technology to induce algebraic states (see the Appendix B) on the algebras

$\mathcal{W}(\mathsf{S}(\mathscr{M}))$ and $\mathcal{W}(\mathsf{S}(\mathscr{W}))$ from states defined, respectively, on the algebras $\mathcal{W}(\mathsf{S}(\mathcal{H})) \otimes \mathcal{W}(\mathsf{S}(\Im^-))$ and $\mathcal{W}(\mathsf{S}(\mathcal{H}^-)) \otimes \mathcal{W}(\mathsf{S}(\Im^-))$. A bit improperly, we shall call **bulk states** the states defined on the algebra $\mathcal{W}(\mathsf{S}(\mathscr{M}))$ (and on the other subalgebra defined in the spacetime as said above) and **boundary states** the states on $\mathcal{W}(\mathsf{S}(\mathcal{H})) \otimes \mathcal{W}(\mathsf{S}(\Im^-))$ (or on the relevant subalgebra defined on null surfaces as previously mentioned). To this end, the main tools are Theorem 2.1 and 2.2.
Let us consider the case of $\mathcal{W}(\mathsf{S}(\mathscr{M}))$ as an example. If the linear functional $\omega : \mathcal{W}(\mathsf{S}(\mathcal{H})) \otimes \mathcal{W}(\mathsf{S}(\Im^-)) \to \mathbb{C}$ is an algebraic state, the isometric $*$-homomorphism $\imath$ constructed in Theorem 2.1 gives rise to a state $\omega_{\mathscr{M}}$ on $\mathcal{W}(\mathsf{S}(\mathscr{M}))$ defined by employing the pullback $\imath^*(\omega)$ of $\imath$ as:

$$\omega_{\mathscr{M}} \doteq \imath^*(\omega)\,, \qquad \text{where } \; (\imath^*(\omega))\,(a) \doteq \omega\,(\imath(a))\;, \quad \text{for every } a \in \mathcal{W}(\mathsf{S}(\mathscr{M})). \tag{39}$$

Similar states can be defined using $\imath^-$ for the corresponding algebra. The situation will now be specialised to *quasifree states*. As discussed in the Appendix B, a quasifree state ω_μ, for instance, defined on $\mathcal{W}(\mathsf{S}(\mathcal{H})) \otimes \mathcal{W}(\mathsf{S}(\Im^-))$, is unambiguously defined by requiring that

$$\omega_\mu\,(W_{\mathcal{H}\cup\Im^-}(\psi)) = e^{-\mu(\psi,\psi)/2}\,, \quad \text{for all } \psi \in \mathsf{S}(\mathcal{H}) \oplus \mathsf{S}(\Im^-)\,, \tag{40}$$

where $\mu : (\mathsf{S}(\mathcal{H}) \oplus \mathsf{S}(\Im^-)) \times (\mathsf{S}(\mathcal{H}) \oplus \mathsf{S}(\Im^-)) \to \mathbb{R}$ is a real scalar product satisfying the requirement (93). It turns out immediately that if ω in (39) is quasifree, then $\omega_{\mathscr{M}}$ is such (and this is valid in general for the quasifree states on subalgebras as previously discussed). Therefore, we turn our attention to quasifree states defined on the boundaries $\mathcal{W}(\mathsf{S}(\Im^\pm))$, $\mathcal{W}(\mathsf{S}(\mathcal{H}))$, and on the possible *composition* of such states in view of the following proposition.

Proposition 3.1. *Let (S_1, σ_1), (S_2, σ_2) be symplectic spaces and ω_1, ω_2 be two quasifree algebraic states on $\mathcal{W}(\mathsf{S}_1, \sigma_1)$ and $\mathcal{W}(\mathsf{S}_2, \sigma_2)$, induced respectively by the real scalar products $\mu_1 : \mathsf{S}_1 \times \mathsf{S}_1 \to \mathbb{R}$ and $\mu_2 : \mathsf{S}_2 \times \mathsf{S}_2 \to \mathbb{R}$. Then the scalar product $\mu_1 \oplus \mu_2 : (\mathsf{S}_1 \oplus \mathsf{S}_2) \times (\mathsf{S}_1 \oplus \mathsf{S}_2) \to \mathbb{R}$ defined by:*

$$\mu_{\mathsf{S}_1 \oplus \mathsf{S}_2}((\psi_1, \psi_2), (\psi_1', \psi_2')) \doteq \mu_1(\psi_1, \psi_1') + \mu_2(\psi_2, \psi_2')\,, \quad \textit{for all } (\psi_1, \psi_2), (\psi_1', \psi_2') \in \mathsf{S}_1 \oplus \mathsf{S}_2\,,$$

uniquely individuates a quasifree state $\omega_1 \otimes \omega_2$ on $\mathcal{W}(\mathsf{S}_1) \otimes \mathcal{W}(\mathsf{S}_2)$ by requiring that:

$$\omega_1 \otimes \omega_2\,(W_{\mathsf{S}_1}(\psi_1) \otimes W_{\mathsf{S}_2}(\psi_2)) = e^{-\mu_1 \oplus \mu_2((\psi_1,\psi_2),(\psi_1',\psi_2'))/2}\,, \quad \textit{for all } (\psi_1, \psi_2) \in \mathsf{S}_1 \oplus \mathsf{S}_2\,. \tag{41}$$

Proof. The only thing to be proved is that the requirement (93) is valid for $\mu_1 \oplus \mu_2$ with respect to $\sigma_1 \oplus \sigma_2$ defined in (22). This fact immediately follows from the definition of $\mu_1 \oplus \mu_2$ and making use of (2) in remark B.1. □

Obviously, we can iterate the procedure in order to consider the composition of three (or more) states on corresponding three (or more) Weyl algebras. In view of the established proposition we may study separately the quasifree states on the Weyl algebras $\mathcal{W}(\mathsf{S}(\mathcal{N}))$ associated to the null surfaces $\mathcal{N}$ (a)-(c) listed in Sec. 2.3.

3.2. *The Kay-Wald quasifree state on $\mathcal{W}(\mathcal{H})$.* We remind the reader that, if μ individuates a quasifree state over $\mathcal{W}(\mathsf{S}, \sigma)$, its **two-point function** is defined as $\lambda_\mu(\psi_1, \psi_2) \doteq \mu(\psi_1, \psi_2) - \frac{i}{2}\sigma(\psi_1, \psi_2)$ (see the Appendix B). When passing to the one-particle space structure (K_μ, H_μ) (see the Appendix B) one has $\lambda_\mu(\psi_1, \psi_2) = \langle K_\mu\psi_1, K_\mu\psi_2\rangle_\mu$, where $\langle\cdot,\cdot\rangle_\mu$ is the scalar product in H_μ. Obviously, the two-point function of a quasifree state on a given Weyl algebra brings is the same information as the scalar product μ itself since the symplectic form is known *a priori*, thus the two-point function individuates the state completely. In [KW91], some properties are discussed for a certain state on $\mathcal{W}(\mathsf{S}(\mathscr{K}))$, (where $\mathscr{K}$ is the whole Kruskal

extension of the Schwarzschild spacetime), assumed to exist, to be Hadamard and invariant under the Killing flux of X. That state, *if any*, was proved to be unique (with respect to certain algebras of observables) and to satisfy the KMS property when working on a suitable algebra of observables in $\mathscr{W}$. Physically speaking that state coincides to the celebrated Hartle-Hawking state when the background is Kruskal spacetime. It is important to remark that in [KW91], general globally-hyperbolic spacetimes with bifurcate Killing horizon are considered, whereas or work focus only on Kruskal spacetime $\mathscr{K}$. As an intermediate step, Kay and Wald showed that the two-point function of the state has a very particular form when restricted to the horizon $\mathcal{H}$. More precisely, the two-point function λ_{KW} of the preferred state can be written in terms of the restrictions of the wavefunctions to $\mathcal{H}$ and it reads

$$\lambda_{KW}(\varphi_1, \varphi_2) = \lim_{\epsilon \to 0^+} -\frac{r_S^2}{\pi} \int_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2} \frac{\varphi_1\restriction_{\mathcal{H}}(U_1, \theta, \phi)\ \ \varphi_2\restriction_{\mathcal{H}}(U_2, \theta, \phi)}{(U_1 - U_2 - i\epsilon)^2}\ dU_1 \wedge dU_2 \wedge d\mathbb{S}^2\,. \tag{42}$$

provided that $\varphi_1\restriction_{\mathcal{H}}, \varphi_2\restriction_{\mathcal{H}} \in C_0^\infty(\mathbb{R}\times\mathbb{S}^2; \mathbb{R})$. It is important to stress that the expression above is valid when $\varphi_1\restriction_{\mathcal{H}}$ and $\varphi_2\restriction_{\mathcal{H}}$ have compact support on $\mathcal{H}$. Actually, the same two-point function was already found in [Sw82] discussing the physical consequences of the Bisognano-Wichmann theorem and in [DK86-87] discussing the various states in the right Schwarzschild wedge $\mathscr{W}$ in the Kruskal manifold, adopting a S-*matrix* point of view. In the latter paper the two-point function in (42) was referred to the Killing horizon in *two-dimensional Minkowski spacetime rather than Kruskal one.* In that case there are smooth solutions of Klein-Gordon equation, for $m \geq 0$, and with compactly supported Cauchy data, which intersect the horizon in a compact set. These solutions of the *characteristic Cauchy problem* can be used in the right-hand side of (42) when the discussion is referred to Minkowski spacetime instead of Kruskal spacetime. These Minkowskian solutions, at least in the case $m = 0$ where asymptotic completeness was proved to hold, are however related with corresponding solutions of the (*i.e.*, corresponding to the above mentioned φ_1, φ_2 in our case) in Schwarzschild spacetime by means of a relevant Moller operator. Unfortunately, in the proper Schwarzschild space, the wavefunctions φ_1 and φ_2 with compact support on $\mathcal{H}$ fail to be smooth in general, since they are *weak solutions* of the characteristic Cauchy problem [DK86-87] so that they do *not* belong to the space $\mathsf{S}(\mathscr{K})$ in general, making difficult the direct use of λ_{KW}. This is an annoying technical drawback of the approach followed in [KW91] which also affects the domain of the validity of the KMS property discussed below (see also the *Note added in proof* in [KW91]).

Now we are going to prove that, actually, such form of the two-point function can be extended in order to work on elements of $\mathsf{S}(\mathcal{H})$ and, with this extension, it defines a quasifree state on $\mathcal{W}(\mathsf{S}(\mathcal{H}))$. This result is by no means trivial, because the space $\mathsf{S}(\mathcal{H})$ contains the restrictions to the horizon of the very elements of $\mathsf{S}(\mathscr{K})$ (i.e. all the smooth wavefunctions with compact support on spacelike Cauchy surfaces). Our result, which is valid for the particular case of the Kruskal spacetime and for $m = 0$, is obtained thanks to the achievements recently presented [DR09]. On the other hand the space $\mathsf{S}(\mathcal{H})$ is just that used in the hypotheses of theorem 2.1, which assures the existence of the $*$-homomorphism $\imath$. As remarked at the end of the previous section, the procedure can be generalized in order to individuate an injective $*$-homomorphism from the algebra of observables on the whole Kruskal space to the algebras on $\Im_L^+$, $\mathcal{H}$ and $\Im^-$, that is $\imath_{\mathscr{K}} : \mathcal{W}(\mathsf{S}(\mathscr{K})) \to \mathcal{W}(\mathsf{S}(\Im_L^+)) \otimes \mathcal{W}(\mathsf{S}(\mathcal{H})) \otimes \mathcal{W}(\mathsf{S}(\Im^-))$. Therefore, the state on $\mathsf{S}(\mathscr{K})$ could be used, together with a couple of states on $\mathcal{W}(\Im^-)$ and on $\mathcal{W}(\Im_L^+)$ to induce a state on the whole algebra of observables $\mathcal{W}(\mathsf{S}(\mathscr{K}))$. This should provide an *existence theorem* for the Hartle-Hawking state on the whole Kruskal manifold $\mathscr{K}$. However we shall not attempt to give such an existence proof here and we rather focus attention on another physically interesting state, the so called *Unruh vacuum* defined in the submanifold $\mathscr{M}$ only. Nevertheless, even in this case we have to tackle the problem of the extension of the two-point function (42) to the whole space $\mathsf{S}(\mathcal{H})$. We shall prove the existence of such an extension that individuates a pure quasifree state on $\mathcal{W}(\mathsf{S}(\mathcal{H}))$, and which turns out to be KMS at inverse Hawking's temperature when restricting on a half horizon $\mathcal{W}(\mathsf{S}(\mathcal{H}^\pm))$ with respect to the Killing

displacements individuated by $X\restriction_{\mathcal{H}}$. The way we follow goes on through several steps. As a first step we introduce a relevant Hilbert space which we show later to be the one-particle space of the quasifree state we wish to define on $\mathcal{W}(\mathsf{S}(\mathcal{H}))$. The proof of the following proposition stays in the Appendix C. ¿From now on,

$$\mathscr{F}(\psi)(K,\theta,\phi) \doteq \int_{\mathbb{R}} \frac{e^{iKU}}{\sqrt{2\pi}}\psi(U,\theta,\phi)dU \ , \tag{43}$$

indicates the U-Fourier transform of ψ, also in the L^2 (Fourier-Plancherel) sense or even in distributional sense if appropriate, whose properties are essentially the same as the standard Fourier transform[2].

Proposition 3.2. *Let $\overline{(C_0^\infty(\mathcal{H};\mathbb{C}),\lambda_{KW})}$ be the Hilbert completion of the complex vector space $C_0^\infty(\mathcal{H};\mathbb{C})$ equipped with the Hermitian scalar product:*

$$\lambda_{KW}(\psi_1,\psi_2) \doteq \lim_{\epsilon\to 0^+} -\frac{r_S^2}{\pi}\int_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2} \frac{\overline{\psi_1(U_1,\theta,\phi)}\psi_2(U_2,\theta,\phi)}{(U_1-U_2-i\epsilon)^2} dU_1\wedge dU_2\wedge d\mathbb{S}^2 \ . \tag{44}$$

where $\mathcal{H}\equiv\mathbb{R}\times\mathbb{S}^2$ adopting the coordinate (U,θ,ϕ) over $\mathcal{H}$. Denote by $\widehat{\psi}_+ \doteq \mathscr{F}(\psi)\restriction_{\{K\geq 0,\theta,\phi\in\mathbb{S}^2\}}$ the restriction to positive values of K of the U-Fourier transform of $\psi\in C_0^\infty(\mathcal{H};\mathbb{C})$. The following facts hold.

(a) *The linear map*

$$C_0^\infty(\mathcal{H};\mathbb{C}) \ni \psi \mapsto \widehat{\psi}_+(K,\theta,\phi) \in L^2(\mathbb{R}_+\times\mathbb{S}^2, 2KdK\wedge r_S^2 d\mathbb{S}^2) \doteq \mathsf{H}_{\mathcal{H}}$$

is isometric and uniquely extends, by linearity and continuity, to a Hilbert space isomorphism of

$$F_{(U)} : \overline{(C_0^\infty(\mathcal{H};\mathbb{C}),\lambda_{KW})} \to \mathsf{H}_{\mathcal{H}} \ . \tag{45}$$

(b) *(Notice the appearance of $\mathbb{R}$ in place of $\mathbb{C}$):*

$$\overline{F_{(U)}\left(C_0^\infty(\mathcal{H};\mathbb{R})\right)} = \mathsf{H}_{\mathcal{H}} \ . \tag{46}$$

As a second step we should prove that there is a natural way to densely embed the space $\mathsf{S}(\mathcal{H})$ into the Hilbert space $\overline{(C_0^\infty(\mathcal{H};\mathbb{C}),\lambda_{KW})}$, that is into $\mathsf{H}_{\mathcal{H}}$, as requested by the definition of quasifree state. However, this is a very delicate stuff because the most straightforward way, consisting of computing the U-Fourier transform of $\psi\in\mathsf{S}(\mathcal{H})$ and checking that it belongs to $L^2(\mathbb{R}_+\times\mathbb{S}^2, 2KdK\wedge r_S^2 d\mathbb{S}^2) = \mathsf{H}_{\mathcal{H}}$, does not work in view of the too slow decay of ψ as $|U|\to+\infty$ obtained in [DR09] and embodied in the very definition of $\mathsf{S}(\mathcal{H})$. As a matter of fact, the idea we intend to exploit is to decompose every $\psi\in\mathsf{S}(\mathcal{H})$ as a sum of three functions, one compactly supported and the remaining ones supported in $\mathcal{H}^+$ and $\mathcal{H}^-$ respectively and to consider each function separately. The following proposition, whose proof stays in the Appendix C, analyses the features of the last two functions. It introduces some results too, which will be very useful later when dealing with the KMS property of the state λ_{KW}.
In the following $H^1(\mathcal{H}^\pm)_u$ are the Sobolev spaces of the functions $\psi:\mathbb{R}\times\mathbb{S}^2\to\mathbb{C}$, referred to the coordinate $(u,\omega)\in\mathbb{R}\times\mathbb{S}^2$ on $\mathcal{H}^\pm$, which are in $L^2(\mathbb{R}\times\mathbb{S}^2, du\wedge d\mathbb{S}^2)$ with their first (distributional) u derivative. Following the same proof as that valid for $C_0^\infty(\mathbb{R};\mathbb{C})$ and $H^1(\mathbb{R})$ (along the procedure of Theorem VIII.6 in [BR87] and employing sequences of regularizing functions which are constant in

[2]For some general properties, see Appendix C of [Mo08] with the caveat that, in this cited paper, $\mathscr{F}$ was inidcated by $\mathscr{F}_+$ and the angular coordinates (θ,ϕ) on the sphere were substituted by the complex ones $(z,\bar{z})$ obtained out of stereographic projection.

the angular variables), one establishes that $C_0^\infty(\mathcal{H}^\pm;\mathbb{C})$ is dense in the corresponding $H^1(\mathcal{H}^\pm)_u$. Every $\psi \in \mathsf{S}(\mathcal{H}^\pm)$ is an element of $H^1(\mathcal{H}^\pm)_u$ as it follows immediately from the definition of $\mathsf{S}(\mathcal{H}^\pm)$.

Proposition 3.3. *The following facts hold, where $u \doteq 2r_S \ln(U) \in \mathbb{R}$ and $u \doteq -2r_S \ln(-U) \in \mathbb{R}$ are the natural global coordinate covering $\mathcal{H}^+$ and $\mathcal{H}^-$, respectively, and $\mu(k)$ is the positive measure on $\mathbb{R}$:*

$$d\mu(k) \doteq 2r_S^2 \frac{ke^{2\pi r_S k}}{e^{2\pi r_S k} - e^{-2\pi r_S k}} dk \,. \tag{47}$$

(a) *If $\widetilde{\psi} = (\mathscr{F}(\psi))(k,\theta,\phi) = \widetilde{\psi}(k,\theta,\phi)$ denotes the u-Fourier transform of either $\psi \in C_0^\infty(\mathcal{H}^+;\mathbb{C})$ or $\psi \in C_0^\infty(\mathcal{H}^-;\mathbb{C})$ the maps*

$$C_0^\infty(\mathcal{H}^\pm;\mathbb{C}) \ni \psi \mapsto \widetilde{\psi} \in L^2(\mathbb{R}\times\mathbb{S}^2, d\mu(k)\wedge d\mathbb{S}^2)$$

are isometric when $C_0^\infty(\mathcal{H}^\pm;\mathbb{C})$ is equipped with the scalar product λ_{KW} and uniquely extend, by continuity, to Hilbert space isomorphisms:

$$F^{(\pm)}_{(u)} : \overline{C_0^\infty(\mathcal{H}^\pm;\mathbb{C})} \to L^2(\mathbb{R}\times\mathbb{S}^2, d\mu(k)\wedge d\mathbb{S}^2) \,, \tag{48}$$

where $\overline{C_0^\infty(\mathcal{H}^\pm;\mathbb{C})}$ are viewed as Hilbert subspaces of the Hilbert space $\overline{(C_0^\infty(\mathcal{H};\mathbb{C}),\lambda_{KW})}$.

(b) *The spaces $\mathsf{S}(\mathcal{H}^\pm)$ are naturally identified with real subspaces of $\overline{C_0^\infty(\mathcal{H};\mathbb{C})}$ in view of the following. If either $\{\psi_n\}_{n\in\mathbb{N}}, \{\psi'_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\mathcal{H}^+;\mathbb{R})$ or $\{\psi_n\}_{n\in\mathbb{N}}, \{\psi'_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\mathcal{H}^-;\mathbb{R})$ and, according to the case, both sequences $\{\psi_n\}_{n\in\mathbb{N}}, \{\psi'_n\}_{n\in\mathbb{N}}$ converge to the same $\psi \in \mathsf{S}(\mathcal{H}^\pm)$ in $H^1(\mathcal{H}^\pm)$, then both sequences are of Cauchy type in $\overline{(C_0^\infty(\mathcal{H};\mathbb{C}),\lambda_{KW})}$ and $\psi_n - \psi'_n \to 0$ in $\overline{(C_0^\infty(\mathcal{H};\mathbb{C}),\lambda_{KW})}$.*
The consequent identification of $\mathsf{S}(\mathcal{H}^\pm)$ with real subspaces of $\overline{C_0^\infty(\mathcal{H};\mathbb{C})}$ is such that:

$$F^{(\pm)}_{(u)} \restriction_{\mathsf{S}(\mathcal{H}^\pm)} = \mathscr{F} \restriction_{\mathsf{S}(\mathcal{H}^\pm)} \,, \tag{49}$$

where $\mathscr{F} : L^2(\mathbb{R}\times\mathbb{S}^2, du\wedge d\mathbb{S}^2) \to L^2(\mathbb{R}\times\mathbb{S}^2, dk\wedge d\mathbb{S}^2)$ is here the standard u-Fourier-Plancherel transform.

We are finally in place to specify how $\mathsf{S}(\mathcal{H})$ is embedded in $\mathsf{H}_\mathcal{H}$. Consider a compactly supported smooth function $\chi \in C^\infty(\mathcal{H})$, such that $\chi = 1$ in a neighbourhood of the bifurcation sphere $\mathcal{B} \in \mathcal{H}$. Every $\psi \in \mathsf{S}(\mathcal{H})$ can now be decomposed as the sum of three functions:

$$\psi = \psi_- + \psi_0 + \psi_+ \,, \quad \text{with } \psi_\pm = (1-\chi)\psi \restriction_{\mathcal{H}^\pm} \in \mathsf{S}(\mathcal{H}^\pm) \text{ and } \psi_0 = \chi\psi \in C_0^\infty(\mathcal{H};\mathbb{R}). \,, \tag{50}$$

Now define the map $\mathsf{K}_\mathcal{H} : \mathsf{S}(\mathcal{H}) \to \mathsf{H}_\mathcal{H} = L^2(\mathbb{R}_+\times\mathbb{S}^2, dK\wedge d\mathbb{S}^2)$ given by:

$$\mathsf{K}_\mathcal{H} : \mathsf{S}(\mathcal{H}) \ni \psi \mapsto F_{(U)}(\psi_-) + F_{(U)}(\psi_0) + F_{(U)}(\psi_+) \in \mathsf{H}_\mathcal{H} \,, \tag{51}$$

where $F_{(U)}(\psi_\pm)$ makes sense in view of the identification of $\mathsf{S}(\mathcal{H})$ with a real subspace of $\overline{(C_0^\infty(\mathcal{H};\mathbb{C}),\lambda_{KW})}$ as established in (b) of Proposition 3.3. The following proposition establishes that, in particular, $\mathsf{K}_\mathcal{H}$ is well-defined and injective and thus it identifies $\mathsf{S}(\mathcal{H})$ to a subspace of $\mathsf{H}_\mathcal{H}$. That identification enjoys a nice interplay with the symplectic form $\sigma_\mathcal{H}$. Furthermore we prove that, $\mathsf{K} : \mathsf{S}(\mathcal{H}) \to \mathsf{H}_\mathcal{H}$ is continuous if viewing $\mathsf{S}(\mathcal{H})$ as a normed space equipped with the norm

$$\|\psi\|^\chi_\mathcal{H} = \|(1-\chi)\psi\|_{H^1(\mathcal{H}^-)_u} + \|\chi\psi\|_{H^1(\mathcal{H})_U} + \|(1-\chi)\psi\|_{H^1(\mathcal{H}^+)_u} \tag{52}$$

where $\|\cdot\|_{H^1(\mathcal{H}^\pm)_u}$ and $\|\cdot\|_{H^1(\mathcal{H})_U}$ are the norms of the Sobolev spaces $H^1(\mathcal{H}^\pm)_u$ and $H^1(\mathcal{H})_U$ respectively. Notice that, the norms $\|\cdot\|^\chi_\mathcal{H}$ and $\|\cdot\|^{\chi'}_\mathcal{H}$ defined with respect of different decompositions generated by

χ and χ' are equivalent, in the sense that there are two positive real numbers C_1 and C_2 such that $C_1\|\psi\|^{\chi}_{\mathcal{H}} \leq \|\psi\|^{\chi'}_{\mathcal{H}} \leq C_2\|\psi\|^{\chi}_{\mathcal{H}}$ for all $\psi \in \mathsf{S}(\mathcal{H})$. The proof of such an equivalence is laborious and it is based on the decomposition of the various integrals appearing in the mentioned norms with respect to both the partitions of the unit $\chi, 1-\chi$ and $\chi', 1-\chi'$. Next one uses the triangular inequality iteratively and the fact that, passing to work in the only variable U, the norms $\|\cdot\|_{H^1(\mathcal{H}^\pm)_u}$ and $\|\cdot\|_{H^1(\mathcal{H})_U}$ are equivalent (because the Jacobian of the change of coordinates is strictly positive and bounded there) when restricting to work with the functions supported in any fixed open, relatively compact set $J \times \mathbb{S}^2$ where $J \subset \mathbb{R}$ is an interval for the variable U whose closure does not include 0. It is helpful, in the proof, noticing that $(\chi - \chi')$ is a compactly supported smooth function on the disjoint union of a pair of sets $J \times \mathbb{S}^2 \subset \mathcal{H}$ as mentioned above.
Because such an equivalence we will often write $\|\psi\|_{\mathcal{H}}$ in place of $\|\psi\|^{\chi}_{\mathcal{H}}$.

Proposition 3.4. *The linear map $\mathsf{K}_{\mathcal{H}} : \mathsf{S}(\mathcal{H}) \to \mathsf{H}_{\mathcal{H}}$ in (51) verifies the following properties:*
(a) *it is independent from the choice of the function χ used in the decomposition (50) of $\psi \in \mathsf{S}(\mathcal{H})$;*
(b) *it reduces to $F_{(U)}$ when restricting to $C_0^\infty(\mathcal{H};\mathbb{R})$;*
(c) *it satisfies*

$$\sigma_{\mathcal{H}}(\psi,\psi') = -2Im\langle \mathsf{K}_{\mathcal{H}}(\psi), \mathsf{K}_{\mathcal{H}}(\psi')\rangle_{\mathsf{H}_{\mathcal{H}}}\,, \quad \text{if } \psi,\psi' \in \mathsf{S}(\mathcal{H}); \tag{53}$$

(d) *it is injective;*
(e) *it holds $\overline{\mathsf{K}_{\mathcal{H}}(\mathsf{S}(\mathcal{H}))} = \mathsf{H}_{\mathcal{H}}$;*
(f) *it is continuous with respect to the norm $\|\cdot\|_{\mathcal{H}}$ defined in (52) (for every choice of the function χ). Consequently, there is $C > 0$ with:*

$$|\langle \mathsf{K}_{\mathcal{H}}(\psi), \mathsf{K}_{\mathcal{H}}(\psi')\rangle_{\mathsf{H}_{\mathcal{H}}}| \leq C^2\|\psi\|_{\mathcal{H}} \cdot \|\psi'\|_{\mathcal{H}} \quad \text{if } \psi,\psi' \in \mathsf{S}(\mathcal{H}).$$

The proof stays in the Appendix C. Collecting all the achievements and presenting some further result, we can now conclude stating the theorem about the state individuated by λ_{KW}.

Theorem 3.1. *The following facts hold referring to $(\mathsf{H}_{\mathcal{H}}, \mathsf{K}_{\mathcal{H}})$.*
(a) *The pair $(\mathsf{H}_{\mathcal{H}}, \mathsf{K}_{\mathcal{H}})$ is the one-particle structure for a quasi-free pure state $\omega_{\mathcal{H}}$ on $\mathcal{W}(\mathsf{S}(\mathcal{H}))$ uniquely individuated by the requirement that its two-point function coincides to the right-hand side of (44) when restricting to $C_0^\infty(\mathcal{H};\mathbb{R})$.*
(b) *The state $\omega_{\mathcal{H}}$ is invariant under the natural action of the one-parameter group of $*$-automorphisms generated by $X\restriction_{\mathcal{H}}$ and those generated by the Killing vectors of $\mathbb{S}^2$.*
(c) *The restriction of $\omega_{\mathcal{H}}$ to $\mathcal{W}(\mathsf{S}(\mathcal{H}^\pm))$ is a quasifree state $\omega^{\beta_H}_{\mathcal{H}^\pm}$ individuated by the one particle structure $(\mathsf{H}^{\beta_H}_{\mathcal{H}^\pm}, \mathsf{K}^{\beta_H}_{\mathcal{H}^\pm})$ with:*

$$\mathsf{H}^{\beta_H}_{\mathcal{H}^\pm} \doteq L^2(\mathbb{R}\times\mathbb{S}^2, d\mu(k)\wedge d\mathbb{S}^2) \quad \text{and } \mathsf{K}^{\beta_H}_{\mathcal{H}^\pm} \doteq \mathscr{F}\restriction_{\mathsf{S}(\mathcal{H}^\pm)} = F^{(\pm)}_{(u)}\restriction_{\mathsf{S}(\mathcal{H}^\pm)}. \tag{54}$$

(d) *The states $\omega^{\beta_H}_{\mathcal{H}^\pm}$ satisfy the KMS condition with respect to one-parameter group of $*$-automorphisms generated by, respectively, $\mp X\restriction_{\mathcal{H}}$, with Hawking's inverse temperature $\beta_H = 4\pi r_S$.*
(e) *If $\{\beta^{(X)}_\tau\}_{\tau\in\mathbb{R}}$ denotes the pull-back action on $\mathsf{S}(\mathcal{H}^-)$ of the one-parameter group generated by $X\restriction_{\mathcal{H}}$ (that is $(\beta_\tau(\psi))(u,\omega) = \psi(u-\tau,\omega)$), for every $\tau \in \mathbb{R}$ and every $\psi \in \mathsf{S}(\mathcal{H}^-)$ it holds:*

$$\mathsf{K}^{\beta_H}_{\mathcal{H}^-}\beta^{(X)}_\tau(\psi) = e^{i\tau\hat{k}}\mathsf{K}^{\beta_H}_{\mathcal{H}^-}\psi \tag{55}$$

where $\hat{k}$ is the k-multiplicative self-adjoint operator on $L^2(\mathbb{R}\times\mathbb{S}^2, d\mu(k)\wedge d\mathbb{S}^2)$. An analog statement holds for $\mathcal{H}^+$.

Proof. (a) In view of Proposition B.1 (and Lemma B.1), the wanted state is that uniquely associated with the real scalar product over $\mathsf{S}(\mathcal{H})$

$$\mu_{\mathcal{H}}(\psi,\psi') \doteq Re\langle \mathsf{K}_{\mathcal{H}}\psi, \mathsf{K}_{\mathcal{H}}\psi'\rangle_{\mathsf{H}_{\mathcal{H}}}\ , \tag{56}$$

and the one-particle structure is just $(\mathsf{H}_{\mathcal{H}}, \mathsf{K}_{\mathcal{H}})$. This holds true provided two conditions are fulfilled, as required in Proposition B.1. As a first condition it must be:

$$|\sigma_{\mathcal{H}}(\psi,\psi')|^2 \leq 4\mu_{\mathcal{H}}(\psi,\psi)\mu_{\mathcal{H}}(\psi',\psi')\ . \tag{57}$$

This fact is an immediate consequence of (c) in Proposition 3.4. The second condition to be satisfied is that $\overline{\mathsf{K}_{\mathcal{H}}(\mathsf{S}(\mathcal{H})) + i\mathsf{K}_{\mathcal{H}}(\mathsf{S}(\mathcal{H}))} = \mathsf{H}_{\mathcal{H}}$. Actually a stronger fact holds: $\overline{\mathsf{K}_{\mathcal{H}}(\mathsf{S}(\mathcal{H}))} = \mathsf{H}_{\mathcal{H}}$, because of (e) in Proposition 3.4. As a consequence, the state $\omega_{\mathcal{H}}$ is pure for (d) in Proposition B.1.
(c) We consider the case of $\mathcal{H}^+$ only, the other case being analogous. The state $\omega^{\beta_H}_{\mathcal{H}^+}$, which is the restriction of $\omega_{\mathcal{H}}$ to $\mathcal{W}(\mathsf{S}(\mathcal{H}^+))$, by definition is completely individuated by requiring that

$$\omega^{\beta_H}_{\mathcal{H}^+}\left(W_{\mathcal{H}^+}(\psi)\right) = e^{-\mu_{\mathcal{H}}(\psi,\psi)/2} \quad \text{for } \psi\in\mathsf{S}(\mathcal{H}^+).$$

One straightforwardly proves the following three facts. (i) If $\psi,\psi'\in\mathsf{S}(\mathcal{H}^+)$, then:

$$\begin{aligned}\mu_{\mathcal{H}}(\psi,\psi') &= Re\lambda_{KW}(\psi,\psi') = Re\langle F^{(+)}_{(u)}\psi, F^{(+)}_{(u)}\psi'\rangle_{\mathsf{H}^{\beta_H}_{\mathcal{H}^+}} = Re\langle\widetilde{\psi},\widetilde{\psi'}\rangle_{L^2(\mathbb{R}\times\mathbb{S}^2, d\mu(k)\wedge d\mathbb{S}^2)}\\ &= Re\langle \mathsf{K}^{\beta_H}_{\mathcal{H}^+}\psi, \mathsf{K}^{\beta_H}_{\mathcal{H}^+}\psi'\rangle_{\mathsf{H}^{\beta_H}_{\mathcal{H}^+}}\ ,\end{aligned}$$

due to (a) and (b) in Proposition 3.3. (ii) The condition (57) is valid also restricting to $\mathsf{S}(\mathcal{H}^+)$ (notice that $\sigma_{\mathcal{H}^+} = \sigma_{\mathcal{H}}\restriction_{\mathsf{S}(\mathcal{H}^+)\times\mathsf{S}(\mathcal{H}^+)}$). (iii) One has $\overline{\mathsf{K}^{\beta_H}_{\mathcal{H}^+}(\mathsf{S}(\mathcal{H}^+)) + i\mathsf{K}^{\beta_H}_{\mathcal{H}^+}(\mathsf{S}(\mathcal{H}^+))} = \mathsf{H}^{\beta_H}_{\mathcal{H}^+}$ by (a) and (b) of Proposition 3.3, noticing that $\mathsf{S}(\mathcal{H}^+) + i\mathsf{S}(\mathcal{H}^+) \supset C_0^\infty(\mathcal{H}^+;\mathbb{C})$. This concludes the proof because (i), (ii) and (iii) entail that $(\mathsf{H}^{\beta_H}_{\mathcal{H}^+}, \mathsf{K}^{\beta_H}_{\mathcal{H}^+})$ is the one-particle structure of $\omega^{\beta_H}_{\mathcal{H}^+}$ in view of Proposition B.1 (and Lemma B.1).
(b) If $\psi\in\mathsf{S}(\mathcal{H})$, the 1-parameter group of symplectomorphisms $\beta^{(X)}_\tau$ generated by X individuates $\beta^{(X)}_\tau(\psi)\in\mathsf{S}(\mathcal{H})$ such that $\beta^{(X)}_\tau(\psi)(U,\omega) = (\psi)\left(e^{-\tau/(4m)}U,\omega\right)$. This is an obvious consequence of $X=-\partial_u$ on $\mathcal{H}^+$, $X=\partial_u$ on $\mathcal{H}^-$ and $X=0$ on the bifurcation at $U=0$. Since $\beta^{(X)}$ preserves the symplectic form $\sigma_{\mathcal{H}}$, there must be a representation $\alpha^{(X)}$ of $\beta^{(X)}$, in terms of $*$-automorphisms of $\mathcal{W}(\mathsf{S}(\mathcal{H}))$. We do not need the explicit form of $\alpha^{(X)}$ now, rather let us focus on $\beta^{(X)}$ again. If $\psi\in C_0^\infty(\mathcal{H};\mathbb{R})$ one has immediately, from the definition of $F_{(U)}$, which coincides to $\mathsf{K}_{\mathcal{H}}$ in the considered case, that $\mathsf{K}_{\mathcal{H}}(\beta^{(X)}_\tau(\psi))(K,\theta,\phi) = e^{\tau/(4m)}\mathsf{K}_{\mathcal{H}}(\psi)\left(e^{\tau/(4m)}K,\theta,\phi\right)$.This result generalises to the case where $\psi\in\mathsf{S}(\mathcal{H})$ has support in the set $U>0$ (or $U<0$) as it can be proved along the lines of the proof of (b) of Proposition 3.3 employing a sequence of smooth functions ψ_n supported in $U>0$ (resp. $U<0$) which converges to ψ in the Sobolev topology of $H^1(\mathcal{H}^\pm, du)$ (see the mentioned proof), and using the fact that $\beta^{(X)}_\tau(\psi_n)$ converges to $\beta^{(X)}_\tau(\psi)$ in the same topology. Summing up, from the definition of $\mathsf{K}_{\mathcal{H}}$ (51), one finally gets that $\mathsf{K}_{\mathcal{H}}(\beta^{(X)}_\tau(\psi))(K,\theta,\phi) = \left(U^{(X)}_\tau\psi\right)(V,\theta,\phi) \doteq e^{\tau/(4m)}\mathsf{K}_{\mathcal{H}}(\psi)\left(e^{\tau/(4m)}K,\theta,\phi\right)$ for every $\psi\in\mathsf{S}(\mathcal{H})$ without further restrictions. Since $U^{(X)}_\tau$ is evidently an isometry of $L^2(\mathbb{R}_+\times\mathbb{S}^2, KdK\wedge d\mathbb{S}^2)$, in view of the definition of $\omega_{\mathcal{H}}$ it results that $\omega_{\mathcal{H}}(W_{\mathcal{H}}(\beta^{(X)}_\tau\psi)) = \omega_{\mathcal{H}}(W_{\mathcal{H}}(\psi))$ for all $\psi\in\mathsf{S}(\mathcal{H})$, and this is enough (by continuity and linearity) to conclude that ω is invariant under the action of the group of

$*$-automorphisms $\alpha^{(X)}$ induced by X. The proof for the Killing vectors of $\mathbb{S}^2$ is similar.
(d) and (e) In $\mathsf{S}(\mathcal{H}^-)$, the natural action one parameter group of isometries generated by $X\restriction_{\mathcal{H}^-}$ is $\beta^{(X)}_\tau : \psi \mapsto \beta^{(X)}_\tau(\psi)$ with $\beta^{(X)}_\tau(\psi)(u,\theta,\phi) \doteq \psi(u-\tau,\theta,\phi)$, for all $u,\tau,\in\mathbb{R}$, $(\theta,\phi)\in\mathbb{S}^2$ and for every $\psi\in\mathsf{S}(\mathcal{H}^-)$. As before, this is an obvious consequence of $X=\partial_u$ on $\mathcal{H}^-$. Since $\beta^{(X)}$ preserves the symplectic form $\sigma_{\mathcal{H}^-}$, there must be a representation $\alpha^{(X)}$ of $\beta^{(X)}$, in terms of $*$-automorphisms of $\mathcal{W}(\mathsf{S}(\mathcal{H}^-))$. Let us prove that $\alpha^{(X)}$ is unitarily implemented in the GNS representation of $\omega^{\beta_H}_{\mathcal{H}^-}$. To this end, we notice that β is unitarily implemented in the one-particle space of $\omega^{\beta_H}_{\mathcal{H}^-}$, $\mathsf{H}_{\mathcal{H}^-}$ by the strongly-continuous one-parameter group of unitary operators V_τ such that $\left(V_\tau\widetilde{\psi}\right)(k,\theta,\phi) = e^{ik\tau}\widetilde{\psi}(k,\theta,\phi)$ describing the time displacements with respect to the Killing vector ∂_u. Thus the self-adjoint generator of V is $h : Dom(\hat{k}) \subset L^2(\mathbb{R}\times\mathbb{S}^2, d\mu(k)\wedge d\mathbb{S}^2) \to L^2(\mathbb{R}\times\mathbb{S}^2, d\mu(k)\wedge d\mathbb{S}^2)$ with $\hat{k}(\phi)(k,\theta,\phi) = k\phi(k,\theta,\phi)$ and

$$Dom(\hat{k}) \doteq \left\{ \phi\in L^2(\mathbb{R}\times\mathbb{S}^2, d\mu(k)\wedge d\mathbb{S}^2) \;\middle|\; \int_{\mathbb{R}\times\mathbb{S}^2} |k\phi(k,\theta,\phi)|^2 d\mu(k)\wedge d\mathbb{S}^2 < +\infty \right\} .$$

By direct inspection, employing the found form for V and exploiting

$$\omega^{\beta_H}_{\mathcal{H}^-}\left(W_{\mathcal{H}^-}(\psi)\right) = e^{-\frac{1}{2}\langle\widetilde{\psi},\widetilde{\psi}\rangle_{L^2(\mathbb{R}\times\mathbb{S}^2, d\mu(k)\wedge d\mathbb{S}^2)}} ,$$

one sees that $\omega^{\beta_H}_{\mathcal{H}^-}$ is invariant under $\alpha^{(X)}$, so that it must admit a unitary implementation [Ar99]. (As a matter of fact it is obtained by tensorialization of V_τ.) To establish that the $\alpha^{(X)}$-invariant quasifree state $\omega^{\beta_H}_{\mathcal{H}^-}$ over the Weyl algebra $\mathcal{W}(\mathsf{S}(\mathcal{H}^-))$ is a KMS state with inverse temperature $\beta_H = 4\pi r_S$ with respect to $\alpha^{(X)}$ which, in turn, is unitarily implemented by $V=\{\exp\{i\tau\hat{k}\}\}_{\tau\in\mathbb{R}}$ in the one particle space $\mathsf{H}^{\beta_H}_{\mathcal{H}^-}$, it is equivalent (see proposition B.3 in the appendix) to prove that $\mathsf{K}^{\beta_H}_{\mathcal{H}^-}(\mathsf{S}(\mathcal{H}^-)) \subset Dom\left(e^{-\frac{1}{2}\beta\hat{k}}\right)$ and $\langle e^{i\tau\hat{k}}\mathsf{K}^{\beta_H}_{\mathcal{H}^-}\psi, \mathsf{K}^{\beta_H}_{\mathcal{H}^-}\psi'\rangle = \langle e^{-\beta_H\hat{k}/2}\mathsf{K}^{\beta_H}_{\mathcal{H}^-}\psi', e^{-\beta_H\hat{k}/2}e^{i\tau\hat{k}}\mathsf{K}^{\beta_H}_{\mathcal{H}^-}\psi\rangle$. These requirements can be proved to hold by direct inspection using the fact that $\mathsf{K}^{\beta_H}_{\mathcal{H}^-}(\psi) = \widetilde{\psi}\in L^2(\mathbb{R}\times\mathbb{S}^2, d\mu(k)\wedge d\mathbb{S}^2)$, the explicit form (47) of the measure $\mu(k)$ and the fact that $\overline{\widetilde{\psi}(-k,\omega)} = \widetilde{\psi}(k,\omega)$ if $\psi\in\mathsf{S}(\mathcal{H}^-)$ because ψ is real-valued. The case of $\mathcal{H}^+$ is strongly analogous with the only difference that $X\restriction_{\mathcal{H}^+} = -\partial_u$. $\square$

We conclude by stating without proof, since it is straightforward in this case, the following proposition concerning the natural X-invariant vacuum states of $\mathcal{H}^-$ and $\mathcal{H}_{ev}$(actually, a *quasifree regular ground states* in the sense of [KW91]).

Proposition 3.5. *If $\mathsf{K}_{\mathcal{H}^-} : \mathsf{S}(\mathcal{H}^-) \to \mathsf{H}_{\mathcal{H}^-} \doteq L^2(\mathbb{R}_+\times\mathbb{S}^2, 2kdk\wedge d\mathbb{S}^2)$ denotes the standard u-Fourier-Plancherel transform, followed by the restriction to $\mathbb{R}_+\times\mathbb{S}^2$, the following facts hold.*

(a) *The pair $(\mathsf{H}_{\mathcal{H}^-}, \mathsf{K}_{\mathcal{H}^-})$ is the one-particle structure for a quasi-free pure state $\omega_{\mathcal{H}^-}$ on $\mathcal{W}(\mathsf{S}(\mathcal{H}^-))$.*

(b) *The state $\omega_{\mathcal{H}^-}$ is invariant under the natural action of the one-parameter group of $*$-automorphisms generated by $X\restriction_{\mathcal{H}^-}$ and those generated by the Killing vectors of $\mathbb{S}^2$.*

Replacing the u-Fourier-Plancherel transform with the v-Fourier-Plancherel transform and analogous state $\omega_{\mathcal{H}_{ev}}$ is defined, which is is invariant under the natural action of the one-parameter group of $*$-automorphisms generated by $X\restriction_{\mathcal{H}_{ev}}$ and those generated by the Killing vectors of $\mathbb{S}^2$.

3.3. *The vacuum state $\omega_{\Im^-}$ on $\mathcal{W}(\Im^-)$.* We now introduce a relevant vacuum state $\omega_{\Im^-}$ on $\mathcal{W}(\Im^-)$ which is invariant with respect to u-displacements and under the isometries of $\mathbb{S}^2$. The idea is, in principle, the same as that used to define $\omega_{\mathcal{H}}$, i.e. starting from a two-point function similar to λ_{KW}, with the important difference that, now, the coordinate U is replaced by v. As a starting point we state the following proposition whose proof is, *mutatis mutandis*, identical to that of proposition 3.2.

Proposition 3.6. *Consider the Hilbert completion* $\overline{\left(C_0^\infty(\Im^-_R;\mathbb{C}),\lambda_{\Im^-}\right)}$ *of the complex vector space* $C_0^\infty(\Im^-;\mathbb{C})$ *equipped with the Hermitian scalar product:*

$$\lambda_{\Im^-}(\psi_1,\psi_2) \doteq \lim_{\epsilon\to 0^+} -\frac{1}{\pi}\int_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2} \frac{\overline{\psi_1(v_1,\theta,\phi)}\psi_2(v_2,\theta,\phi)}{(v_1-v_2-i\epsilon)^2} dv_1\wedge dv_2\wedge d\mathbb{S}^2\ , \tag{58}$$

where $\Im^- \equiv \mathbb{R}\times\mathbb{S}^2$ *adopting the coordinate* (v,ω) *over* $\Im^-$*. The following facts hold.*

(a) *If* $\widehat{\psi}_+(k,\theta,\phi) \doteq \mathscr{F}(\psi)\restriction_{\{k\geq 0,(\theta,\phi)\in\mathbb{S}^2\}}(k,\theta,\phi)$ *denotes the* v*-Fourier transform of* $\psi\in C_0^\infty(\Im^-;\mathbb{C})$ *(see the Appendix C in [Mo08]) restricted to* $k\in\mathbb{R}_+$*, the map*

$$C_0^\infty(\Im^-;\mathbb{C}) \ni \psi \mapsto \widehat{\psi}_+(k,\theta,\phi) \in L^2(\mathbb{R}_+\times\mathbb{S}^2, 2kdk\wedge d\mathbb{S}^2) =: \mathsf{H}_{\Im^-}$$

is isometric and uniquely extends, by continuity, to a Hilbert space isomorphism of

$$F_{(v)} : \overline{(C_0^\infty(\Im^-;\mathbb{C}),\lambda_{\Im^-})} \to \mathsf{H}_{\Im^-}\ . \tag{59}$$

(b) *(Notice the appearance of* $\mathbb{R}$ *in place of* $\mathbb{C}$*):*

$$\overline{F_{(v)}\left(C_0^\infty(\Im^-;\mathbb{R})\right)} = \mathsf{H}_{\Im^-}\ . \tag{60}$$

We have now to state and prove the corresponding of the Proposition 3.4, establishing that there is a state $\omega_{\Im^-}$ which is completely determined by $\lambda_{\Im^-}$ and such that the one-particle space coincides with $\mathsf{H}_{\Im^-}$. The delicate point is to construct the corresponding of the $\mathbb{R}$-linear map $\mathsf{K}_{\mathcal{H}}$, which now has to be thought of as $\mathsf{K}_{\Im^-} : \mathsf{S}(\Im^-) \to \mathsf{H}_{\Im^-}$. Notice that $\mathsf{K}_{\Im^-}$ cannot be defined as the v-Fourier transform (neither the Fourier-Plancherel transform), since the elements of $\mathsf{S}(\Im^-)$ do not decay rapidly enough. Similarly to that done before, a suitable extension with respect to the topology of $\overline{(C_0^\infty(\Im^-;\mathbb{C}),\lambda_{\Im^-})}$ is necessary. To this end, we are going to prove that the real subspace of the functions of $\mathsf{S}(\Im^-)$ supported in the region $v>0$ can be naturally identified with a real subspace of $\overline{(C_0^\infty(\Im^-;\mathbb{C}),\lambda_{\Im^-})}$. This is stated in the following proposition whose proof is in the Appendix C. In the following, we pass to the coordinate over $\mathbb{R}$ defined by $x\doteq\sqrt{v}$ if $v\geq 0$ and $x\doteq -\sqrt{-v}$ if $v\leq 0$. Then, adopting the coordinate x over the factor $\mathbb{R}$ of $\Im^-\equiv\mathbb{R}\times\mathbb{S}^2$, the Sobolev space $H^1(\Im^-)_x$, is that of the functions which belong to $L^2(\mathbb{R}\times\mathbb{S}^2, dx\wedge d\mathbb{S}^2)$ with their (distributional) first x derivative. Notice that, in view of the very definition of $\mathcal{S}(\Im^-)$, if ψ is supported in the subset of $\Im^-$ with $v<0$ (i.e. $x<0$) and $\psi\in\mathsf{S}(\Im^-)$, then $\psi\in H^1(\Im^-)_x$.

Proposition 3.7. *If* $\psi\in\mathsf{S}(\Im^-)$ *and supp* $(\psi)\subset\mathbb{R}^*_-\times\mathbb{S}^2$ *(where* $\mathbb{R}^*_-\doteq(-\infty,0)$*), the following holds.*

(a) *Every sequence* $\{\psi_n\}_{n\in\mathbb{N}}\subset C_0^\infty(\mathbb{R}^*_-\times\mathbb{S}^2;\mathbb{R})$ *such* $\psi_n\to\psi$ *as* $n\to+\infty$ *in* $H^1(\Im^-)_x$ *is necessarily of Cauchy type in* $\overline{(C_0^\infty(\Im^-;\mathbb{C}),\lambda_{\Im^-})}$*.*

(b) *There is* $\{\psi_n\}_{n\in\mathbb{N}}\subset C_0^\infty(\mathbb{R}^*_-\times\mathbb{S}^2;\mathbb{R})$ *such* $\psi_n\to\psi$ *as* $n\to+\infty$ *in* $H^1(\Im^-)_x$ *and, if* $\{\psi'_n\}_{n\in\mathbb{N}}\subset C_0^\infty(\mathbb{R}^*_-\times\mathbb{S}^2;\mathbb{R})$ *converges to the same* ψ *in* $H^1(\Im^-)_x$*, then* $\psi'_n-\psi_n\to 0$ *in* $\overline{(C_0^\infty(\Im^-;\mathbb{C}),\lambda_{\Im^-})}$*.*
As a consequence every $\psi\in\mathsf{S}(\Im^-)$ *with supp*$(\psi)\subset\mathbb{R}^*_+\times\mathbb{S}^2$ *can be naturally identified with a corresponding element of* $\overline{(C_0^\infty(\Im^-;\mathbb{C}),\lambda_{\Im^-})}$*, which we indicate with the same symbol* ψ*.*
With this identification it holds

$$F_{(v)}\restriction_{\mathsf{S}(\Im^-)} = \Theta\cdot\mathscr{F}\restriction_{\mathsf{S}(\Im^-)}\ , \tag{61}$$

and, for $\psi,\psi'\in\mathsf{S}(\Im^-)$*,*

$$\lambda_{\Im^-}(\psi,\psi') = \int_{\mathbb{R}_+\times\mathbb{S}^2}\left(\overline{\mathscr{F}(\psi')}\,(I+C)\,\mathscr{F}(\psi)\right)(h,\omega)\,2hdh\wedge d\mathbb{S}^2(\omega)\ , \tag{62}$$

where, $\Theta(h) = 0$ if $h \leq 0$ and $\Theta(h) = 1$ otherwise, $\mathscr{F} : L^2(\mathbb{R} \times \mathbb{S}^2, dx \wedge d\mathbb{S}^2) \to L^2(\mathbb{R} \times \mathbb{S}^2, dh \wedge d\mathbb{S}^2)$ is the x-Fourier-Plancherel transform ($x \doteq -\sqrt{-v}$ if $v \leq 0$ and $x \doteq \sqrt{v}$ if $v \geq 0$) and C is the standard complex conjugation.

We are in place to define the map $\mathsf{K}_{\Im^-}$ similarly to that done for $\mathsf{K}_{\mathcal{H}}$. Let χ be a non-negative smooth function on $\Im^-$ whose support is contained in $\mathbb{R}^*_- \times \mathbb{S}^2$, and such that $\eta(v, \theta, \phi) = 1$ for $v < v_0 < 0$. Consider $\psi \in \mathsf{S}(\Im^-)$ and decompose it as:

$$\psi = \psi_0 + \psi_- \text{ , where } \psi_0 = (1 - \eta)\psi \text{ and } \psi_- = \eta\psi \in \mathsf{S}(\Im^-) \tag{63}$$

Obviously, $\psi_0 \in C_0^\infty(\Im^-; \mathbb{R})$ and supp $(\psi_-) \subset \mathbb{R}^*_- \times \mathbb{S}^2$, where $\mathbb{R}^*_-$ is referred to the coordinate v on $\mathbb{R}$. Finally, define

$$\mathsf{K}_{\Im^+}(\psi) \doteq F_{(v)}(\psi_0) + F_{(v)}(\psi_-) \, , \quad \forall \psi \in \mathsf{S}(\Im^-) \, , \tag{64}$$

where, ψ_- in the second term is considered an element of $\overline{(C_0^\infty(\Im^-; \mathbb{C}), \lambda_{\Im^-})}$ in view of Proposition 3.7. Finally $\mathsf{K}_{\Im^+} : \mathsf{S}(\Im^-) \to \mathsf{H}_{\Im^-}$ is continuous when the domain is equipped with the norm

$$\|\psi\|^\eta_{\Im^-} = \|\psi_-\|_{H^1(\Im^-)_x} + \|\psi_0\|_{H^1(\Im^-)_v} \tag{65}$$

where $\|\cdot\|_{H^1(\Im^-)_x}$ and $\|\cdot\|_{H^1(\Im^-)_v}$ are the norms of the Sobolev spaces $H^1(\Im^-)_x$ and $H^1(\Im^-)_v$ respectively, the latter being that of the functions which belong to $L^2(\mathbb{R} \times \mathbb{S}^2, dv \wedge d\mathbb{S}^2)$ with their (distributional) first v derivative. Notice that, as before, different η and η' produce equivalent norms $\|\cdot\|^\eta_{\Im^-}$ and $\|\cdot\|^{\eta'}_{\Im^-}$, for this reason we shall drop the index η in $\|\cdot\|^\eta_{\Im^-}$ if not strictly necessary. The following proposition, whose proof relying upon Propositions 3.6 and 3.7, is very similar to that of Proposition 3.4 and will be discussed in the Appendix C, states that the definition of $\mathsf{K}_{\Im^-}$ given above is meaningful.

Proposition 3.8. *The linear map $\mathsf{K}_{\Im^-} : \mathsf{S}(\Im^-) \to \mathsf{H}_{\Im^-}$ in (64) enjoys the following properties:*

(a) *it is well-defined, i.e., it is independent from the chosen decomposition (63) for a fixed $\psi \in \mathsf{S}(\Im^-)$;*

(b) *it reduces to $F_{(v)}$ when restricting to $C_0^\infty(\Im^-; \mathbb{R})$;*

(c) *it satisfies:*

$$\sigma_{\Im^-}(\psi, \psi') = -2Im\langle \mathsf{K}_{\Im^-}(\psi), \mathsf{K}_{\Im^-}(\psi')\rangle_{\mathsf{H}_{\Im^-}} \, , \quad \text{if } \psi, \psi' \in \mathsf{S}(\Im^-); \tag{66}$$

(d) *it is injective;*

(e) *it holds $\overline{\mathsf{K}_{\Im^-}(\mathsf{S}(\Im^-))} = \mathsf{H}_{\Im^-}$;*

(f) *it is continuous with respect to the norm $\|\cdot\|_{\Im^-}$ defined in (65) (for every choice of the function η). Consequently, there is a constant $C > 0$ such that:*

$$|\langle \mathsf{K}_{\Im^-}(\psi), \mathsf{K}_{\Im^-}(\psi')\rangle_{\mathsf{H}_{\Im^-}}| \leq C^2 \|\psi\|_{\Im^-} \cdot \|\psi'\|_{\Im^-} \quad \text{if } \psi, \psi' \in \mathsf{S}(\Im^-).$$

We can now define the state $\omega_{\Im^-}$ collecting all the achieved results.

Theorem 3.2. *The following facts hold referring to $(\mathsf{H}_{\Im^-}, \mathsf{K}_{\Im^-})$.*

(a) *The pair $(\mathsf{H}_{\Im^-}, \mathsf{K}_{\Im^-})$ is the one-particle structure for a quasi-free pure state $\omega_{\Im^-}$ on $\mathcal{W}(\mathsf{S}(\Im^-))$ which is uniquely determined by the requirement that its two-point function coincides to the right-hand side of (58) when restricting to $C_0^\infty(\Im^-; \mathbb{R})$.*

(b) *The state $\omega_{\Im^-}$ is invariant under the natural action of the one-parameter group of $*$-automorphisms generated by $X\restriction_{\Im^-}$ and those generated by the Killing vectors of $\mathbb{S}^2$.*

(c) *If $\{\beta_\tau^{(X)}\}_{\tau\in\mathbb{R}}$ denotes the pull-back action on $\mathsf{S}(\Im^-)$ of the one-parameter group generated by $X\restriction_{\Im^-}$ (that is $(\beta_\tau(\psi))(v,\omega)=\psi(v-\tau,\omega)$), for every $\tau\in\mathbb{R}$ and every $\psi\in\mathsf{S}(\Im^-)$ it holds:*

$$\mathsf{K}_{\Im^-}\beta_\tau^{(X)}(\psi)=e^{i\tau\hat{h}}\mathsf{K}_{\Im^-}\psi \tag{67}$$

where $\hat{h}$ is the h-multiplicative self-adjoint operator on $\mathsf{H}_{\Im^-}=L^2(\mathbb{R}\times\mathbb{S}^2,2hdh\wedge d\mathbb{S}^2)$.
Analogous statements hold for $\mathcal{W}(\mathsf{S}(\Im^\pm_L))$ and for $\mathcal{W}(\mathsf{S}(\Im^+))$ and corresponding analogous states $\omega_{\Im^\pm_L}$ and $\omega_{\Im^+}$ exist.

Proof. The proof of (a) and (b) is essentially identical to that of the corresponding items in Theorem 3.1. In particular, the proof of item (b) is a trivial consequence of Lemma C.3. □

4 The extended Unruh state ω_U.

When a spherically-symmetric black hole forms, the metric of the spacetime outside the event horizon (and that inside the region containing the singularity away from the collapsing matter) must be of Schwarzschild type due to the Birkhoff theorem (see [WR96, Wa94] for a more mathematically detailed discussion) and a model of this spacetime can be realized selecting a relevant subregion of $\mathscr{M}$ in Kruskal manifold, *i.e.*, the so called regions I and II of the Kruskal diagram as depicted in chapter 6.4 of [Wa84]. A quantum state that accounts for Hawking's radiation was heuristically defined by Unruh in $\mathscr{M}$, using a mode decomposition approach [Un76, Ca80, Wa94]. A rigorous, though indirect, definition of ω_U restricted to $\mathscr{W}$ has been subsequently proposed by Kay and Dimock in terms of S-matrix interpretation assuming (and proving in the massless case) asymptotic completeness [DK86-87]. It is imperative to stress that, in the last cited papers, the restriction to the static region $\mathscr{W}$ was crucial to employ the mathematical techniques used to describe the scattering in stationary spacetimes and, as a byproduct, the algebras $\mathcal{W}(\mathsf{S}(\mathcal{H}))$ and $\mathcal{W}(\mathsf{S}(\Im^-))$ were introduced and used with some differences with respect to our approach.

4.1. *The states ω_U, ω_B and their basic properties.* We are in place to give a rigorous definition of the Unruh state employing the technology introduced previously. Our definition is valid for the whole region $\mathscr{M}$ (not only in the static region $\mathscr{W}$), it does not require any S-matrix interpretation, nor formal manipulation of distributional modes as in the more traditional presentations (see [Ca80]). Our prescription is a possible rigorous version of Unruh's original idea according to which the state is made of thermal modes propagating in $\mathscr{M}$ from the white hole and of vacuum modes entering $\mathscr{M}$ from $\Im^-$. Together the Unruh state ω_U on $\mathcal{W}(\mathsf{S}(\mathscr{M}))$ we define the *Boulware vacuum*, ω_B on $\mathcal{W}(\mathsf{S}(\mathscr{W}))$, since it will be useful later.

Definition 4.1. Consider the states $\omega_{\Im^+}$, $\omega_{\Im^-}$, $\omega_{\mathcal{H}}$ and $\omega_{\mathcal{H}_{ev}}$ as defined in Theorem 3.2, Theorem l3.1, Proposition 3.5.
The **Unruh state** is the unique state $\omega_U:\mathcal{W}(\mathsf{S}(\mathscr{M}))\to\mathbb{C}$ such that:

$$\omega_U\left(W_{\mathscr{M}}(\varphi)\right)=\omega_{\mathcal{H}}\left(W_{\mathcal{H}}(\varphi_{\mathcal{H}})\right)\omega_{\Im^-}\left(W_{\Im^-}(\varphi_{\Im^-})\right)\quad\text{for all }\varphi\in\mathsf{S}(\mathscr{M}). \tag{68}$$

The **Boulware vacuum** is the unique state $\omega_B:\mathcal{W}(\mathsf{S}(\mathscr{W}))\to\mathbb{C}$ such that:

$$\omega_B\left(W_{\mathscr{W}}(\varphi)\right)=\omega_{\mathcal{H}_{ev}}\left(W_{\mathcal{H}_{ev}}(\varphi_{\mathcal{H}_{ev}})\right)\omega_{\Im^+}\left(W_{\Im^+}(\varphi_{\Im^+})\right)\quad\text{for all }\varphi\in\mathsf{S}(\mathscr{W}). \tag{69}$$

In other words $\omega_U \doteq (\imath)^* (\omega_{\mathcal{H}} \otimes \omega_{\Im^-})$ and $\omega_B \doteq (\imath^+)^* (\omega_{\mathcal{H}^+} \otimes \omega_{\Im^-})$.

We study now the interplay between ω_U, ω_B and the action of X. The Killing field X individuates a one-parameter group of (active) symplectomorphisms $\{\beta_t^{(X)}\}_{t\in\mathbb{R}}$ on $\mathsf{S}(\mathscr{M})$ which leaves $\mathsf{S}(\mathscr{M})$ and $\mathsf{S}(\mathscr{W})$ invariant. As X is defined on the whole manifold $\widetilde{\mathscr{M}}$, similarly, a one-parameter group of (active) symplectomorphisms are induced on $\mathsf{S}(\Im^\pm)$, $\mathsf{S}(\mathcal{H})$, $\mathsf{S}(\mathcal{H}^-)$, $\mathsf{S}(\mathcal{H}_{ev})$ and, henceforth, we shall use the same symbol $\{\beta_t^{(X)}\}_{t\in\mathbb{R}}$ for all these groups. In turn, $\{\beta_t^{(X)}\}_{t\in\mathbb{R}}$ induces a one-parameter group of $*$-automorphisms, $\{\alpha_t^{(X)}\}_{t\in\mathbb{R}}$, on $\mathcal{W}(\mathscr{M})$ unambiguously individuated by the requirement:

$$\alpha_t^{(X)} (W_{\mathscr{M}}(\varphi)) \doteq W_{\mathscr{M}} \left(\beta_t^{(X)}(\varphi)\right) , \quad \text{for all } \varphi \in \mathsf{S}(\mathscr{M}). \tag{70}$$

Whenever $\{\alpha_t^{(X)}\}_{t\in\mathbb{R}}$ acts on $\mathcal{W}(\mathsf{S}(\mathscr{M}))$ and $\mathcal{W}(\mathsf{S}(\mathscr{W}))$, it leaves these algebras fixed and the second one in particular represents the time-evolution, with respect to the Schwarzschild time, of the observables therein. Analogous one-parameter groups of $*$-automorphisms, indicated with the same symbol, are defined on $\mathcal{W}(\Im^\pm)$, $\mathcal{W}(\mathcal{H})$, $\mathcal{W}(\mathcal{H}^-)$, $\mathcal{W}(\mathcal{H}_{ev})$ by X. The following relations hold true, for all $t \in \mathbb{R}$ and $\varphi \in \mathsf{S}(\mathscr{M})$:

$$\Gamma \left(\beta_t^{(X)}(\varphi)\right) = \left(\beta_t^{(X)}(\varphi_{\mathcal{H}}) \, , \, \beta_t^{(X)}(\varphi_{\Im^-})\right) . \tag{71}$$

The same result is valid replacing $\mathscr{M}$ with $\mathscr{W}$, $\mathcal{H}$ with $\mathcal{H}^-$ or $\mathcal{H}_{ev}$ and, in the second case $\Im^-$ with $\Im^+$, so that Γ is replaced by Γ_- or Γ_+ respectively, and $\imath$ is replaced by $\imath^-$ or $\imath^+$ correspondingly. The proof is a consequence of the invariance of the Klein-Gordon equation under $\beta^{(X)}$. Similar identities hold concerning the remaining Killing $\mathbb{S}^2$-symmetries of $\mathscr{M}$ and $\mathscr{W}$.

Proposition 4.1. *The following facts hold,*
(a) *ω_U and ω_B are invariant under the action of $\{\alpha_t^{(X)}\}_{t\in\mathbb{R}}$ and under that of the remaining Killing $\mathbb{S}^2$-symmetries of the metric of $\mathscr{M}$ and $\mathscr{W}$ respectively.*
(b) *ω_B is a* **regular quasifree ground state** *(i.e., the unitary one-parameter groups implementing $\{\alpha_t\}_{t\in\mathbb{R}}$ are strongly continuous and the self-adjoint generators have positive spectrum with no zero eigenvalues in the one-particle spaces) and so it coincides to the analogous vacuum state defined with respect to the past null boundary of $\mathscr{W}$, i.e., $\omega_B = (\imath^-)^* (\omega_{\mathcal{H}^-} \otimes \omega_{\Im^-})$.*

Proof. (a) It follows immediately from (70), (71) as well as from the analogy for $\mathscr{W}$ together with the definitions (68) and (69), taking into account that the states $\omega_{\mathcal{H}}$, $\omega_{\Im^-}$, $\omega_{\mathcal{H}_{ev}}$, $\omega_{\Im^+}$, are invariant under the action of $\{\alpha_t^{(X)}\}_{t\in\mathbb{R}}$ and under the action of the remaining Killing symmetries, as established in theorems 3.1, 3.2 and proposition 3.5.
(b) By direct inspection one sees that, in the GNS representation space of the quasifree states, ω_B and $(\imath^-)^* (\omega_{\mathcal{H}^-} \otimes \omega_{\Im^-})$ are quasifree regular ground states with respect to $\{\alpha_t\}_{t\in\mathbb{R}}$. Thus *Kay's uniqueness theorem* [Ka79] implies that $\omega_B = (\imath^-)^* (\omega_{\mathcal{H}^-} \otimes \omega_{\Im^-})$. $\square$

If $\varphi, \varphi' \in \mathsf{S}(\mathscr{W})$, the function $F_{\varphi,\varphi'}(t) \doteq \omega_H \left(W_{\mathscr{W}}(\varphi)\alpha^{(X)} (W_{\mathscr{W}}(\varphi))\right)$ decomposes in a product

$$F_{\varphi,\varphi'}(t) = F^{(\beta_H)}_{\varphi,\varphi'}(t) F^{(\infty)}_{\varphi,\varphi'}(t) \, .$$

Referring to the Schwarzschild-time evolution, the first factor fulfils the KMS requirements (see definition B.2), whereas the second factor enjoys the properties of a ground state two-point function: it can be extended to an analytic functions for $Imt > 0$ which is continuous and bounded in $Imt \geq 0$ and tends to 1

as $\mathbb{R} \ni t \to \pm\infty$. The term $F^{(\beta_H)}_{\varphi,\varphi'}(t)$ (which evaluates only the part $\varphi_{\mathcal{H}^-}$ and $\varphi'_{\mathcal{H}^-}$ of the wavefunctions) represents the components of the wavefunction bringing the thermal radiation entering $\mathscr{W}$ through the white hole, the latter (which evaluates only the components $\varphi_{\mathfrak{I}^-}$ and $\varphi'_{\mathfrak{I}^-}$ of the wavefunctions) represents the part of the wavefunction associated with the Boulware vacuum.

4.2 On the Hadamard property

Consider a quasifree state ω on the Weyl algebra of the real Klein-Gordon scalar field $\mathcal{W}(\mathscr{N})$ for a globally hyperbolic spacetime $(\mathscr{N}, g)$ and let $(\mathsf{H}_\omega, \mathsf{K}_\omega)$ its one-particle structure determining the Fock GNS representation $(\mathfrak{H}_\omega, \Pi_\omega, \Psi_\omega)$ of ω. Finally introduce the field operators $\Phi_\omega(f)$ as discussed in sec. 2.2. The **two-point function** of ω is the bilinear form $\lambda : \mathsf{S}(\mathscr{N}) \times \mathsf{S}(\mathscr{N}) \to \mathbb{C}$ where $\lambda_\omega(\psi, \psi') \doteq \langle \mathsf{K}_\omega \psi, \mathsf{K}_\omega \psi' \rangle_{\mathsf{H}_\omega}$. Equivalently, following Sec. 2.2 and the Appendix B, it turns out that

$$\lambda_\omega(\psi, \psi') = \langle \Psi_\omega, \Phi_\omega(f)\Phi_\omega(f')\Psi_\omega \rangle \ , \quad \psi = E_{P_g} f \ , \quad \psi' = E_{P_g} f' \ ,$$

where the expectation value of the product of two field operators $\Phi_\omega(f)$ and $\Phi_\omega(f')$ is computed with respect to the cyclic vector Ψ_ω of the GNS representation of ω and $E_{P_g} : C_0^\infty(\mathscr{N}; \mathbb{C}) \to \mathsf{S}(\mathscr{N})$ is the causal propagator. Therefore a **four-smeared two-point function** can equivalently be defined as a bilinear map $\Lambda_\omega : C^\infty(\mathscr{N}; \mathbb{R}) \times C^\infty(\mathscr{N}; \mathbb{R}) \to \mathbb{C}$ associated with the formal integral kernel $\Lambda_\omega(x, x')$ with

$$\Lambda_\omega(f, g) \doteq \int_{\mathscr{N}\times\mathscr{N}} \Lambda_\omega(x, x') f(x) g(x') d\mu_g(x) d\mu_g(x') \ \doteq \langle \Psi_\omega, \Phi_\omega(f)\Phi_\omega(f')\Psi_\omega \rangle \ .$$

Obviously:

$$\Lambda_\omega(f, g) = \lambda_\omega(E_{P_g} f, E_{P_g} g) \quad \text{if } f, g \in C^\infty(\mathscr{N}; \mathbb{R}) \ . \tag{72}$$

In this framework, the state ω is said to satisfy the (local) **Hadamard property** when, in a geodetically convex neighborhood of any point the two-point (Wightmann) function $\omega(x, x')$ of the state has a structure

$$\Lambda_\omega(x, x') = \frac{\Delta(x, x')}{\sigma(x, x')} + V(x, x') \ln \sigma(x, x') + w(x, x') \ ,$$

where $\Delta(x, x')$ and $V(x, x')$ are determined by the local geometry, $\sigma(x, x')$ is the signed squared geodesical distance of x and x' , while w is a smooth function determining the quasifree state. The precise definition, also at global level and specifying the regularization procedure enclosed in the definition of σ, was stated in [KW91]. Knowledge of the singularity part of the two-point function (and thus of all n-point functions in view of Wick expansion procedure), permits the definition of a suitable renormalization procedure of several physically interesting quantities as the stress energy tensor and more complicated objects [Wa84, Mo03, HW04] and it has been the starting point of a full renormalization procedure in curved spacetime and other very important developments of the general theory [BF00, HW01, BFV03]. A fundamental technical achievement was obtainjed by Radzikowski [Ra96a, Ra96b] who, among other results, proved in a pair of remarkable papers [Ra96a, Ra96b], that, referring to the Klein-Gordon scalar field, the global Hadamard condition for a quasifree state ω whose two-point function is, properly speaking, a distribution $\Lambda_\omega \in \mathscr{D}'(\mathscr{N} \times \mathscr{N})$ where $(\mathscr{N}, g)$ is globally hyperbolic and time-oriented, is equivalent to the following constraint of the *wavefront set* [Hö89] of Λ_ω.

$$WF(\Lambda_\omega) = \{(x, y, k_x, k_y) \in T^*(\mathscr{N} \times \mathscr{N}) \setminus \{0\} \mid (x, k_x) \sim (y, -k_y) \ , \ k_x \triangleright 0\} \ , \tag{73}$$

Above 0 *denotes the null section of* $T^*(\mathscr{N}\times\mathscr{N})$ *and* $(x,k_x)\sim(z,k_z)$ *means that there exists a light-like geodesic* γ *connecting* x *to* z *with* k_x *and* k_z *as (co)tangent vectors of* γ *respectively at* x *and at* z *(in particular if* $x=z$*, it must hold that* $k_x=k_z$*,* k_z *being of null type). The symbol* $\triangleright$ *indicates that* k_x *must lie in the future-oriented light cone.*

Radzikowski's breakthrough allows one to employ the powerful machinery of microlocal analysis as we shall do in the rest of the paper.

The aim of this subsection is to prove that the two-point function associated to the state (68) on $\mathcal{W}(\mathscr{M})$ fulfils the Hadamard property empoying the microlocal approach based on the condition (73). To this avail, the general strategy, we shall follow, consists in combining in a new non trivial way the results presented in [SV00] and in [Mo08, DMP09b]. Since we intepret the two-point function as maps from $C_0^\infty(\mathscr{M};\mathbb{C})\times C_0^\infty(\mathscr{M};\mathbb{C})\to\mathbb{C}$ a useful tool is the map $\Gamma:\mathsf{S}(\mathscr{M})\to\mathsf{S}(\mathcal{H})\oplus\mathsf{S}(\mathfrak{I}^-)$ introduced in the statement of Theorem 2.1, and we shall combine it with the causal propagator to obtain

$$(\varphi^f_{\mathcal{H}},\varphi^f_{\mathfrak{I}^-})\doteq\Gamma Ef\ . \tag{74}$$

We can now state the following proposition, whose last statement permit us to check Radzikowski's microlocal condition (73), as the two point function of ω_U determines a proper distribution of $\mathscr{D}'(\mathscr{M}\times\mathscr{M})$.

Proposition 4.2. *The four-smeared two-point function* $\Lambda_U:C_0^\infty(\mathscr{M};\mathbb{R})\times C_0^\infty(\mathscr{M};\mathbb{R})\to\mathbb{C}$ *of the Unruh state* ω_U *can be written as the sum*

$$\Lambda_U=\Lambda_{\mathcal{H}}+\Lambda_{\mathfrak{I}^-}, \tag{75}$$

with $\Lambda_{\mathcal{H}}$ *and* $\Lambda_{\mathfrak{I}^-}$ *defined out of the following relations for* $\lambda_{\mathcal{H}}$ *and* $\lambda_{\mathfrak{I}^-}$ *as in* (44) *and* (58)*:*

$$\Lambda_{\mathcal{H}}(f,g)\doteq\lambda_{\mathcal{H}}(\varphi^f_{\mathcal{H}},\varphi^g_{\mathcal{H}})\ ,\qquad \Lambda_{\mathfrak{I}^-}(f,g)\doteq\lambda_{\mathfrak{I}^-}(\varphi^f_{\mathfrak{I}^-},\varphi^g_{\mathfrak{I}^-})\ ,\quad \textit{for every } f,g\in C_0^\infty(\mathscr{M};\mathbb{R}), \tag{76}$$

Separately, $\Lambda_{\mathcal{H}}$*,* $\Lambda_{\mathfrak{I}^-}$ *and* Λ_U *individuate elements of* $\mathscr{D}'(\mathscr{M}\times\mathscr{M})$ *we shall indicate with the symbols* $\Lambda_{\mathcal{H}}$*,* $\Lambda_{\mathfrak{I}^-}$ *and* Λ_U *again, which are uniquely individuated by* $\mathbb{C}$*-linearity and continuity assuming (75) and:*

$$\Lambda_{\mathcal{H}}(f\otimes g)\doteq\lambda_{\mathcal{H}}(\varphi^f_{\mathcal{H}},\varphi^g_{\mathcal{H}})\ ,\qquad \Lambda_{\mathfrak{I}^-}(f\otimes g)\doteq\lambda_{\mathfrak{I}^-}(\varphi^f_{\mathfrak{I}^-},\varphi^g_{\mathfrak{I}^-})\ ,\quad \textit{for every } f,g\in C_0^\infty(\mathscr{M};\mathbb{R}). \tag{77}$$

The proof is in Appendix C.

In the remaining part of this section we shall prove one of the main theorems of this paper, namely that Λ_U satisfies the microlocal spectral condition (73) and thus the Unruh state ω_U is Hadamard.

Theorem 4.1. *The two-point function* $\Lambda_U\in\mathscr{D}'(\mathscr{M}\times\mathscr{M})$ *associated with the Unruh state* ω_U *satisfies the microlocal spectral condition:*

$$WF(\Lambda_U)=\{(x,y,k_x,k_y)\in T^*(\mathscr{M}\times\mathscr{M})\setminus\{0\},\ (x,k_x)\sim(y,-k_y),\ k_x\triangleright 0\}\ , \tag{78}$$

consequently ω_U *is of Hadamard type.*

Proof. As it is often the case with identities of the form (78), the best approach, to prove them, is to show that two inclusions $\supset$ and $\subset$ hold separately, hence yielding the desired equality. Nonetheless, in this case, we should keep in mind that Λ_U is a two-point function, hence it satisfies in a weak sense the equation of motion (13) with respect to P_g, a properly supported, homogeneous of degree 2, hyperbolic operator of real principal part. This entails a simplification of the problem, we are tackling, since, as a further consequence, the antisymmetric part of Λ_U must correspond to the causal propagator E introduced

in subsection 2.2 and, in this framework, all the hypothesis to apply the theorem of propagation of singularities (PST) – see Theorem 6.1.1 in [DH72] – are met. Hence one has all the ingredients necessary to proceed as in the proof of Theorem 5.8 in [SV01], to conclude that the inclusion $\supset$ holds true once $\subset$ has been established. Therefore to prove (78), it is enough establishing the inclusion $\subset$ only. That is the goal of the remaining part of the proof and we shall divide our reasoning in two different sequential logical steps. In the first part, below indicated as *part 1*, we shall prove that the microlocal spectrum condition is fulfilled in the static region $\mathscr{W}$. In the second, displayed as *part 2*, we apply this result extending it to the full $\mathscr{M}$, mostly by means of the PST which strongly constraints the form of $WF(\Lambda_U)$ in the full background. The left-over terms, not fulfilling (78), are eventually excluded by means of a case by case analysis.

Part 1. In order to establish the validity of the microlocal spectral condition in $\mathscr{W}$, our overall idea is to restrict Λ_U to a distribution in $\mathscr{D}'(\mathscr{W}\times\mathscr{W})$ and to apply/adapt to our case the result on the wave-front set of the two-point function of passive quantum states, as devised in [SV00].
As a starting point, let us rememind that $\mathscr{W}$ is a static spacetime with respect to the Schwarzschild Killing vector X, and the state Λ_U is invariant under the associated time translation, as established in Proposition 4.1. However, despite this set-up, it is not possible to prove that Λ_U is passive in the strict sense given in [SV00] and, hence, we cannot directly conclude that the Hadamard property is fulfilled in $\mathscr{W}$, *i.e.*, in other words, Theorem 5.1 in [SV00] does not straightforwardly go through. Nonetheless, luckily enough, a closer look at the proof of the mentioned statement reveals that it can be repeated slavishly with the due exception of the second step in which the passivity condition is explicitly employed. Yet, this property is not used to its fullest extent and, actually, a weaker one suffice to get the wanted result; in other words, the mentioned "step 2", or more precisely formula (5.2) in the last mentioned paper, can be recast as the following lemma for Λ_U.

Lemma 4.1. *The wave front set of the restriction to $\mathscr{D}(\mathscr{W}\times\mathscr{W})$ of Λ_U, satisfies the following inclusion*

$$WF(\Lambda_U\restriction_{\mathscr{D}(\mathscr{W}\times\mathscr{W})})\subset\{(x,y,k_x,k_y)\in T^*(\mathscr{W}\times\mathscr{W})\setminus\{0\},\ k_x(X)+k_y(X)=0,\ k_y(X)\geq 0\}\ , \tag{79}$$

where X is the generator of the Killing time translation.

Proof. As a first step we recall the invariance of Λ_U as well as of $\Lambda_{\Im^-}$ and $\Lambda_{\mathcal{H}}$ under the action of X, an assertion which arises out of part (b) of both Theorem 3.1 and 3.2. Furthermore, out of (77), it is manifest that both $\Lambda_{\Im^-}$ and $\Lambda_{\mathcal{H}}$ satisfy in a weak sense and in both entries the equation of motion, since they are constructed out of the causal propagator (77). Yet their antisymmetric part does not correspond to the causal propagator and this lies at the heart of the impossibility to directly apply the proof of theorem 5.1 as it appears in [SV00].
Nevertheless, if, as before, we indicate the pull-back action of one-parameter group of isometries generated by X on elements in $C_0^\infty(\mathscr{W};\mathbb{R})$ by $\beta_t^{(X)}$ $(t\in\mathbb{R})$, we can employ (77), as well as the definition of both $\lambda_{\Im^-}$ and $\lambda_{\mathcal{H}}$, to infer the following. $\Lambda_{\Im^-}$, which we shall refer as **vacuum like**, fulfils formula (A1) in [SV00]:

$$\int_{\mathbb{R}}\widehat{f}(t)\Lambda_{\Im^-}(h_1\otimes\beta_t^{(X)}(h_2))dt=0,\quad h_1,\ h_2\in C_0^\infty(\mathscr{W};\mathbb{R})$$

for all $\widehat{f}(t)\doteq\int_{\mathbb{R}}e^{-ikt}f(k)dk$ such that $f\in C_0^\infty(\mathbb{R}^*_-;\mathbb{C})$. At the same time $\Lambda_{\mathcal{H}}$ fulfils formula (A2) in the same mentioned paper, which implies that it is **KMS like** at inverse temperature β_H, *i.e.*

$$\int_{\mathbb{R}}\widehat{f}(t)\Lambda_{\mathcal{H}}(h_1\otimes\beta_t^{(X)}(h_2))dt=\int_{\mathbb{R}}\widehat{f}(t+i\beta_H)\Lambda_{\mathcal{H}}(\beta_t^{(X)}(h_2)\otimes h_1)dt,\quad h_1,\ h_2\in C_0^\infty(\mathscr{W};\mathbb{R})\ ,$$

for every $f \in C_0^\infty(\mathbb{R};\mathbb{R})$. The former identity arises exploiting the Fubini-Tonelli's theorem and basic properties of Fourier-Plancherel transform, from the fact that in view if the definition of $\Lambda_{\Im^-}$, $\omega_{\Im^-}$ and the explicit expression of $\mathsf{H}_{\Im^-} = L^2(\mathbb{R}_+ \times \mathbb{S}^2; 2kdk \wedge d\mathbb{S}^2)$ and (c) of Theorem 3.2:

$$\Lambda_{\Im^-}(h_1 \otimes \alpha_t(h_2)) = \int_{\mathbb{S}^2} d\mathbb{S}^2(\omega) \int_0^{+\infty} \overline{\psi_1(k,\omega)} e^{itk} \psi_2(k,\omega) 2kdk$$

for suitable functions ψ_1 and $\psi_2 \in L^2(\mathbb{R}\times\mathbb{S}^2; 2kdk\wedge d\mathbb{S}^2)$ corresponding to h_1 and h_2, and where we stress that the k integration is extended to the *positive real axis* only, whereas the support of f is contained in $\mathbb{R}_-$. Noticing that if $h \in C_0^\infty(\mathscr{W};\mathbb{R})$, then $h_{\mathcal{H}} \in \mathsf{S}(\mathcal{H}^-)$, The second identity follows similarly from Theorem 3.1 in view if the definition of $\Lambda_{\mathcal{H}}$, $\omega^{\beta_H}_{\mathcal{H}^-}$, the explicit expression of the measure $\mu(k)$ employied to define $\mathsf{H}^{\beta_H}_{\mathcal{H}^-} = L^2(\mathbb{R} \times \mathbb{S}^2; \mu(k) \wedge d\mathbb{S}^2)$, and (e) of Theorem 3.1.
The validity of this pair of identities suffices to establish the statement of Proposition 2.1 in [SV00], whose proof can be slavishly repeated with our slightly weaker assumptions (paying attention to the different conventions in our definition of the Fourier transform). From this point onwards, one can follow, in our framework and step by step, the calculations leading to the second step in the proof of Theorem 5.1 in [SV00], which is nothing but the statement of our lemma. We shall not reproduce all the details here, since it would lead to no benefit for the reader. □

Equipped with the proved lemma, and following the remaining steps of the proof of Theorem 5.1 in [SV00] the last statement in the thesis of Theorem 5.1 in [SV00] can be achieved in our case, too. As remarked immediately after the proof of the mentioned theorem in [SV00], that statement entails the validity of the microlocal microlocal spectral condition for the considered two-point function. Thus we can claim that.

Proposition 4.3. *The two-point function $\Lambda_U \in \mathscr{D}'(\mathscr{M} \times \mathscr{M})$ of the Unruh state, restricted on $C_0^\infty(\mathscr{W} \times \mathscr{W};\mathbb{C})$, satisfies the microlocal spectral condition (73) with $\mathscr{N} = \mathscr{W}$ and thus $\omega_U \restriction_{\mathcal{W}(\mathscr{W})}$ is a Hadamard state.*

Part 2. Our goal, now, is to establish that the microlocal spectrum condition for $\Lambda_U(x,x')$ holds true also considering pairs $(x,x') \in \mathscr{M} \times \mathscr{M}$ which do not belong to $\mathscr{W} \times \mathscr{W}$. The overall strategy, we shall employ, mainly consists of a careful use of the propagation of singularity theorem which shall allow us to divide our analysis in simpler specific subcases.
To this avail, we introduce the following bundle of null cones $\mathscr{N}_g \subset T^*\mathscr{M} \setminus \{0\}$ constructed out of the principal symbol of P_g, as in (13):

$$\mathscr{N}_g \doteq \{(x,k_x) \in T^*\mathscr{M} \setminus \{0\} \ , \ g^{\mu\nu}(x)(k_x)_\mu(k_x)_\nu = 0\} \ .$$

and define the **bicharacteristic strips** generated by $(x,k_x) \in \mathscr{N}_g$

$$B(x,k_x) \doteq \{(x',k'_x) \in \mathscr{N}_g \mid (x',k_{x'}) \sim (x,k_x)\} \, , \tag{80}$$

where $\sim$ was introduced in (73). The operator P_g is such that we can apply to the weak-bisolution Λ_U the theorem of propagation of singularities (PST), as devised in Theorem 6.1.1 of [DH72]. This guarantees that, on a hand:

$$WF(\Lambda_U) \subset (\{0\} \cup \mathscr{N}_g) \times (\{0\} \cup \mathscr{N}_g) \ , \tag{81}$$

on the other hand:

$$\text{if } (x,y,k_x,k_y) \in WF(\Lambda_U) \text{ then } \quad B(x,k_x) \times B(x',k_{x'}) \subset WF(\Lambda_U), \tag{82}$$

A pair of technical results we shall profitably use in the proof is given by the following lemma and proposition whose proofs can be found in appendix C.
The lemma has a statement which closely mimics an important step in the analysis of the Hadamard form of two-point functions, first discussed in [SV01]. It establishes the the right-hand side of (81) can be restricted further.

Lemma 4.2. *Isolated singularities do not enter the wave-front set of Λ_U, namely*

$$(x, y, k_x, 0) \notin WF(\Lambda_U) \ , \quad (x, y, 0, k_y) \notin WF(\Lambda_U) \qquad \text{if } x, y \in \mathscr{M}, k_x \in T^*_x\mathscr{M}, k_y \in T^*_y\mathscr{M} \ .$$

Thus, as a consequence of (81), it has to hold:

$$WF(\Lambda_U) \subset \mathscr{N}_g \times \mathscr{N}_g \ . \tag{83}$$

The pre-announced proposition characterizes the decay property, with respect to $p \in T^*_x\mathscr{M}$, of the distributional Fourier transforms (notice that in [Hö89] the opposit convention concerning the sign in gront of $i\langle p, \cdot\rangle$ is adopted):

$$\varphi^{f_p}_{\Im^-} := \lim_{\rightarrow \Im^-} E_{P_g}(fe^{i\langle p,\cdot\rangle}) \ , \qquad \varphi^{f_p}_{\mathcal{H}} := E_{P_g}(fe^{i\langle p,\cdot\rangle})\restriction_{\mathcal{H}}$$

where we have used the straightforwardly complexfied version of causal propagator, which enjoys the same causal and topological properties as those of the real one, and where and henceforth $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^4$ and $|\cdot|$ the associated norm, thus and referring to the identification of $\mathscr{M}$ with $\mathbb{R}^4$ due coordinate patch including the support of f, to give sense to the function $(p, x) \mapsto e^{i\langle p,x\rangle}$. ¿From now on we shall assume to fix such a coordinate patch whenever necessary, all the results being independent from the choice of these coordinates as discussed after Theorem 8.2.4 in [Hö89]. We remind the reader that, given a function $F : \mathbb{R}^n \to \mathbb{C}$, an element $k \in \mathbb{R}^n \setminus \{0\}$ is said to be of **rapid decreasing** for F if there is an open conical set V_k (i.e., an open set such that, if $p \in V_k$ then $\lambda p \in V_k$ for all $\lambda > 0$) such that, $V_k \ni k$ and, for every $n = 1, 2, \ldots$, there is $C_n \geq 0$ with $|F(p)| \leq C_n/(1 + |p|^n)$ for all $p \in V_k$.

Proposition 4.4. *Take $(x, k_x) \in \mathscr{N}_g$ such that (i) $x \in \mathscr{M} \setminus \mathscr{W}$ and (ii) the unique inextendible geodesic γ (co-)tangent to k_x at x intersects $\mathcal{H}$ in a point whose U coordinate is nonnegative, and fix $\chi' \in C^\infty(\mathcal{H}; \mathbb{R})$ with $\chi' = 1$ in $U \in (-\infty, U_0]$ and $\chi' = 0$ if $U \in [U_1, +\infty)$ for $U_0 < U_1 < 0$ constant.*
For any $f \in C^\infty_0(\mathscr{M})$ with $f(x) = 1$ and sufficiently small support, k_x is a direction of rapidly decreasing for both $p \mapsto \|\varphi^{f_p}_{\Im^-}\|_{\Im^-}$ and $p \mapsto \|\chi'\varphi^{f_p}_{\mathcal{H}}\|_{\mathcal{H}}$.

The next step in our proof consists of the analysis of $WF(\Lambda_U)$, in order to establish the validity of (78), with $=$ replaced with $\subset$, for the remaining cases left untreated by the statement of Proposition 4.3. As previously discussed, this is enough to conclude the proof of the validity of the Hadamard property for ω_U.
The remaining cases amount to the points in $WF(\Lambda_U)$ such that $(x, y, k_x, k_y) \in \mathscr{N}_g \times \mathscr{N}_g$ (in view of Lemma 4.2) with either x or y or both in $\mathscr{M} \setminus \mathscr{W}$. Therefore, we shall divide the forthcoming analysis in two parts, **case A**, where only one point of x, y is in $\mathscr{M} \setminus \mathscr{W}$, and **case B**, where both lie in $\mathscr{M} \setminus \mathscr{W}$.

Case A. Let us consider an arbitrary $(x, y, k_x, k_y) \in \mathscr{N}_g \times \mathscr{N}_g$ which belongs to $WF(\Lambda_U)$ and such that $x \in \mathscr{M} \setminus \mathscr{W}$ and $y \in \mathscr{W}$, the symmetric case being treated analogously. If a representative of the equivalence class $B(x, k_x)$ has its basepoint in $\mathscr{W}$, (82) entails that the portion of $B(x, k_y) \times B(y, k_y)$ inclosed in $T^*(\mathscr{W} \times \mathscr{W})$ must belong to $WF(\Lambda_U\restriction_{C^\infty_0(\mathscr{W}\times\mathscr{W};\mathbb{C})})$ and thus it must have the shape stated in Proposition 4.3. By uniqueness of a geodesic passing through a point with a given (co-)tangent vector,

it implies that $(x, k_x) \sim (y, k_y)$ and $k_x \triangleright 0$ as wanted.
Let us consider the remaining subcase where no representative of $B(x, k_x)$ has basepoint in $\mathscr{W}$. Our goal is to prove that, actually, this case is not possible since, under the said conditions, it results $(x, y, k_x, k_y) \notin WF(\Lambda_U)$ for every k_y. This will be established by proving that there a re two compactly supported smooth functions f and g with $f(x) = 1$ and $g(y) = 1$ such that the said (k_x, k_y) individuate directions of rapid decreasing of $(p_x, p_y) \mapsto \Lambda_U((fe^{i\langle p_x,\rangle} \otimes he^{i\langle p_y,\rangle})$.
If $B(x, k_k)$ does not meet $\mathscr{W}$, there must exist $(q, k_q) \in B(x, k_x)$, such that $q \in \mathcal{H}$ and the Kruskal null coordinate $U = U_q$ is nonnegative. Consider, then, the two-point function

$$\Lambda_U(f \otimes h) = \Lambda_{\mathcal{H}}(f \otimes h) + \Lambda_{\Im^-}(f \otimes h), \quad f, h \in C_0^\infty(\mathscr{M}; \mathbb{R}),$$

where $\Lambda_{\mathcal{H}}$ and $\Lambda_{\Im^-}$ are as in (77). If the supports of the chosen f and h are sufficiently small, we can always engineer a function $\chi \in C_0^\infty(\mathcal{H})$ in such a way that $\chi(U_q, \omega) = 1$ for all $\omega \in \mathbb{S}^2$ and $\chi = 0$ on the intersection of $J^-(supp\ h)$ and $\mathcal{H}$. Furthermore, using a coordinate patch which identifies an open neighborhood of $supp(f)$ with $\mathbb{R}^4$ and defining $\chi' \doteq 1 - \chi$, we can arrange a conical neighborhood $\Gamma_{k_x} \in \mathbb{R}^4 \setminus \{0\}$ of k_x such that all the bicaracteristics $B(s, k_s)$ with $s \in supp(f)$ and $k_s \in \Gamma_{k_x}$ do not meet any point of $supp\chi'$ on $\mathcal{H}$. Referring to (74), we can now divide $\Lambda_{\mathcal{H}}(f \otimes h)$ as:

$$\Lambda_{\mathcal{H}}(f \otimes h) = \lambda_{\mathcal{H}}(\chi\varphi^f_{\mathcal{H}}, \varphi^h_{\mathcal{H}}) + \lambda_{\mathcal{H}}(\chi'\varphi^f_{\mathcal{H}}, \varphi^h_{\mathcal{H}}),$$

and we separately analyze the large (k_x, k_y) beahviour of the three contributions:

$$\lambda_{\mathcal{H}}(\chi\varphi^{f_{k_x}}_{\mathcal{H}}, \varphi^{h_{k_y}}_{\mathcal{H}}) \ , \qquad \lambda_{\mathcal{H}}(\chi'\varphi^{f_{k_x}}_{\mathcal{H}}, \varphi^{h_{k_y}}_{\mathcal{H}}) \qquad \text{and} \qquad \lambda_{\Im^-}(\varphi^{f_{k_x}}_{\Im^-}, \varphi^{h_{k_y}}_{\Im^-}) \ , \tag{84}$$

each seen as the action of a corresponding distribution of $\mathscr{D}'(\mathscr{M} \times \mathscr{M})$ and where, again, $f_{k_x} \doteq fe^{i\langle k_x,\cdot\rangle}$ and $\varphi^{f_{k_x}} \doteq E_{P_g} f_{k_x}$ (using the standard complexification of the $\mathbb{R}$-linear map E_{P_g} due to the fact that f_{k_x} is complex).
The scenario, we face, is less complicated than it looks at first glance since, on a hand, we know that neither $(x, y, k_x, 0)$ nor $(x, y, 0, k_y)$ can be contained in $WF(\Lambda_U)$, as Lemma 4.2 yields, and hence this implies that, in the splitting we are considering in (84), we can concentrate on the points (x, y, k_x, k_y) where both k_x and k_y are not zero, only. If we were able to prove that these points are not contained in the wave front set of any of the three distributions considered in (84) above, we could conclude that they cannot be contained in the wave front set of the their sum Λ_U, because the wave front set of the sum of distributions is contained in the union of the wave front set of the single component (this is equivalent to the self-evident statement that the intersection of the complements of the wavefront sets of n distributions is included in the complement of the wavefront set of the sum of those distributions).
On the other hand, the second and third distribution in the right-hand side of (84) turn out to be dominated by $C\|\chi'\varphi^{f_{-k_x}}_{\mathcal{H}}\|_{\mathcal{H}}\|\varphi^{h_{k_y}}_{\mathcal{H}}\|_{\mathcal{H}}$ and $C'\|\varphi^{f_{-k_x}}_{\Im^-}\|_{\Im^-}\|\varphi^{h_{k_y}}_{\Im^-}\|_{\Im^-}$, respectively, C and C' being suitable positive constants, where $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\Im^-}$ stand for the norm introduced in (52) and (65). This comes out from continuity property presented in points (f) of both the propositions 3.4 and 3.8, which can straightforwardly be proved to be valid for complex functions, too. Furthermore, by Proposition 4.4, both $\|\chi'\varphi^{f_{k_x}}_{\mathcal{H}}\|_{\mathcal{H}}$ and $\|\varphi^{f_{k_x}}_{\Im^-}\|_{\Im^-}$ are rapidly decreasing in $k_x \in T^*\mathscr{M} \setminus \{0\}$ for an f with sufficiently small support and if k_x is in a open conical neighborhood of any null direction, while the remaining two terms $\|\varphi^{h_{k_y}}_{\mathcal{H}}\|_{\mathcal{H}}$ and $\|\varphi^{h_{k_y}}_{\Im^-}\|_{\Im^-}$, appearing in (84), can at most grow polynomially in k_y. The last property can be proved as follows. Starting from the bounds for the behavior of the wavefunctions restricted to on $\mathcal{H}^-$ and $\Im^-$, as appearing in Proposition 2.1, one can estimate the norms $\|\varphi^{h_{k_y}}_{\Im^-}\|_{\Im^-}$, $\|\varphi^{h_{k_y}}_{\mathcal{H}}\|_{\mathcal{H}}$ embodying the dependence on k_y in the explicit expression of the coefficients C_i appearing in Proposition 2.1, which depend on the considered wavefunction and thus on the used $he^{i\langle k_y,\cdot\rangle}$. Then, by an argument similar

to that exploited in the proof of Proposition 4.4, for fixed k_y and h, those coefficients, up to the factor C or C', can be bounded by $\sqrt{|\tilde{E}_5(\varphi^{h_{k_y}})|}$ as in (127) where $\tilde{E}_5(\varphi^{h_{k_y}})$ is the integral of a polynomial of derivatives of $\varphi^{h_{k_y}}$ on a certain Cauchy surface Σ of $\mathscr{W}$. Since $\varphi^{h_{k_y}}(z) = (E(h_{k_y}))(z)$ is smooth and has compact support when restricted to Cauchy surfaces, and varying k_y all the supports can be included in a common compact $K \subset \Sigma$ in view of the causal as it happens for the supports of all h_{k_y}, one can exploit the continuity of the (complexfied) causal propagator E. It yields that, for every fixed multi-index α, $\sup_K |\partial^\alpha E(h_{k_y})|$ is bounded by a corresponding polynomial in the absolute values of the components of k_y, whose coefficients are the supremum of derivatives of $h \in C_0^\infty(\mathscr{M};\mathbb{R})$ up to a certain order. This implies immediately (especially noticing that the integral of the Cauchy surface necessary to compute $\tilde{E}_5(\varphi^{h_{k_y}})$ has to be performed on the compact set K) that $\tilde{E}_5(\varphi^{h_{k_y}})$ is polynomially bounded in k_y, and $\|\varphi^{h_{k_y}}_{\Im^-}\|_{\Im^-}$, $\|\varphi^{h_{k_y}}_{\mathcal{H}}\|_{\mathcal{H}}$ are such.)

We now remind the reader that we have identified $\mathscr{K} \times \mathscr{K}$ with $\mathbb{R}^4 \times \mathbb{R}^4$ by means of a suitable pair of coordinate frames thus, in particular, cotangent vectors at different points x and y can be though of as elements of the same $\mathbb{R}^4$ and compared. This fact allow us to define the following open cone in $\mathbb{R}^4$, $\Gamma \subset \mathbb{R}^4 \times \mathbb{R}^4$, for a fixed strictly positive $\epsilon < 1$,

$$\Gamma_{k_x} = \left\{ (p_x, p_y) \in \mathbb{R}^4 \times \mathbb{R}^4 \;\middle|\; \epsilon|p_x| < |p_y| < \frac{1}{\epsilon}|p_x| \;\; , -p_x \in U_{-k_x} \right\} \tag{85}$$

where U_{k_x} is an open cone about the null vector $k_x \neq 0$ where $p \mapsto \|\chi' \varphi^{f_{p_x}}_{\mathcal{H}}\|^\eta_{\mathcal{H}^-}$ and $p \mapsto \|\varphi^{f_{p_x}}_{\Im^-}\|_{\Im^-}$ decrease rapidly, and the norm $|\cdot|$ on $\mathbb{R}^4$ is defined as follows. We adopt two coordinate frames about x and y, such that the matrices of metric g_x and g_y take the canonical form $diag(-1,+1,+1,+1)$ in both cases exactly at x and y. Then $|\cdot|$ is the standard Euclidean metric referred to those coordinates.

By construction for any direction (k_x, k_y) with both $k_x \neq 0$ and $k_y \neq 0$ of null type like those we are considering there is a cone Γ_{k_x} containing it. Moreover all the directions contained in Γ_{k_x} (in particular any direction (k_x, k_y) with both $k_x \neq 0$ and $k_y \neq 0$ of null type) are of rapid decreasing for $\lambda_{\mathcal{H}}(\chi\varphi^{f_{k_x}}_{\mathcal{H}}, \varphi^{h_{k_y}}_{\mathcal{H}})$ and $\lambda_{\mathcal{H}}(\chi'\varphi^{f_{k_x}}_{\mathcal{H}}, \varphi^{h_{k_y}}_{\mathcal{H}})$ because the rapid decreasing of $\|\chi'\varphi^{f_{-p_x}}_{\mathcal{H}}\|_{\mathcal{H}}$ and $\|\varphi^{f_{-p_x}}_{\Im^-}\|_{\Im^-}$ controls the polynomial growth in $|p_y|$ of $\|\varphi^{h_{p_y}}_{\mathcal{H}}\|_{\mathcal{H}}$ and $\|\varphi^{h_{p_y}}_{\Im^-}\|_{\Im^-}$ respectively, just in view of the shape of Γ_{k_x} (which is also invariant under $p_y \mapsto -p_y$).

We are thus left off only with the first term in (84) and, also in this case, if the support of f and h are chosen sufficiently small, $\lambda_{\mathcal{H}}(\chi\varphi^{f_{k_x}}_{\mathcal{H}}, \varphi^{h_{k_y}}_{\mathcal{H}})$ can be shown to be rapidly decreasing in both k_x and k_y. To this end, let us thus choose $\chi'' \in C^\infty(\mathcal{H};\mathbb{R})$ such that both $\chi''(p) = 1$ for every p in supp $(\varphi^{h_{k_y}}_{\mathcal{H}})$ (for every k_y) and $\chi'' \cap \chi = \emptyset$. Then we write

$$\lambda_{\mathcal{H}}(\chi\varphi^{f_{k_x}}_{\mathcal{H}}, \varphi^{h_{k_y}}_{\mathcal{H}}) = \int_{\mathcal{H}\times\mathcal{H}} \chi(x')\,(E(f_{kx}))\,(x')\,T(x',y')\chi''(y')\varphi^{h_{k_y}}_{\mathcal{H}}(y') \quad dU_{x'}d\mathbb{S}^2(\omega_{x'})\,dU_{y'}d\mathbb{S}^2(\omega_{y'})$$

Theorem 8.2.14 of [Hö89] guarantees us that

$$(x', y', k_{x'}, k_{y'}) \notin WF\left((T\chi'') \circ (\chi E\restriction_{\mathcal{H}})\right) \qquad \forall (y', k_{y'}) \in T^*\mathscr{M},$$

where T is the integral kernel of $\lambda_{\mathcal{H}}$ seen as a distribution on $\mathscr{D}'(\mathcal{H}\times\mathcal{H})$, while $\circ$ stands for the composition on $\mathcal{H}$ *with the E on the left of T*. Finally $E\restriction_{\mathcal{H}}$ means that the left entry of the causal propagator as been restricted on the horizon $\mathcal{H}$, an allowed operation thanks to theorem 8.2.4 in [Hö89] (one can convince himself, out of a direct construction, that the set of normals associated to the map embedding $\mathcal{H}$ in $\mathscr{W}$ does not intersect the wave front set of E), which furthermore provides us with a full control on $WF(E\restriction_{\mathcal{H}})$. The integral kernel of $(\chi T\chi'')(x', y')$, with the entry x' restricted on the support of χ and the

entry y' restricted on that of χ'', moreover, is always smooth and, keeping x' fixed, it is dominated by a smooth function whose H^1-norm in y' is, uniformly in x', finite. This also yields that, the $H^1(\mathcal{H})_U$-norm of $\|(T\chi'')\circ\chi E f_{k_x}\|_{H^1(\mathcal{H})_U}$ is dominated by the product of two integrals one over x' and one over y'. The presence of the compactly supported function χ and the absence of points of the form $(x, y, k_x, 0)$ and $(y, x, 0, k_y)$ in $WF(E)$ assures thus that the integral kernel of $\chi T\chi''$ is rapid decreasing in k_x. Summing up we have that

$$|\lambda_{\mathcal{H}}(\chi\varphi^{f_{k_x}}_{\mathcal{H}}, \varphi^{h_{k_y}}_{\mathcal{H}})| \leq C\| \left((T\chi'')\circ(\chi E)\right)(f_{-k_x})\|_{H^1(\mathcal{H})_U} \ \|\varphi^{h_{k_y}}_{\mathcal{H}}\|_{\mathcal{H}} \ , \tag{86}$$

where the second norm in the right-hand side is given in (52). This bound proves that, for a fixed k_y, $k_x \to \lambda_{\mathcal{H}}(\chi\varphi^{f_{k_x}}_{\mathcal{H}}, \varphi^{h_{k_y}}_{\mathcal{H}})$ is rapidly decreasing.
To conclude, looking again at the (86), if we introduce a cone as in (85) exploiting Lemma 4.2 we can control the (at most) polynomial growth of $\|\varphi^{h_{k_y}}_{\mathcal{H}}\|_{\mathcal{H}}$ using the rapid decreasing of the map $k_x \mapsto \| \left((T\chi'')\circ(\chi E)\right)(f_{-k_x})\|_{H^1(\mathcal{H})_U}$, establishing that (k_x, k_y) is a direction of fast decreasing of $\lambda_{\mathcal{H}}(\chi\varphi^{f_{k_x}}_{\mathcal{H}}, \varphi^{h_{k_y}}_{\mathcal{H}})$.

Case B. We shall now tackle the case in which we consider an arbitrary but fixed $(x, y, k_x, k_y) \in \mathscr{N}_g \times \mathscr{N}_g$, with both x and y lying $\mathscr{M} \setminus \mathscr{W}$.
Assuming that $(x, y, k_x, k_y) \in WF(\Lambda_U)$ we have to prove that $(x, k_x) \sim (y, k_y)$ and $k_x \triangleright 0$ have to be valid. If $B(x, k_x)$ and $B(y, k_y)$ are such that both admit representatives in $\mathscr{W}$, making use of (82) and of the fact that elements in the wavefront set of the restriction of Λ_U to $\mathscr{W}$ fulfils $(x', k'_{x'}) \sim (y', k'_{y'})$ and $k'_{x'} \triangleright 0$ one extend this property to (x, y, k_x, k_y) following the same reasoning as that at the beginning of the Case A. If, instead, only one representative, either of $B(x, k_x)$ or of $B(y, k_y)$ lies in $\mathscr{W}$, then we fall back in the *Case A* studied above making use of (82) again. Thus, we need only to establish the wanted behaviour of the wave front set when it is possible to find representatives of both $B(x, k_x)$ and $B(y, k_y)$ which intersects $\mathcal{H}$ at a nonnegative value of U. We shall follow a procedure similar to that already employed in [Mo08].
In this framework, let us consider the following decomposition of $\Lambda_U(\varphi^{f_{k_x}} \otimes \varphi^{h_{k_y}})$:

$$\lambda_U(\varphi^{f_{k_x}}, \varphi^{h_{k_y}}) = \lambda_{\mathcal{H}}(\varphi^{f_{k_x}}, \varphi^{h_{k_y}}) + \lambda_{\Im^-}(\varphi^{f_{k_x}}, \varphi^{h_{k_y}}),$$

where $f, h \in C^\infty_0(\mathscr{M})$ and they attain the value 1 respectively at the point x and y.
As before, we start decomposing the first term in the preceding expression by means of a partition of unit χ, χ' on $\mathcal{H}$ – where $\chi, \chi' \in C^\infty_0(\mathcal{H})$ satisfy $\chi + \chi' = 1 : \mathcal{H} \to \mathbb{R}$ – obtaining:

$$\begin{aligned}\lambda_{\mathcal{H}}(f_{k_x}, h_{k_y}) = \lambda_{\mathcal{H}}(\chi\varphi^{f_{k_x}}_{\mathcal{H}}, \chi\varphi^{h_{k_y}}_{\mathcal{H}}) + \lambda_{\mathcal{H}}(\chi'\varphi^{f_{k_x}}_{\mathcal{H}}, \chi\varphi^{h_{k_y}}_{\mathcal{H}}) + \\ \lambda_{\mathcal{H}}(\chi\varphi^{f_{k_x}}_{\mathcal{H}}, \chi'\varphi^{h_{k_y}}_{\mathcal{H}}) + \lambda_{\mathcal{H}}(\chi'\varphi^{f_{k_x}}_{\mathcal{H}}, \chi'\varphi^{h_{k_y}}_{\mathcal{H}}).\end{aligned} \tag{87}$$

Furthermore, the above functions χ, χ' can be engineered in such a way that the inextensible null geodesics γ_x and γ_y starting respectively at x and y with cotangent vectors k_x and k_y, intersect $\mathcal{H}$ in u_x and u_y (possibly $u_x = u_y$), respectively, included in two corresponding open neighbourhoods O_x and O_y (possibly $O_x = O_y$) where χ' vanishes. Let us start by examining the first term in the right hand side of (87) and, particularly, we shall focus on the wave front set of the unique extension of $f \otimes g \mapsto \lambda_{\mathcal{H}}(\chi\varphi^f_{\mathcal{H}}, \chi\varphi^h_{\mathcal{H}})$ to a distribution in $\mathscr{D}'(\mathscr{M} \times \mathscr{M})$. If we indicate as T the integral kernel of $\lambda_{\mathcal{H}}$, intepreted as distribution of $\mathscr{D}'(\mathcal{H} \times \mathcal{H})$, we notice that, as an element in $\mathscr{D}'(\mathscr{M} \times \mathscr{M})$, $\lambda_{\mathcal{H}}$ can be written as:

$$\lambda_{\mathcal{H}}(\chi\varphi^f_{\mathcal{H}}, \chi\varphi^h_{\mathcal{H}}) \doteq \ \chi T\chi \left(E|_{\mathcal{H}} \otimes E|_{\mathcal{H}} \left(f \otimes h\right)\right),$$

where $E|_{\mathcal{H}}$ is the causal propagator with one entry restricted on the horizon $\mathcal{H}$ and $\chi T \chi \in \mathcal{E}'(\mathcal{H} \times \mathcal{H})$. Thanks to the insertion of the compactly supported smooth functions χ, and with the knowledge that $WF(E \otimes E)_{\mathcal{H}\times\mathcal{H}} = \emptyset$ (see [Mo08]), we can make sense of the previous expression as an application of Theorem 8.2.13 in [Hö89], of which we also employ the notation. The wave front set of T has been already explicitly written in Lemma 4.4 of [Mo08] and, hence, still Theorem 8.2.13 in [Hö89] guarantees us that if (x, y, k_x, k_y) is contained in the wave front set of the resulting distribution then $(x, k_x) \sim (y, -k_y)$ and $k_x \rhd 0$ hold.
Coming back to the remaining terms in (87), it is possible to show that all them together with $\lambda_{\Im^-}$ are rapidly decreasing in both k_x and k_y, provided that f and h have sufficiently small support giving no contribution to $WF(\Lambda_U)$ and concluding the proof.
Here we analyse in details only the second term in (87) since the others can be treated exactly in the same way. To start with, notice that, due to (f) in proposition 3.4 $|\lambda_{\mathcal{H}}(\chi'\varphi_{\mathcal{H}}^{f_{k_x}}, \chi\varphi_{\mathcal{H}}^{h_{k_y}})|$ is bounded by $C\|\chi'\varphi_{\mathcal{H}}^{f_{-k_x}}\|_{\mathcal{H}^-}\|\chi\varphi_{\mathcal{H}}^{h_{k_y}}\|_{\mathcal{H}}$, where $\|\cdot\|_{\mathcal{H}}$ is the norm introduced in (52) and $C > 0$ is a constant. Due to Proposition 4.4, $\|\chi'\varphi_{\mathcal{H}}^{f_{-k_x}}\|_{\mathcal{H}}$ is rapidly decreasing in k_x for some f with sufficiently small support. Finally, the rapid decreasing of $\|\chi'\varphi_{\mathcal{H}}^{f_{-k_x}}\|_{\mathcal{H}}$ can control the at-most polynomial growth of $\|\chi\varphi_{\mathcal{H}}^{h_{k_y}}\|_{\mathcal{H}}$, as discussed above in the analysis of (84) using the fact that $(x, y, k_x, 0)$ and $(x, y, 0, k_y)$ cannot be contained in the wave front set of Λ_U leading to the construction of the open cone Γ_{k_x}.
Collecting all the pieces of information we have got about the shape of Λ_U, we can state that:

$$WF(\Lambda_U) \subset \{(x, y, k_x, k_y) \in T^*(\mathscr{M} \times \mathscr{M}) \setminus \{0\},\ (x, k_x) \sim (y, -k_y),\ k_x \rhd 0\}$$

and this concludes the proof. $\square$

5 Conclusions

In this paper we employed a bulk-to-boundary reconstruction procedure to rigorously and unambiguously construct and characterise on $\mathscr{M}$ (*i.e.*, the static joined with the black hole region of Schwarzschild spacetime, event horizon included) the so-called Unruh state ω_U. That state plays the role of natural candidate to be used in the quantum description of the radiation arising during a stellar collapse. Furthermore we proved that ω_U fulfils the so-called Hadamard condition, hence it can be considered a genuine ground state for a massless scalar field theory living on the considered background. Overall, the achieved result can be seen as a novel combination of earlier approaches [DMP06, Mo08, DMP09a, DMP09b] with the theorems proved in [SV00] as well as with the powerful results obtained by Rodnianski and Dafermos in their recent analysis on the peeling behaviour both at the horizon and at infinity of the solutions of the Klein-Gordon equation in Schwarzschild spacetime.

Therefore we can safely claim that it is now possible to employ the Unruh vacuum in order to enhance the analysis in [FH90] studying quantum effects such as the role of the back reaction of Hawking's radiation, a phenomenon which was almost always discarded as negligible.

At the same time it would be certainly interesting to try to enhance the results of this paper since, as one can readily infer from the main body of the manuscript, ω_U has been here constructed only on $\mathscr{M}$ since, as already suggested in [Ca80], a Hadamard extension to the full Kruskal spacetime appears to be impossible due to bad divergencies of the renormalized stress-energy tensor on the past horizon $\mathcal{H}$. It is worth stressing that it is however possible to extend ω_U to the whole Kruskal manifold following our induction procedure, defining a further part of the state on $\Im_L^+$. The obtained state on the whole Weyl algebra $\mathcal{W}(\mathscr{K})$ would be invariant under the group of Killing isometries generated by X and without zero modes referring to the one-parameter group of isometries. The problem with this extension is related

with the Hadamard constraint. Indeed, we do not expect that this extension is of Hadamard form on $\mathcal{H}$, due to a theoretical obstruction beyond Candelas' remarks. In view of the uniqueness and KMS-property theorem proved in [KW91] for a large class of spacetimes including Kruskal one, the validity of the Hadamard property on the whole spacetime together with the invariance under X and the absence of zero-modes, imply that the state is unique on a certain enlarged algebra of observables $\mathscr{A}$ on $\mathscr{K}$, and that it coincides to a KMS state with respect to the Killing vector X at the Hawking temperatire, *i.e.* to the Hartle-Hawking state, on a certain subalgebra of observables $\mathscr{A}_{0I} \subset \mathscr{A}$ supported in the wedge $\mathscr{W}$. These algebras are obtained by two steps. (1) Enlarging $\mathcal{W}(\mathscr{K})$ to a larger algebra $\mathscr{A}$, where Weyl generators are smeared with the standard solutions of KG equation (with compactly supported Cauchy data in $\mathscr{K}$) and a certain class of weak solutions of Klein-Gordon equation. (2) Restricting this enlarged algebra to a certain sub algebra of observables $\mathscr{A}_{0I}$ supported in $\mathscr{W}$ in a suitable sense related with properties of the supports of the smearing distributions accross the Killing horizon. Concerning our state we know that the KMS property is not verified about $\Im^-$, so we do not expect that any extension of that state satisfies the KMS property there. However the issue is not completely clear since the extension we are discussing and the failure of the KMS condition are both referred to $\mathcal{W}(\mathscr{W})$ rather than $\mathscr{A}_{0I}$, and further investigation in that direction is desirable.
A further and certainly enticing possible line of research consists of using the very same approach discussed in this paper in order both to rigorously define the very Hartle-Hawking state and to prove its Hadamard property; although, from a physical perspective, this is certainly a very interesting problem, from a mathematical perspective, it amounts to an enhancement of the peeling behaviours for the solutions of the Klein-Gordon equation discussed by Rodnianski and Dafermos. Although there is no proof that the obtained ones are sharp conditions, the high degree of mathematical specialisation, needed to obtain the present results, certainly makes the proposed programme a challenging line of research, which we hope to tackle in future papers.
As an overall final remark it is important stressing that all our results are valid for the masselss case only. It would be very interesting to extend our analysis to the massive case.

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A Further details on the geometric setup.

In this paper, the extension of the underlying background to include null infinities as well as a region beyond them, plays a pivotal role and we shall now dwell into a few more details. To this end, one follows [SW83] (though only for the part of the Kruskal extension we are interested in) and rescales the global metric g in (11) by a factor $1/r^2$ after which one can notice that the obtained manifold $(\mathscr{M}, g/r^2)$ admits a smooth larger extension $(\widetilde{\mathscr{M}}, \widetilde{g})$. We have to notice that, in this case, the singularity present at $r = 0$ in $(\mathscr{M}, g)$ is pushed at infinity (in the sense that the non-null geodesics takes an infinite amount of affine parameter to reach a point situated at $r = 0$). The extension of $(\widetilde{\mathscr{M}}, \widetilde{g})$ obtained in this way does not cover the points (actually sets) indicated by $i^\pm$ and i_0 in figure 1 and figure 2, though it includes the boundaries $\Im^\pm$, called **future** and **past null infinity** respectively and representing subsets of $\widetilde{\mathscr{M}}$ which are null 3-submanifold of $\widetilde{\mathscr{M}}$ formally localised at $r = +\infty$.

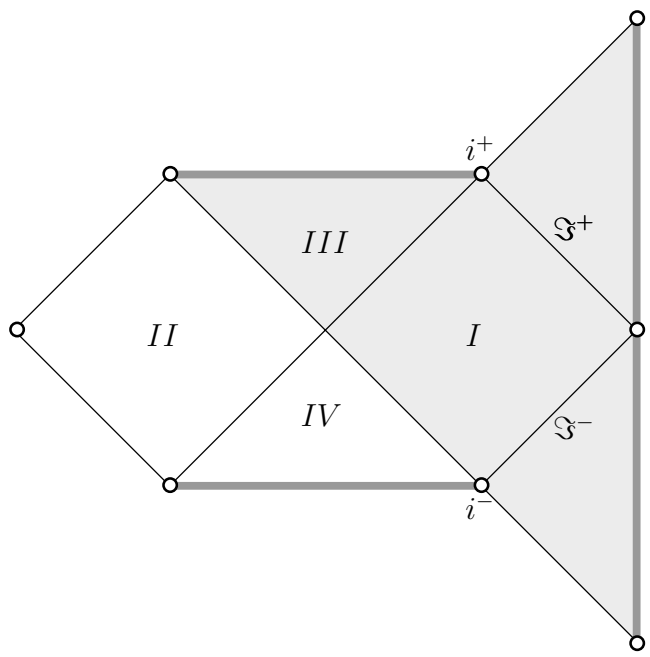

Figure 2: The Kruskal spacetime $\mathscr{K}$ is the union of the open regions I,II,III, and IV including their common boundaries. $\mathscr{M}$ is the union of I and III including the common boundary $\mathcal{H}_{ev}$. The conformal extension $\widetilde{\mathscr{M}}$ of $\mathscr{M}$ beyond $\Im^+$ and $\Im^-$ is the gray region. The thick lines denote the metric singularities at $r = 0$.

Let us now examine the form of the rescaled extended metric restricted to the Killing horizon $\mathcal{H}$ as well as to the null infinities $\Im^\pm$. Per direct inspection, one finds that, if one fixes $\Omega \doteq 2V$, which vanishes on $\mathcal{H}$,

$$\widetilde{g}\restriction_{\mathcal{H}} = r_S^2 \left(-d\Omega \otimes dU - dU \otimes d\Omega + h_{\mathbb{S}^2}(\theta, \phi)\right) \ . \tag{88}$$

In this case $V \in \mathbb{R}$ is the complete affine parameters of the null $\widetilde{g}$-geodesics generating $\mathcal{H}$ and $\mathcal{H}$ itself is obtained setting $U = 0$. This form of the metric is called **geodetically complete Bondi form**.
The same structure occurs on $\Im^+$, formally individuated by $v = +\infty$ and on $\Im^-$, formally individuated by $u = -\infty$, where the metric $\tilde{g}$ has still a **geodetically complete Bondi form**, namely

$$\widetilde{g}\restriction_{\Im^+} = -d\Omega \otimes du - du \otimes d\Omega + h_{\mathbb{S}^2}(\theta, \phi) \ , \tag{89}$$

where $\Omega \doteq -2/v$ individuates $\Im^+$ for $\Omega = 0$. Similarly

$$\tilde{g}\restriction_{\Im^-} = -d\Omega \otimes dv - dv \otimes d\Omega + h_{\mathbb{S}^2}(\theta, \phi) \ , \tag{90}$$

where $\Omega \doteq -2/u$ individuates $\Im^+$ for $\Omega = 0$.

$$X = \partial_u \text{ on } \Im^+, \quad X = \partial_v \text{ on } \Im^-. \tag{91}$$

In both cases the coordinates u and v are well defined and they coincide with the complete affine parameters of the null $\widetilde{g}$-geodesics forming $\Im^+$ and $\Im^-$ respectively.
With respect to Killing symmetries, we notice that the g-Killing vector X is a Killing vector for $\widetilde{g}$, too and it extends to a $\widetilde{g}$-Killing vector X defined on $\widetilde{\mathscr{M}}$. Particularly, in $\partial\mathscr{M}$ it satisfies

B Weyl algebras, quasifree states, KMS condition.

A C^*-algebra $\mathcal{W}(\mathsf{S})$ is called **Weyl algebra** associated with a (real) symplectic space (S,σ) (the symplectic form σ being nondegenerate) if it contains a class of non-vanishing elements $W(\psi)$ for all $\psi\in\mathsf{S}$, called **Weyl generators**, satisfying **Weyl relations**[3]:

$$(W1)\qquad W(-\psi)=W(\psi)^*\,,\qquad\qquad (W2)\qquad W(\psi)W(\psi')=e^{i\sigma(\psi,\psi')/2}W(\psi+\psi')\,;$$

and $\mathcal{W}(\mathsf{S})$ coincides with the closure of the $*$-algebra (finitely) generated by Weyl generators. As a consequence of (W1) and (W2), one gets: $W(0)=\mathbb{I}$ (the unit element), $W(\psi)^*=W(\psi)^{-1}$, $||W(\psi)||=1$ and, using non degenerateness of σ, $W(\psi)=W(\psi')$ iff $\psi=\psi'$.
$\mathcal{W}(\mathsf{S})$ is *uniquely* determined by (S,σ) (theorem 5.2.8 in [BR022]): Two different realizations admit a unique $*$-isomorphism which transform the former into the latter, preserving Weyl generators, and the norm on $\mathcal{W}(\mathsf{S})$ is unique, since $*$ isomorphisms of C^* algebras are isometric. This results implies that every GNS $*$-representation of a Weyl algebra is always faithful and isometric. It is worth also mentioning that, by constructions, any GNS $*$-representation of a Weyl algebra is such that the generators are always represented by unitary operators, but it is not the case for other $*$-representations in Hilbert spaces.
$\mathcal{W}(\mathsf{S})$ can always be realized in terms of bounded operators on $\ell^2(\mathsf{S})$, viewing S as a Abelian group and defining the generators as $(W(\psi)F)(\psi')\doteq e^{-i\sigma(\psi,\psi')/2}F(\psi+\psi')$ for every $F\in\ell^2(\mathsf{S})$. In this realization (and thus in every realization) it turns out evident that generators $W(\psi)$ are *linearly independent*. A state ω on $\mathcal{W}(\mathsf{S})$, with GNS triple $(\mathfrak{H}_\omega,\Pi_\omega,\Omega_\omega)$, is called **regular** if the maps $\mathbb{R}\ni t\mapsto\Pi_\omega(W(t\psi))$ are strongly continuous. (In general, strong continuity of the unitary group implementing a $*$-automorphism representation β of a topological group $G\ni g\mapsto\beta_g$ for a β-invariant state ω on a Weyl algebra $\mathcal{W}(\mathsf{S})$, is equivalent to $\lim_{g\to\mathbb{I}}\omega(W(-\psi)\beta_gW(\psi))=1$ for all $\psi\in\mathsf{S}$. The proof follows immediately from $||\Pi_\omega\left(\beta_{g'}W(\psi)\right)\Omega_\omega-\Pi_\omega\left(\beta_gW(\psi)\right)\Omega_\omega||^2=2-\omega\left(W(-\psi)\beta_{g'^{-1}g}W(\psi)\right)-\omega\left(W(-\psi)\beta_{g^{-1}g'}W(\psi)\right)$ and $\overline{\Pi_\omega(\mathcal{W}(\mathsf{S}))\Omega_\omega}=\mathfrak{H}_\omega$.)
In ω is regular, in accordance with Stone theorem, one can write $\Pi_\omega(W(\psi))=e^{i\sigma(\psi,\Phi_\omega)}$, $\sigma(\psi,\Phi_\omega)$ being the (self-adjoint) **field operator symplectically-smeared** with ψ.
When $\mathcal{W}(\mathsf{S})=\mathcal{W}(\mathsf{S}(\mathscr{N}))$ is the Weyl algebra on the space of Klein-Gordon equation solutions as in Sec. 2.2, the field operator $\Phi_\omega(f)$ introduced in that section, smeared with smooth compactly supported functions $f\in C_0^\infty(\mathscr{N};\mathbb{R})$, is related with $\sigma(\psi,\Phi_\omega)$ by

$$\Phi_\omega(f)\doteq\sigma(E_{P_g}(f),\Phi_\omega)\quad\text{for all } f\in C_0^\infty(\mathscr{N};\mathbb{R}),\tag{92}$$

exploiting notations used in Sec. 2.2. In this way field operators enters the theory in Weyl algebra scenario. Working formally, by Stone theorem (W2) implies $\mathbb{R}$-linearity and standard CCR:

$$(L)\quad\sigma(a\psi+b\psi',\Phi_\omega)=a\sigma(\psi,\Phi_\omega)+b\sigma(\psi',\Phi_\omega)\,,\qquad(CCR)\quad[\sigma(\psi,\Phi_\omega),\sigma(\psi',\Phi_\omega)]=-i\sigma(\psi,\psi')I\,,$$

for $a,b\in\mathbb{R}$ and $\psi,\psi'\in\mathsf{S}$. Actually (L) and (CCR) hold rigorously in an invariant dense set of analytic vectors by Lemma 5.2.12 in [BR022] (it holds if ω is quasifree by proposition B.1).
In the standard approach of QFT, based on bosonic real scalar field operators Φ a, *either vector or density matrix*, state is *quasifree* if the associated n-point functions (expectation values of a product of n fields) satisfy (i) $\langle\sigma(\psi,\Phi)\rangle=0$ for all $\psi\in\mathsf{S}$ and (ii) the n-point functions $\langle\sigma(\psi_1,\Phi)\cdots\sigma(\psi_n,\Phi)\rangle$ are determined from the functions $\langle\sigma(\psi_i,\Phi)\sigma(\psi_j,\Phi)\rangle$, with $i,j=1,2,\cdots,n$, using standard Wick's expansion. A technically different but substantially equivalent definition, completely based on the Weyl algebra was presented in [KW91]. It relies on the following three observations. (a)Working formally with

[3]Notice that in [KW91] a different convention for the sign of σ in (W2) is employed.

(i) and (ii), one finds that it holds $\langle e^{i\sigma(\psi,\Phi)}\rangle = e^{-\langle\sigma(\psi,\Phi)\sigma(\psi,\Phi)\rangle/2}$. In turn, at least formally, that identity determines the n-point functions (reproducing Wick's rule) by Stone theorem and (W2). (b) From (CCR) it holds $\langle\sigma(\psi,\Phi)\sigma(\psi',\Phi)\rangle = \mu(\psi,\psi') - (i/2)\sigma(\psi,\psi')$, where $\mu(\psi,\psi')$ is the symmetrised two-point function $(1/2)(\langle\sigma(\psi,\Phi)\sigma(\psi',\Phi)\rangle + \langle\sigma(\psi',\Phi)\sigma(\psi,\Phi)\rangle)$ which defines a symmetric positive-semidefined bilinear form on S. (c) $\langle A^\dagger A\rangle \geq 0$ for elements $A \doteq [e^{i\sigma(\psi,\Phi)} - I] + i[e^{i\sigma(\psi,\Phi)} - I]$ entails:

$$|\sigma(\psi,\psi')|^2 \leq 4\,\mu(\psi,\psi)\mu(\psi',\psi')\,, \qquad \text{for every } \psi,\psi' \in \mathsf{S}\,, \tag{93}$$

which, in turn, implies that μ *is strictly positive defined* because σ is non degenerate. Reversing the procedure, the general definition of quasifree states on Weyl algebras is the following.

Definition B.1. *Let $\mathcal{W}(\mathsf{S})$ be a Weyl algebra and μ a real scalar product on S satisfying (93). A state ω_μ on $\mathcal{W}(\mathsf{S})$ is called the* **quasifree state** *associated with μ if*

$$\omega_\mu(W(\psi)) \doteq e^{-\mu(\psi,\psi)/2}\,, \quad \textit{for all } \psi \in \mathsf{S}. \tag{94}$$

The following technical lemma is useful to illustrate the GNS triple of a quasifree state as established in the subsequent theorem. The last statement in the lemma arises by Cauchy-Schwarz inequality and the remaining part being in Proposition 3.1 in [KW91].

Lemma B.1. *Let (S) be a real symplectic space with σ non degenerate and μ a real scalar product on S satisfying (93). There is a complex Hilbert space H_μ and a map $K_\mu : \mathsf{S} \to \mathsf{H}_\mu$ with:*
(i) K_μ is $\mathbb{R}$-linear with dense complexified range, i.e. $\overline{K_\mu(\mathsf{S}) + iK_\mu(\mathsf{S})} = \mathsf{H}_\mu$,
(ii) for all $\psi,\psi' \in \mathsf{S}$, $\langle K_\mu\psi, K_\mu\psi'\rangle = \mu(\psi,\psi') - (i/2)\sigma(\psi,\psi')$.
Conversely, if the pair (H, K) satisfies (i) and $\sigma(\psi,\psi') = -2Im\langle K\psi, K\psi'\rangle_{\mathsf{H}}$, with $\psi,\psi' \in \mathsf{S}$, the unique real scalar product μ on S satisfying (ii) verifies (93).

An existence theorem for quasifree states can be proved using the lemma above with the following proposition relying on Lemma A.2, Proposition 3.1 and a comment on p.77 in [KW91]).

Proposition B.1. *For every μ as in definition B.1 the following hold.*
(a) *there is a unique quasifree state ω_μ associated with μ and it is regular.*
(b) *The GNS triple $(\mathfrak{H}_{\omega_\mu}, \Pi_{\omega_\mu}, \Omega_{\omega_\mu})$ is determined as follows with respect to (H_μ, K_μ) as in lemma B.1. (i) $\mathfrak{H}_{\omega_\mu}$ is the symmetric Fock space with one-particle space H_μ. (ii) The cyclic vector Ω_{ω_μ} is the vacuum vector of $\mathfrak{H}_\omega$. (iii) Π_{ω_μ} is determined by $\Pi_{\omega_\mu}(W(\psi)) = e^{i\overline{\sigma(\psi,\Phi_{\omega_\mu})}}$, the bar denoting the closure, where*[4]

$$\sigma(\psi,\Phi_{\omega_\mu}) \doteq ia(K_\mu\psi) - ia^\dagger(K_\mu\psi)\,, \quad \textit{for all } \psi \in \mathsf{S} \tag{95}$$

$a(\phi)$ and $a^\dagger(\phi)$, $\phi \in \mathsf{H}_\mu$, being the usual annihilation (antilinear in ϕ) and creation operators defined in the dense linear manifold spanned by the states with finite number of particles.
(c) *A pair $(\mathsf{H}, K) \neq (\mathsf{H}_\mu, K_\mu)$ satisfies (i) and (ii) in lemma B.1 for μ (thus determining the same quasifree state ω_μ), if and only if there is a unitary operator $U : \mathsf{H}_\mu \to \mathsf{H}$ such that $UK_\mu = K$.*
(d) *ω_μ is pure (i.e. its GNS representation is irreducible) if and only if $\overline{K_\mu(\mathsf{S})} = \mathsf{H}_\mu$. In turn, this is equivalent to $4\mu(\psi',\psi') = \sup_{\psi\in\mathsf{S}\setminus\{0\}} |\sigma(\psi,\psi')|/\mu(\psi,\psi)$ for every $\psi' \in \mathsf{S}$.*

Remark B.1.
(1) K_μ is always injective due to (ii) and non degenerateness of σ.

[4]The field operator $\Phi(f)$, with f in the complex Hilbert space $\mathfrak{h}$, used in [BR022] in propositions 5.2.3 and 5.2.4 is related to $\sigma(\psi,\Phi)$ by means of $\sigma(\psi,\Phi) = \sqrt{2}\Phi(iK_\mu\psi)$ assuming $\mathsf{H} \doteq \mathfrak{h}$.

(2) Consider the real Hilbert space obtained by taking the completion of S with respect to μ. The requirement (93) is equivalent to the fact that there is is a bounded operator S everywhere defined over the mentioned Hilbert space, with $S = -S^*$, $||S|| \leq 1$ and such that $\frac{1}{2}\sigma(\psi, \psi') = \mu(\psi, S\psi')$, for all $\psi, \psi' \in \mathsf{S}$.
(3) The pair (H_μ, K_μ) is called the **one-particle structure** of the quasifree state ω_μ.

Let us pass to discuss the KMS condition [Hu72, Ha92, BR022]. KMS state are the algebraic corresponding, for infinitely extended systems, of thermal states of standard statistic mechanics. There are several different equivalent definitions of KMS states, see [BR022] for a list of various equivalent definitions. Comparing Definition 5.3.1 and Proposition 5.3.7 in [BR022] we adopt the following one.

Definition B.2. A state ω on a C^*-algebra $\mathscr{A}$ is said to be a **KMS state at inverse temperature** $\beta \in \mathbb{R}$ with respect to a one-parameter group of $*$-automorphisms $\{\alpha_t\}_{t\in\mathbb{R}}$ (representing, from the algebraic point of view, some notion of time-evolution) if, for every pair $A, B \in \mathscr{A}$, and referring to the function $\mathbb{R} \ni t \mapsto \omega\left(A\alpha_t(B)\right) =: F^{(\omega)}_{A,B}(t)$, the following facts hold.
(a) $F^{(\omega)}_{A,B}$ extends to a continuous complex function $F^{(\omega)}_{A,B} = F^{(\omega)}_{A,B}(z)$ with domain

$$\overline{D_\beta} \doteq \{z \in \mathbb{C} \,|\, 0 \leq Imz \leq \beta\} \quad \text{if } \beta \geq 0, \text{ or} \quad \overline{D_\beta} \doteq \{z \in \mathbb{C} \,|\, \beta \leq Imz \leq 0\} \quad \text{if } \beta \leq 0,$$

(b) $F^{(\omega)}_{A,B} = F^{(\omega)}_{A,B}(z)$ is analytic in the interior of $\overline{D_\beta}$;
(c) it holds, and this identity is – a bit improperly – called the **KMS condition**:

$$F^{(\omega)}_{A,B}(t + i\beta) = \omega\left(\alpha_t(B)A\right) \ , \quad \text{for all } t \in \mathbb{R}. \tag{96}$$

With the given definition, an $\{\alpha_t\}_{t\in\mathbb{R}}$-KMS state ω turns out to be invariant under $\{\alpha_t\}_{t\in\mathbb{R}}$ [BR022]; the function $\overline{D_\beta} \ni z \mapsto F^{(\omega)}_{A,B}(z)$ is uniquely determined by its restriction to real values of z (by the "edge of the wedge theorem") and $\sup_{\overline{D_\beta}} |F^{(\omega)}_{A,B}| = \sup_{\partial D_\beta} |F^{(\omega)}_{A,B}|$ (by the "three lines theorem") [BR022].
Equivalent definitions of KMS states are obtained by the following propositions, the second for quasifree states, due to Kay [Ka85b, KW91] and relying upon earlier results by Hugenholtz [Hu72]. We sketch the proofs since they are very spread in the literature.

Proposition B.2. *An algebraic state ω, on the C^*-algebra $\mathscr{A}$, which is invariant under the one-parameter group of $*$-automorphisms $\{\alpha_t\}_{t\in\mathbb{R}}$ is a KMS state at the inverse temperature $\beta \in \mathbb{R}$ if and only if its GNS triple $(\mathcal{H}_\omega, \Pi_\omega, \Omega_\omega)$ satisfies the following three requirements.*
(1) The unique unitary group $\mathbb{R} \ni t \mapsto U_t$ which leaves Ω_ω invariant and implements $\{\alpha_t\}_{t\in\mathbb{R}}$ – i.e. $\Pi_\omega\left(\alpha_t(A)\right) = U_t \Pi_\omega(A) U_t^*$ for all $A \in \mathscr{A}$ and $t \in \mathbb{R}$ – is strongly continuous, so that $U_t = e^{itH}$ for some self-adjoint operator H on $\mathcal{H}_\omega$.
(2) $\Pi_\omega\left(\mathscr{A}\right)\Omega_\omega \subset Dom\left(e^{-\beta H/2}\right)$.
(3) There exists an antilinear operator $J : \mathcal{H}_\omega \to \mathcal{H}_\omega$ with $JJ = I$ such that:

$$Je^{-itH} = e^{-itH}J \quad \textit{for all } t \in \mathbb{R}, \textit{ and} \quad e^{-\beta H/2}\Pi_\omega(A)\Omega_\omega = J\Pi_\omega(A^*)\Omega_\omega \quad \textit{for all } A \in \mathscr{A}.$$

Proof. A $\{\alpha_t\}_{t\in\mathbb{R}}$-KMS state with inverse temperature β is $\{\alpha_t\}_{t\in\mathbb{R}}$-invariant and fulfils the conditions (1), (2) and (3) due to Theorem 6.1 in [Hu72]. Conversely, consider an $\{\alpha_t\}_{t\in\mathbb{R}}$-invariant state ω on $\mathscr{A}$ fulfilling the conditions (1), (2) and (3). When A and B are entire analytic elements of $\mathscr{A}$ (see [BR022]), $\mathbb{R} \ni t \mapsto F^{(\omega)}_{A,B}(t)$ uniquely extends to an analytic function on the whole $\mathbb{C}$ and thus (a) and (b) in def.

B.2 are true. (1), (2), (3) and $e^{zH}\Omega_\omega = \Omega_\omega$, for all $z \in \overline{D_\beta}$ (following from (2) and (3)) entail (c), too:

$$\omega(\alpha_t(B)A) = \langle \Omega_\omega,\, U_t \Pi_\omega(B) U_t^* \Pi_\omega(A)\Omega_\omega\rangle = \langle \Pi_\omega(B^*)\Omega_\omega,\, U_t^*\Pi_\omega(A)\Omega_\omega\rangle = \langle J U_t^* \Pi_\omega(A)\Omega_\omega,\, J\Pi_\omega(B^*)\Omega_\omega\rangle$$

$$= \langle U_t^* e^{-\beta H/2}\Pi_\omega(A^*)\Omega_\omega,\, e^{-\beta H/2}\Pi_\omega(B)\Omega_\omega\rangle = \langle \Omega_\omega,\, \Pi_\omega(A) e^{i(t+i\beta)H}\Pi_\omega(B) e^{-i(t+i\beta)H}\Omega_\omega\rangle = F^{(\omega)}_{A,B}(t+i\beta)\,.$$

The validity of conditions (a), (b) and (c) for entire analytic elements $A, B \in \mathscr{A}$ implies the validity for all $A, B \in \mathscr{A}$, as established in [BR022] (compare Definition 5.3.1 and Proposition 5.3.7 therein). □

Proposition B.3. *Consider a quasifree algebraic state ω_μ on the Weyl-algebra $\mathcal{W}(\mathsf{S})$, with one-particle structure $(\mathsf{H}_\mu, \mathsf{K}_\mu)$. Assume that ω_μ is invariant under the one-parameter group of $*$-automorphisms $\{\alpha_t\}_{t\in\mathbb{R}}$, which is implemented by the strongly continuous unitary one-parameter group $\mathbb{R} \ni t \mapsto V_t \doteq e^{i\tau h}$ in the one particle space H_μ (so that U_t in proposition B.2 is the tensorialization of V_t). The following facts are equivalent.*

(a) *ω_μ is a KMS at the inverse temperature $\beta \in \mathbb{R}$ with respect to $\{\alpha_t\}_{t\in\mathbb{R}}$.*

(b) *There is an anti-unitary operator $j : \mathsf{H}_\mu \to \mathsf{H}_\mu$ with $jj = I$ and the following facts hold:*
(i) $\mathsf{K}_\mu(\mathsf{S}) \subset Dom\left(e^{-\frac{1}{2}\beta h}\right)$, (ii) $[j, V_t] = 0$ for all $t \in \mathbb{R}$, (iii) $e^{-\frac{1}{2}\beta h}\mathsf{K}_\mu\psi = -j\mathsf{K}_\mu\psi$ for all $\psi \in \mathsf{S}(\mathcal{H}_\mu)$.

(c) *$\mathsf{K}_\mu(\mathsf{S}) \subset Dom\left(e^{-\frac{1}{2}\beta h}\right)$ and $\langle e^{-ith}x, y\rangle = \langle e^{-\beta h/2}y, e^{-ith}e^{-\beta h/2}x\rangle$ if $x, y \in \mathsf{K}_\mu(\mathsf{S})$ and $t \in \mathbb{R}$.*

Proof. (a) is equivalent to (b) as proved on pages 80-81 in [KW91]. (b) entails (c) straightforwardly. Assuming (c) and exploiting (i) of lemma B.1, $j : \mathsf{H}_\mu \to \mathsf{H}_\mu$ fulfilling (b) is completely individuated by continuity and anti-linearity if requiring that $j\mathsf{K}_\mu\psi = -e^{-\frac{1}{2}\beta h}\mathsf{K}_\mu\psi$ when $\psi \in \mathsf{S}$. □

C Proofs of some propositions.

Proof of Lemma 2.1. First of all, as in the Appendix A, consider the conformal extension $(\widetilde{\mathscr{M}}, \widetilde{g})$ of the spacetime $(\mathscr{M}, g)$ determined in [SW83] where $\widetilde{g} = g/r^2$ in $\mathscr{M}$ (see figure 2). In view of the previously illustrated properties of E_{P_g}, if $\varphi \in \mathsf{S}(\mathscr{M})$, there is a smooth function f_φ with support contained in $\mathscr{M}$ and such that $\varphi = E_{P_g} f_\varphi$, so that $supp\varphi \in J^+(suppf_\varphi; \mathscr{M}) \cup J^-(suppf_\varphi; \mathscr{M})$. Since $J^\pm(suppf_\varphi; \mathscr{M}) \subset J^\pm(suppf_\varphi; \widetilde{\mathscr{M}})$, it is obvious from the structure of $\widetilde{\mathscr{M}}$ (see figure 2) that, if the smooth extension $\widetilde{\varphi}$ of $r\varphi$ to a neighborhood of $\Im^\pm$ in $\widetilde{\mathscr{M}}$ exists, as asserted in the thesis, it must have support bounded by constants $v^{(\varphi)}, u^{(\varphi)} \in (-\infty, \infty)$ adopting the relevant null coordinates in that neighborhood: Ω, u, θ, ϕ or Ω, v, θ, ϕ respectively, where $\Omega = 1/r$ in $\mathscr{W}$. Furthermore, in view of the shape of $J^\pm(suppf_\varphi; \widetilde{\mathscr{M}})$, the analogous property holds true for the support of φ in $\mathscr{W}$. The existence of $\widetilde{\varphi}$ can be established as follows examining the various possible cases. First assume that $suppf_\varphi \subset \mathscr{W}$. Let $p \in \mathscr{W}$ be in the chronological past of $suppf_\varphi$ sufficiently close to i^-. Next consider a second point q beyond $\Im^+$ sufficiently close to $\Im^+$ so that the closure of $\mathscr{N}_{p,q} \doteq I^+(p; \widetilde{\mathscr{M}}) \cap I^-(q; \widetilde{\mathscr{M}})$ does not meet the timelike singularity in the conformal extension of $\mathscr{M}$ on the right of $\Im^+$. Consider $\mathscr{N}_{p,q}$ as a spacetime on its own right equipped with the metric $\tilde{g}$. As that spacetime is globally hyperbolic (it can be proved by direct inspection veryfying that the diamonds $J^+(r; \mathscr{N}_{p,q}) \cap J^-(s; \mathscr{N}_{p,q})$ are empty or compact for $r, s \in \mathscr{N}_{p,q}$ and that the spacetime itself is causal) $E_{P_{\tilde{g}}}$ is well defined and individuates a solution $\widetilde{\varphi} \doteq E_{P_{\tilde{g}}} f_\varphi$ of the Klein-Gordon equation associated with $P_{\tilde{g}}$, defined as in (14) with $\widetilde{g} = g/r^2$. By known preperties of Klein-Gordon equation related with conformal rescaling [Wa84] one has $\widetilde{\varphi} = r\varphi$ in $\mathscr{M}$ because $\widetilde{g} = g/r^2$ therein. Keeping p and moving q paralelly to $\Im^+$ towards i^+, one obtains an increasing class of larger and larger globally-hyperbolic spacetimes $\mathscr{N}_{p,q}$ and, correspondingly, a class of analogous extensions $\widetilde{\varphi}$ on corresponding $\mathscr{N}_{p,q}$. Notice that considering two of these extensions, they coincide in the intersections

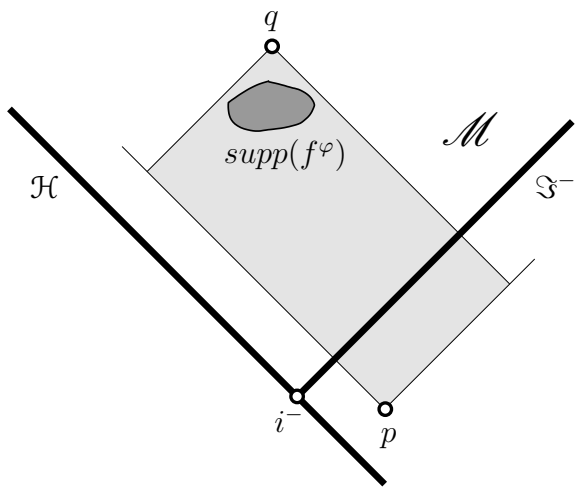

Figure 3: The gray region indicates the globally hyperbolic subspacetime $\mathscr{N}_{p,q}$ of $\widetilde{\mathscr{M}}$, the point p eventually tends to i^-.

of their domains (see figure 3). That is in view of the uniqueness of the solution the Cauchy problem as any compact portion of a spacelike Cauchy surface of a globally hyperbolic spacetime can be extended to a smooth spacelike Cauchy surface of any larger globally hyperbolic spacetime [BS06], and the initial Cauchy data can be viewed as Cauchy data on this larger spacelike Cauchy surface, since any spacelike Cauchy surface is acausal it being acronal (Lemma 42 from Chap. 14 in [O'N83]). In this way a smooth extension of $r\varphi$ turns out to be defined in a neighborhood of $\Im^+$. An analogous argument proves the existence of the analogous extension about $\Im^-$. Let us now suppose that $supp f_\varphi \subset \mathscr{B}$. In that case φ cannot reach $\Im^+$ and the only extension of $r\varphi$ we have to prove to exist is that about $\Im^-$. The procedure is similar to that employed above, but now the class of globally hyperbolic spacetime is constructed as follows. Take a point q beyond $\Im^-$ sufficiently close to i^- and consider the intersection $\mathscr{N}_q \doteq I^+(q;\widetilde{\mathscr{M}}) \cap I^-(\mathscr{M};\widetilde{\mathscr{M}})$. Moving q parallelly to $\Im^-$ and closer and closer to i^- one obtains an increasing class of globally hyperbolic spacetimes, equipped with the metric $\widetilde{g} = g/r^2$, and corresponding class of solutions of rescaled Klein-Gordon equation defining the smooth extension $\widetilde{\varphi}$ of $r\varphi$ about an open neighborhood of $\Im^-$. Now consider the case where $supp f_\varphi$ is concentrated in an open neighborhood of $\mathcal{H}_{ev}$ we can fix arbitrarily shrunk about $\mathcal{H}_{ev}$. The smooth extension $\widetilde{\varphi}$ of $r\varphi$ about an open neighborhood of $\Im^-$ is constructed exactly as the previously examined case. Concerning the analogous extension about $\Im^+$, the relevant class of globally hyperbolic spacetimes is obtained as follows. Fix a point $q \in \mathscr{W}$ in the chronological past of $supp f_\varphi$ sufficiently close to $\mathcal{H}$. Next consider a smooth spacelike surface Σ in the chronological future of $supp f_\varphi$ but in the past of i^+ in $\widetilde{\mathscr{M}}$ which intersects $\Im^+$ for some $u = u_\Sigma$. The relevant class of globally hyperbolic spacetimes is now made of the sets $\mathscr{N}_{q,\Sigma} \doteq J^-(\Sigma;\widetilde{\mathscr{M}}) \cap J^+(q;\widetilde{\mathscr{M}})$ when Σ moves towards i^+. It remains to consider the case where $supp f_\varphi$ intersect $\mathcal{H}_{ev}$, but it is not confined in a small neighborhood of $\mathcal{H}_{ev}$. In this case, taking into account the linearity of the causal propagator and $P_{\widetilde{g}}$, we can reduce to work with a combination of the three above considered cases. Decomposing the constant function 1 in $\mathscr{W}$ as the sum of three non-negative smooth functions $1 = f_1 + f_2 + f_3$, with f_1 supported in $\mathscr{B}$, f_2 supported in $\mathscr{W}$ and f_3 supported in a open neighborhood of $\mathcal{H}_{ev}$ that we can fix arbitrarily shrunk about $\mathcal{H}_{ev}$, we have a corresponding decomposition f_φ as $f_\varphi = f_\varphi \cdot f_1 + f_\varphi \cdot f_2 + f_\varphi \cdot f_3$. Defining $r\varphi_i = rE_{P_g}(f_\varphi \cdot f_i)$, $i = 1, 2, 3$ each wavefunction can be treated separately as discussed above, obtaining corresponding extensions $\widetilde{\varphi}_i$ to $\Im^+$. The sum of those extensions is the wanted extension $\widetilde{\varphi}$ of

$r\varphi$ by construction. The same procedure applies to the case of $\Im^-$. $\square$

Proof of Proposition 2.1. (a) We consider the proof for the case of $t > 0$ (i.e. the behaviour of the wavefunctions about $\mathcal{H}_{ev}$ and $\Im^+$ only), the remaining case being then an immediate consequence of the symmetry $X \to -X$ of the Kruskal geometry.
First of all it is worth noticing that each of our coordinates u, v amounts to twice the corresponding coordinate defined in [DR09] and the difference of our r^* and that defined in [DR09] is $3m + 2m \ln m$.
The bounds concerning the constants C_1 and C_3 are proved in Theorem 1.1 of [DR09]. There, sufficiently regular solutions of the massless Klein-Gordon equation are considered with Cauchy data on a smooth complete spacelike Cauchy surfaces of the full Kruskal extension of $\mathscr{M}$ which is asymptotically flat at spatial infinity, moreover the Cauchy data are supposed to vanish fast enough at space infinity. In our case these requirements are fulfilled because the elements of $\mathsf{S}(\mathscr{M})$ are smooth and have compact support on every smooth spacelike Cauchy surface of $\mathscr{M}$ and employing the results in [BS06] (and using the fact that spacelike Cauchy surface are acausal [O'N83]) we can view these as Cauchy data on a smooth spacelike Cauchy surface full Kruskal extension. The bound concerning C_2 has the same proof as that concerning C_1 because $X(\varphi) \in \mathsf{S}(\mathscr{M})$ when $\varphi \in \mathsf{S}(\mathscr{M})$ because X is a smooth Killing vector field. To conclude it is enough proving the last bound, related with the constant C_4. To this end fix $\varphi \in \mathsf{S}(\mathscr{M})$ and re-define, if necessary, the origin of the killing time t in $\mathscr{W}$ in order that $u^{(\varphi)} \geq 2$, where $u^{(\varphi)}$ is the constant defined in Lemma 2.1. Now we focus on the proof contained in sec. 13.2 of [DR09] (see the part called "*decay in* $r \geq \hat{R}$", concerning the bound associated with C_3. We want to adapt that proof for our case, replacing the solution ϕ considered there with our $X(\varphi)$ (which still is a solution as noticed above), so that $r\phi$ is replaced by $rX(\varphi) = \widetilde{X}(r\varphi)$ in $\mathscr{W}$ which smoothly extends to $X(\widetilde{\varphi})$ on $\mathscr{W} \cup \Im^+$. It is sufficient to prove the bound in the region $\{r > \hat{R}\} \cap \{t > 0\}$ in $\mathscr{W}$, since the bound would then hold on $\Im^+$ by continuity.
Notice that only the region $\{r \geq \hat{R}\} \cap \{t > 0\} \cap \{u \geq 2\}$ has to be considered. Indeed, in the set $\{u < 2\} \cap \{v > v_0^{(\varphi)}\}$ for some $v_0^{(\varphi)} \in \mathbb{R}$, $X(\widetilde{\varphi})$ vanishes due to Lemma 2.1. So $X(\widetilde{\varphi})$ vanishes in $\{r \geq \hat{R}\} \cap \{t > 0\} \cap \{u < 2\} \cap \{v > v_0^{(\varphi)}\}$, satisfying the wanted bound trivially. Moreover, the region individuated by $\{r \geq \hat{R}\} \cap \{t \geq 0\} \cap \{u \leq 2\} \cap \{v \leq v_0^{(\varphi)}\}$ is compact so $X(\widetilde{\varphi})$ is bounded therein and it satisfies the wanted bound trivially.
In the region $\{r \geq \hat{R}\} \cap \{t > 0\} \cap \{u \geq 2\}$, following the way outlined on p.916-917 for ϕ replaced by $\varphi' \doteq \widetilde{X}(\varphi)$ [DR09] we achieve, employing a Sobolev inequality on the sphere

$$r^2 \left|\varphi'(u,v,\theta,\phi)\right|^2 \leq C \int_{\mathbb{S}^2} r^2 \left|\varphi'\right|^2 d\mathbb{S}^2 + C \int_{\mathbb{S}^2} \left|r \not\nabla \varphi'\right|^2 r^2 d\mathbb{S}^2 + C \int_{\mathbb{S}^2} \left|r^2 \not\nabla \not\nabla \varphi'|^2\right) r^2 d\mathbb{S}^2 \,,$$

where $\not\nabla$ denotes the covariant derivative with respect to metric induced on the sphere of radius r and $d\mathbb{S}^2$ is the volume form on the sphere of radius 1. If the squared angular momentum operator is denoted by $\Omega^2 \doteq r^2 \not\nabla \not\nabla$ the inequality above can be re-written as:

$$r^2 \left|\varphi'(u,v,\theta,\phi)\right|^2 \leq C \int_{\mathbb{S}^2} \left|\Omega^0 \varphi'\right|^2 r^2 d\mathbb{S}^2 + C \int_{\mathbb{S}^2} \left|\Omega^0 \varphi'\right| \left|\Omega^1 \varphi'\right| r^2 d\mathbb{S}^2 + C \int_{\mathbb{S}^2} \left|\Omega^2 \varphi'\right|^2 r^2 d\mathbb{S}^2 \,. \tag{97}$$

To conclude it is sufficient to prove that, for $k = 0, 1, 2$ and if $r \geq \hat{R}$, $u \geq 2$, $t > 0$:

$$\int_{\mathbb{S}^2} \left|\Omega^k \varphi'\right|^2 r^2 d\mathbb{S}^2 \leq B_k / u^2 \tag{98}$$

for some constants $B_k \geq 0$. Notice that the second integral in the right hand side of (97) is bounded by the product of the square root of the integrals with $k = 0$ and $k = 1$ in the left-hand side of (98) in

view of Cauchy-Schwartz inequality. If we follow [DR09] and if we pass to the coordinates (t, r^*, θ, ϕ) (see Section A), and for some constant $D \geq 0$:

$$\int_{\mathbb{S}^2} \left|\Omega^k \varphi'\right|^2 r^2(t, r^*, \theta, \phi) d\mathbb{S}^2 \leq \int_{\mathbb{S}^2} \left|\Omega^k \varphi'\right|^2 r^2(t, \tilde{r}^*, \theta, \phi) d\mathbb{S}^2$$
$$+D \int_{\tilde{r}^*}^{r^*} \int_{\mathbb{S}^2} |\partial_\rho \Omega^k \varphi'||\Omega^k \varphi'| r^2(t, \rho, \theta, \phi) d\rho d\mathbb{S}^2 + D \int_{\tilde{r}^*}^{r^*} \int_{\mathbb{S}^2} |\Omega^k \varphi'|^2 r(t, \rho, \theta, \phi) d\rho d\mathbb{S}^2 \,. \tag{99}$$

Following [DR09], the parameter $\tilde{r}^* \geq \hat{R}^*$ can be fixed so that the first integral in the right-hand side satisfies

$$\int_{\mathbb{S}^2} \left|\Omega^k \varphi'\right|^2 r^2(t, \tilde{r}^*, \theta, \phi) d\mathbb{S}^2 \leq \bar{E}_2/t^2 \leq \bar{E}_2/u^2 \tag{100}$$

where the constant $\bar{E}_2$ was defined in [DR09] and depends on $\Omega^k \varphi' \in \mathsf{S}(\mathscr{M})$ and we have used the fact that $u = t - r^* \geq 2$ with $r^* > 0$ and $t > 0$, so that $u \leq t$. Here our procedure departs form that followed in [DR09]. Concerning the third integral in the right-hand side of (99) it can be re-written

$$\int_{\tilde{r}^*}^{r^*} \int_{\mathbb{S}^2} (\partial_t \Omega^k \varphi)^2 r(t, \rho, \theta, \phi) d\rho d\mathbb{S}^2 \leq const. \int_{\tilde{r}^*}^{r^*} \int_{\mathbb{S}^2} (\partial_t \Omega^k \varphi)^2 r^2(t, \rho, \theta, \phi) d\rho d\mathbb{S}^2 \leq F(\mathcal{S}) \tag{101}$$

where $\mathcal{S}$ is the achronal hypersurface individuated by the time t, the interval $[\tilde{r}^*, r^*]$ and the coordinates ω varying over $\mathbb{S}^2$. $F(\mathcal{S})$ is the flux of energy through $\mathcal{S}$ associated with the Klein-Gordon field $\Omega^k \varphi$. Theorem 1.1 in [DR09] assures now that, for some constant C', depending on φ,

$$F(\mathcal{S}) \leq C'/v_+(\mathcal{S})^2 + C'/u_+(\mathcal{S})^2 \,,$$

where $v_+(\mathcal{S}) = \max\{\inf_{\mathcal{S}} v, 2\}$ and $u_+(\mathcal{S}) = \max\{\inf_{\mathcal{S}} u, 2\}$. In our case, by construction, we have $\max\{\inf_{\mathcal{S}} v, 2\} \geq t + \hat{R}^*$ and $\max\{\inf_{\mathcal{S}} u, 2\} = u(t, r^*, \phi, \theta)$. As a conclusion, for $t > 0$, $r \geq \hat{R}$, $u > 2$:

$$\int_{\tilde{r}^*}^{r^*} \int_{\mathbb{S}^2} |\Omega^k \varphi'|^2 r(t, \rho, \theta, \phi) d\rho d\mathbb{S}^2 \leq \frac{C'}{(t + \hat{R}^*)^2} + \frac{C'}{u^2} = \frac{C'}{(u + r^* + \hat{R}^*)^2} + \frac{C'}{u^2} \leq \frac{2C'}{u^2} \,,$$

and thus

$$\int_{\tilde{r}^*}^{r^*} \int_{\mathbb{S}^2} |\Omega^k \varphi'|^2 r(t, \rho, \theta, \phi) d\rho d\mathbb{S}^2 \leq const. \int_{\tilde{r}^*}^{r^*} \int_{\mathbb{S}^2} |\Omega^k \varphi'|^2 r(t, \rho, \theta, \phi)^2 d\rho d\mathbb{S}^2 \leq \frac{K}{u^2} \tag{102}$$

Let us finally consider the second integral in the right-hand side of (99). We preventively notice that

$$\int_{\tilde{r}^*}^{r^*} \int_{\mathbb{S}^2} (\partial_\rho \Omega^k \varphi')^2 r(t, \rho, \theta, \phi)^2 d\rho d\mathbb{S}^2 \leq F'(\mathcal{S}) \tag{103}$$

where $F'(\mathcal{S})$ is the flux of energy through $\mathcal{S}$ associated with the Klein-Gordon field $\Omega^k \varphi'$. Dealing with as before, we obtain the bound

$$\int_{\tilde{r}^*}^{r^*} \int_{\mathbb{S}^2} (\partial_\rho \Omega^k \varphi')^2 r(t, \rho, \theta, \phi)^2 d\rho d\mathbb{S}^2 \leq \frac{K'}{u^2} \,. \tag{104}$$

Cauchy-Schwartz inequality, (102) and (104) lead to

$$\int_{\tilde{r}^*}^{r^*} \int_{\mathbb{S}^2} |\partial_\rho \Omega^k \varphi'||\Omega^k \varphi'| r^2(t, \rho, \theta, \phi) d\rho d\mathbb{S}^2 \leq \frac{K''}{u^2} \,. \tag{105}$$

Putting all together in the right-hand side of (99), the bounds (100), (102) and (105) yield (98).
(b) Fix Σ as any smooth spacelike Cauchy surface of $\mathscr{M}$. Notice that if the sequence of Cauchy data converge to zero in the test function topology on Σ, there is a compact C in Σ containing all the supports of the Cauchy datata of the sequence by definition. In view of [BS06], we can construct a smooth spacelike Cauchy surface Σ' of the complete Kruskal manifold $\mathscr{K}$, which includes that compact. Thus, the sequence of Cauchy data tends to 0 in the test function topology of Σ' as well. The Cauchy data on Σ can be interpreted as Cauchy data on Σ' since the supports of the solutions cannot further intersect Σ' as it is acausal (it being achronal and spacelike [O'N83]). From standard results of continuous dependence from compactly-supported Cauchy data of the smooth solutions of hyperbolic equations in globally hyperbolic spacetimes (see Theorem 3.2.12 in [BGP96]), if the Cauchy data on a fixed spacelike Cauchy surface Σ' tend to 0 in the test function topology, then the solution tends to 0 in the topology of $C^\infty(\mathscr{K};\mathbb{R})$. On the other hand, as one can prove by standard result of the topology of causal sets (e.g. see cap 8 in [Wa84] and use theorems 8.3.11 and 8.3.12 and the fact that the open double cones form a base of the topology) $J^+(C;\mathscr{K}) \cup J^-(C;\mathscr{K})$ has compact intersection with every spacelike Cauchy surface of $\mathscr{K}$, since C is compact in Σ'. So all the Cauchy data of on Σ'' of the considered sequence of solutions are contained in a compact, too. ¿From this pair of results we conclude that, if the Cauchy data tend to 0 in the test function topology on Σ', the Cauchy data of the solutions tend to 0 in the test function topology on any other spacelike Cauchy surface Σ'' of $\mathscr{K}$. For convenience, we fix Σ'' as an extension of the spacelike Cauchy surface of $\mathscr{W}$ (whose closure intersects $\mathcal{B}$) individuated in $\mathscr{W}$ by the locus $t = 1$. Referring to (a), one sees that the coefficients C_i are obtained as the product of universal constants and integrals of derivatives of the compactly supported Cauchy data of φ, and $X(\varphi)$ where appropriate, over $\Sigma'' \cap \overline{\mathscr{W}}$ as explained in Theorem 1.1, Theorem 7.1 and in the formulae appearing in sec. 4 of [DR09] (it is convenient to reformulate those formulae employing global coordinates U and V instead u and v and passing from the variable r^* to the variable r). From these formulas it follows immediately that the constants C_i vanish provided that the Cauchy data tend to 0 in the test function topology on Σ'', and this requirement is valid in our hypotheses. $\square$

Proof of Proposition 3.2. (a) First of all, we notice that, by direct inspection as proved in [KW91] and [Mo08] (making use of the result presented in the Appendix C of [Mo08] with the caveat that, in this cited paper, the angular coordinates (θ, ϕ) are substituted by the complex ones $(z, \bar{z})$ obtained out of stereographic projection), it turns out that:

$$\begin{aligned}\langle \widehat{\psi}_+, \widehat{\psi'}_+ \rangle_{\mathsf{H}_{\mathcal{H}}} &\doteq \int_{\mathbb{R}_+ \times \mathbb{S}^2} \overline{\widehat{\psi}_+}(K,\theta,\phi) \widehat{\psi'}_+(K,\theta,\phi)\, 2K dK \wedge r_S^2 d\mathbb{S}^2 \\ &= \lim_{\epsilon \to 0^+} -\frac{r_S^2}{\pi} \int_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2} \frac{\overline{\psi(U_1,\theta,\phi)}\psi'(U_2,\omega)}{(U_1 - U_2 - i\epsilon)^2} dU_1 \wedge dU_2 \wedge d\mathbb{S}^2 \end{aligned} \tag{106}$$

for $\psi, \psi' \in C_0^\infty(\mathcal{H};\mathbb{C})$.
As a consequence we have obtained that the map $M : C_0^\infty(\mathcal{H};\mathbb{C}) \ni \psi \mapsto \widehat{\psi}_+(K,\omega) \in \mathsf{H}_{\mathcal{H}}$ is isometric and thus, by continuity, it uniquely extends to a Hilbert space isomorphism $F_{(U)}$ of $\overline{(C_0^\infty(\mathcal{H};\mathbb{C}), \lambda_{KW})}$ onto the closed Hilbert space $\overline{M(C_0^\infty(\mathcal{H};\mathbb{C}))} \subset \mathsf{H}_{\mathcal{H}}$. To conclude the proof of the first statement in (a), it is enough to establish that $\overline{M(C_0^\infty(\mathcal{H};\mathbb{C}))} = \mathsf{H}_{\mathcal{H}}$. This immediately follows from the two lemma proved below.

Lemma C.1. *$\overline{M(C_0^\infty(\mathcal{H};\mathbb{C}))}$ includes the space $\mathscr{S}_0$ whose elements $f = f(K,\omega)$ are the restrictions to $\mathbb{R}_+ \times \mathbb{S}^2$ of the functions in $\mathscr{S}(\mathbb{R}\times\mathbb{S}^2)$ and vanish in a neighbourhood of $K = 0$ depending on f.*

Above and henceforth $\mathscr{S}(\mathbb{R}\times\mathbb{S}^2)$ denotes the complex **Schwartz space** on $\mathbb{R}\times\mathbb{S}^2$, *i.e.* the space of complex-valued smooth functions on $\mathbb{R}\times\mathbb{S}^2$ which vanish, with all their K-derivatives of every order, as $|K|\to+\infty$ uniformly in the angles and faster than every inverse power of $|K|$. This space can be equipped with the usual topology induced by seminorms (see Appendix C of [Mo08]).

Lemma C.2. *$\mathscr{S}_0$ is dense in $\mathsf{H}_{\mathcal{H}}$.*

Concerning (b), we notice that, if $f\in\mathscr{S}_0$, $if\in\mathscr{S}_0$ and both vanish in a neighbourhood of $K=0$. Therefore, it is possible to arrange two *real* functions in $\mathscr{S}(\mathcal{H})$, g_1 and g_2 such that $\widehat{g_1}_+=f$ and $\widehat{g_2}_+=if$. With the same proof of Lemma C.1 one can prove that g_i are the the limits, in the topology of λ_{KW}, of sequences $\{f_{(i)n}\}\subset C_0^\infty(\mathcal{H};\mathbb{R})$. We have obtained that every complex element of the dense subspace $\mathscr{S}_0\subset\mathsf{H}_{\mathcal{H}}$ is the limit of elements of $F_{(U)}\left(C_0^\infty(\mathcal{H};\mathbb{R})\right)$. $\square$

Proof of Lemma C.1. Take $f\in\mathscr{S}_0$. As a consequence, it can be written as the restriction to $\mathbb{R}_+\times\mathbb{S}^2$ of $F\in\mathscr{S}(\mathbb{R}\times\mathbb{S}^2)$. In turn, $F=\mathscr{F}_+(g)$ for some $g\in\mathscr{S}(\mathbb{R}\times\mathbb{S}^2)$, since the Fourier transform is bijective from $\mathscr{S}(\mathbb{R}\times\mathbb{S}^2)$ onto $\mathscr{S}(\mathbb{R}\times\mathbb{S}^2)$ (see Appendix C of [Mo08]). Since $C_0^\infty(\mathbb{R}\times\mathbb{S}^2;\mathbb{C})$ is dense in $\mathscr{S}(\mathbb{R}\times\mathbb{S}^2)$ in the topology of the latter, there is a sequence $\{g_n\}\subset C_0^\infty(\mathbb{R}\times\mathbb{S}^2;\mathbb{C})$ with $g_n\to g$ in the sense of $\mathscr{S}(\mathbb{R}\times\mathbb{S}^2)$. Since the Fourier transform is continuous with respect to that topology, we conclude that $\mathscr{F}_+(g_n)\to F$ in the sense of $\mathscr{S}(\mathbb{R}\times\mathbb{S}^2)$. By direct inspection one finds that the achieved result implies that $\mathscr{F}_+(g_n)\restriction_{\mathbb{R}_+\times\mathbb{S}^2}\to F\restriction_{\mathbb{R}_+\times\mathbb{S}^2}$ in the topology of every $L^2(\mathbb{R}_+\times\mathbb{S}^2,cK^ndK\wedge r_S^2d\mathbb{S}^2)$ for every power $n=0,1,2,\ldots$ and $c>0$. In particular it happens for $n=c=2$. We have found that, for every $f\in\mathscr{S}_0$, there is a sequence in $M(C_0^\infty(\mathbb{R}\times\mathbb{S}^2;\mathbb{C}))$ which tends to f in the topology of $\mathsf{H}_{\mathcal{H}_R}$ and thus $\mathscr{S}_0\subset\overline{M(C_0^\infty(\mathbb{R}\times\mathbb{S}^2;\mathbb{C}))}$. $\square$

Proof of Lemma C.2. In this proof $\mathbb{R}_+^*\doteq(0,+\infty)$ and $\mathbb{N}^*=\{1,2,\ldots\}$. A well-known result is that $C_0^\infty((a,b);\mathbb{C})$ is dense on $L^2((a,b),dx)$ so that, in particular, $C_0^\infty((1/n,n);\mathbb{C})$ is dense in $L^2((1/n,n),dx)$, and thus, passing to the new variable $K=\sqrt{x}$, $C_0^\infty((1/\sqrt{n},\sqrt{n});\mathbb{C})$ is dense in $L^2((1/\sqrt{n},\sqrt{n}),2KdK)$. Since, in the sense of the Hilbertian direct sum, $\oplus_{n\in\mathbb{N}}L^2((1/\sqrt{n},\sqrt{n}),2KdK)=L^2(\mathbb{R}_+^*,2KdK)$ (for instance making use of Lebesgue's dominated convergence theorem), we conclude that $C_0^\infty(\mathbb{R}_+^*;\mathbb{C})=\cup_{n\in\mathbb{N}^*}C_0^\infty((1/n,n);\mathbb{C})$ is dense in $L^2(\mathbb{R}_+^*,2KdK)=L^2(\mathbb{R}_+,2KdK)$ and thus there must be a Hilbert base $\{f_n\}_{n\in\mathbb{N}}\subset C_0^\infty(\mathbb{R}_+^*;\mathbb{C})$.
By known theorems on Hilbert spaces with product measure, we know that a Hilbert base of the space $L^2(\mathbb{R}_+\times\mathbb{S}^2,2KdK\wedge r_S^2d\mathbb{S}^2)$ is $\{f_nY_m\}_{n,m\in\mathbb{N}}$, provided $\{Y_m\}_{m\in\mathbb{N}}$ is a Hilbert base for $L^2(\mathbb{S}^2,r_S^2d\mathbb{S}^2)$ and $\{f_n\}_{n\in\mathbb{N}}$ is a Hilbert base of $L^2(\mathbb{R}_+,2KdK)$. The Y_m can be chosen as harmonic functions so that they are smooth and compactly supported (since $\mathbb{S}^2$ is compact). Therefore, if $\{f_n\}_{n\in\mathbb{N}}\subset C_0^\infty((0,+\infty);\mathbb{C})$, it results that $\{f_nY_m\}_{n,m\in\mathbb{N}}\subset C_0^\infty(\mathbb{R}_+^*\times\mathbb{S}^2;\mathbb{C})$ and thus, trivially, the space $C_0^\infty(\mathbb{R}_+^*\times\mathbb{S}^2;\mathbb{C})$ is dense in $L^2(\mathbb{R}_+\times\mathbb{S}^2,2KdK\wedge r_S^2d\mathbb{S}^2)$. Since it holds $C_0^\infty(\mathbb{R}_+^*\times\mathbb{S}^2;\mathbb{C})\subset\mathscr{S}_0$, the achieved result proves the thesis. $\square$

Proof of Proposition 3.3. We consider the case of $\mathcal{H}^+$ only, the proof for the case of $\mathcal{H}^-$ being identical.
(a) If $\psi_1,\psi_2\in C_0^\infty(\mathcal{H}^+;\mathbb{C})$, then:

$$\lambda_{KW}(\psi_1,\psi_2)=\lim_{\epsilon\to0^+}-\frac{1}{4\pi}\int_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2}\frac{\overline{\psi_1(u_1,\theta,\phi)}\psi_2(u_2,\theta,\phi)}{\left[\sinh\left(\frac{u_1-u_2}{4r_S}\right)-i\epsilon\right]^2}du_1\wedge du_2\wedge d\mathbb{S}^2\,. \tag{107}$$

It follows from the definition of λ_{KW} (42), passing to coordinates u_1,u_2 and and making appropriate use of Sokhotsky's formula $1/(x-i0^+)^2=1/x^2-i\delta'(x)$ (where $1/x^2$ is the derivative of the distribution

$-1/x$ intepreted in the sense of the principal value) to cancel a bounded strictly positive factor which apperars in front of ϵ (boundedness arises form the fact that the used test functions are supported in $\mathcal{H}^+$, so that they have compact support in the variables $(u,\omega)\in\mathbb{R}\times\mathbb{S}^2$). In spite of the different relation between the coordinate U and u, the same result arises referring to $\mathcal{H}^-$ instead of $\mathcal{H}^+$. We notice that the u-Fourier transform of the distribution $-\frac{1}{4\pi}\frac{1}{\left[\sinh\left(\frac{u}{4r_S}\right)-i0^+\right]^2}$ in the sense of distributions turns out to be just $\frac{1}{\sqrt{2\pi}}\frac{d\mu(k)}{dk}$. Then, the limit as $\epsilon\to 0^+$ of the integral in the right-hand side of (107) can be interpreted as the $L^2(\mathbb{R}\times\mathbb{S}^2, dv\wedge d\mathbb{S}^2)$ scalar product of ψ_1 and the the $L^2(\mathbb{R}\times\mathbb{S}^2, dk\wedge d\mathbb{S}^2)$ function obtained by the u-convolution of the Schwartz distribution *const.*/ $\left[\sinh\left(\frac{u}{4r_S}\right)-i0^+\right]$ and the compactly-supported function ψ_2. (The convolution makes sense interpreting ψ_2 as a distribution with compact support; it produces a distribution which is the antitransform of $\widetilde{\psi_2}d\mu/dk$ which, in turn belongs to the Schwartz space by construction, and thus, antitransforming, the said convolution has to be an element of $L^2(\mathbb{R}\times\mathbb{S}^2, du\wedge d\mathbb{S}^2)$ as previously stated). In this sense we can apply first the convolution theorem for Fourier transforms and, afterwards, the fact that the Fourier transform is an isometry, achieving:

$$\lim_{\epsilon\to 0^+} -\frac{1}{4\pi}\int_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2}\frac{\overline{\psi_1(u_1,\theta,\phi)}\psi_2(u_2,\theta,\phi)}{\left[\sinh\left(\frac{u_1-u_2}{4r_S}\right)-i\epsilon\right]^2}du_1\wedge du_2\wedge d\mathbb{S}^2=\int_{\mathbb{R}\times\mathbb{S}^2}\overline{\widetilde{\psi}(k,\theta,\phi)}\widetilde{\psi}(k,\theta,\phi)\frac{d\mu}{dk}dk\wedge d\mathbb{S}^2\,, \quad (108)$$

which implies that the map $C_0^\infty(\mathcal{H}^+;\mathbb{C})\ni\psi\mapsto\widetilde{\psi}\in L^2(\mathbb{R}\times\mathbb{S}^2, d\mu(k)\wedge d\mathbb{S}^2)$ is isometric, when the domain is equipped with the scalar product λ_{KW}. The fact that this map extends to a Hilbert space isomorphism $F^{(+)}_{(u)}:\overline{C_0^\infty(\mathcal{H}^+;\mathbb{C})}\to L^2(\mathbb{R}\times\mathbb{S}^2, d\mu(k)\wedge d\mathbb{S}^2)$ is very similar to the proof of the analogue for $F_{(U)}$ and the details are left to the reader.

(b) Let us indicate by $\widetilde{\psi}$ the Fourier-Plancherel transform of ψ, also indicated by $\mathscr{F}(\psi)$, computed with respect to the coordinate u. By definition, if $\psi\in\mathsf{S}(\mathcal{H}^+)$, one has $\psi,\partial_u\psi\in L^2(\mathbb{R}\times\mathbb{S}^2, du\wedge d\mathbb{S}^2)$, so that ψ belongs to the Sobolev space $H^1(\mathcal{H}^+)_u$ and, equivalently, $\widetilde{\psi}\in L^2(\mathbb{R}\times\mathbb{S}^2, dk\wedge d\mathbb{S}^2)\cap L^2(\mathbb{R}\times\mathbb{S}^2, k^2dk\wedge d\mathbb{S}^2)$. The last inclusions implies also that $\widetilde{\psi}$ belongs to $L^2(\mathbb{R}\times\mathbb{S}^1, |k|dk\wedge d\mathbb{S}^2)$ and $L^2(\mathbb{R}\times\mathbb{S}^1, d\mu\wedge d\mathbb{S}^2)$. Since $C_0^\infty(\mathcal{H}^+;\mathbb{C})$ is dense in $H^1(\mathcal{H}^+)_u$, if $\psi\in\mathsf{S}(\mathcal{H}^+)$, there is a sequence of functions $\psi_n\in C_0^\infty(\mathcal{H}^+;\mathbb{R})$ with $F^{(+)}_{(u)}(\psi_n)=\mathscr{F}(\psi_n)\to\widetilde{\psi}$, in the topology of $L^2(\mathbb{R}\times\mathbb{S}^2, dk\wedge d\mathbb{S}^2)$ and $L^2(\mathbb{R}\times\mathbb{S}^2, k^2dk\wedge d\mathbb{S}^2)$ which, in turn imply the convergence in the topology of $L^2(\mathbb{R}\times\mathbb{S}^1, d\mu\wedge d\mathbb{S}^2)$. Since $L^2(\mathbb{R}\times\mathbb{S}^1, d\mu\wedge d\mathbb{S}^2)$ is isometric to $\overline{C_0^\infty(\mathcal{H}^+;\mathbb{C})}$, the sequence $\{\psi_n\}$ is of Cauchy type in $\overline{(C_0^\infty(\mathcal{H};\mathbb{C}),\lambda_{KW})}$. For the same reason, it is clear that any other $\{\psi'_n\}\in C_0^\infty(\mathcal{H}^+;\mathbb{R})$ which converges to the same ψ, is such that $\psi_n-\psi'_n\to 0$ in $\overline{(C_0^\infty(\mathcal{H};\mathbb{C}),\lambda_{KW})}$. Therefore ψ is naturally identified with an element of $\overline{C_0^\infty(\mathcal{H}^+;\mathbb{C})}$, which we shall denote with the same symbol ψ. With this identification, for $\psi\in\mathsf{S}(\mathcal{H}^+)$, the fact that $F^{(+)}_{(u)}(\psi_n)\to\widetilde{\psi}=\mathscr{F}(\psi)$ in the topology of $L^2(\mathbb{R}\times\mathbb{S}^2, kdk\wedge d\mathbb{S}^2)$ implies that $F^{(+)}_{(u)}(\psi)=\mathscr{F}(\psi)$ by continuity of $F_{(u)}$. $\square$

Proof of Proposition 3.4. The map $\mathsf{K}_{\mathcal{H}}$ is obviously linear. Let us prove that (a) is valid, i.e., $\mathsf{K}_{\mathcal{H}}$ does not depend on the particular decomposition (50) for a fixed $\psi\in\mathsf{S}(\mathcal{H})$. Consider a different analogous decomposition $\psi=\psi'_-+\psi'_0+\psi'_+$. We have that the two definitions of $\mathsf{K}_{\mathcal{H}}\psi$ coincides because their difference is:

$$\begin{aligned}
&F_{(U)}(\psi_-)-F_{(U)}(\psi'_-)+F_{(U)}(\psi_0)-F_{(U)}(\psi'_0)+F_{(U)}(\psi_+)-F_{(U)}(\psi'_+)\\
&=F_{(U)}(\psi_--\psi'_-)+F_{(U)}(\psi_0-\psi'_0)+F_{(U)}(\psi_+-\psi'_+)=\widehat{\psi_--\psi'_-}+\widehat{\psi_0-\psi'_0}+\widehat{\psi_+-\psi'_+}\\
&=\widehat{\psi-\psi}=0\,,
\end{aligned} \quad (109)$$

Here we have used the fact that, per construction, $\psi_\pm - \psi'_\pm$ and $\psi_0 - \psi_0$ belongs to $C_0^\infty(\mathcal{H};\mathbb{R})$ and thus $F_{(U)}$, acting on each of them, produces the standard U-Fourier transform indicated by $\widehat{\cdot}$.
(b) The statement is valid by definition of $\mathsf{K}_\mathcal{H}$. Let us prove (c). ¿From now on we write σ instead of $\sigma_\mathcal{H}$. Take $\psi, \psi' \in \mathsf{S}(\mathcal{H})$ and decompose them as $\psi = \psi_1 + \psi_2 + \psi_3$ and $\psi' = \psi'_1 + \psi'_2 + \psi'_3$ where $\psi_1, \psi'_1 \in \mathsf{S}(\mathcal{H}^+)$, $\psi_2, \psi'_2 \in C_0^\infty(\mathcal{H};\mathbb{R})$ and $\psi_3, \psi'_3 \in \mathsf{S}(\mathcal{H}^-_R)$. In this way we have:

$$\begin{aligned}\sigma(\psi,\psi') &= \sigma(\psi_1,\psi'_1) + \sigma(\psi_2,\psi'_2) + \sigma(\psi_3,\psi'_3) + \sigma(\psi_1,\psi'_2) + \sigma(\psi_1,\psi'_3)\\ &\quad + \sigma(\psi_2,\psi'_1) + \sigma(\psi_2,\psi'_3) + \sigma(\psi_3,\psi'_1) + \sigma(\psi_3,\psi'_2)\,. \end{aligned} \tag{110}$$

Let us examine each term separately. Consider $\sigma(\psi_1,\psi'_1)$. ¿From now on $\widetilde{\psi}$ is the Fourier-Plancherel transform of ψ also indicated by $\mathscr{F}(\psi)$, computed with respect to the coordinate u. Notice that $\widetilde{\psi}_1(-k,\theta,\phi) = \overline{\widetilde{\psi}_1(k,\theta,\phi)}$ since ψ_1 and ψ'_1 are real. By direct inspection, using these ingredients and the definition of $d\mu(k)$ one gets immediately he first identity in the following:

$$\begin{aligned}\sigma(\psi_1,\psi'_1) &= -2Im\langle \widetilde{\psi}_1, \widetilde{\psi'}_1\rangle_{L^2(\mathbb{R}\times\mathbb{S}^1, d\mu\wedge d\mathbb{S}^2)} = -2Im\langle F_{(U)}\circ (F^{(+)}_{(u)})^{-1}(\widetilde{\psi_1}), F_{(u)}\circ (F^{(+)}_{(u)})^{-1}(\widetilde{\psi'_1})\rangle_{L^2(\mathbb{R}_+\times\mathbb{S}^1, dK\wedge d\mathbb{S}^2)}\\ &= -2Im\langle \mathsf{K}_\mathcal{H}\psi_1, \mathsf{K}_\mathcal{H}\psi'_1\rangle_{\mathsf{H}_\mathcal{H}}\,, \end{aligned} \tag{111}$$

The second identity arises form the fact that $F_{(U)}\circ (F^{(+)}_{(u)})^{-1}$ is an isometry as follows from (b) in Proposition 3.3 and (a) in Proposition 3.2. The last identity is nothing but the definition of $\mathsf{K}_\mathcal{H}$. With the same procedure we similarly have

$$\sigma(\psi_3,\psi'_3) = -2Im\langle \mathsf{K}_\mathcal{H}\psi_1, \mathsf{K}_\mathcal{H}\psi'_1\rangle_{\mathsf{H}_\mathcal{H}}\,. \tag{112}$$

Referring to $\sigma(\psi_2,\psi'_2)$, we can employ the coordinate U instead that U taking into account that the support of those smooth functions is compact when refereed to the coordinates (U,θ,ϕ) over $\mathcal{H}$. Trivially, $\psi_2, \psi'_2, \partial_U\psi_2, \partial_U\psi'_2 \in L^2(\mathbb{R}\times\mathbb{S}^2, dU\wedge d\mathbb{S}^2)$ so that, concerning the U-Fourier transforms, it holds $\widehat{\psi}_{2+}, \widehat{\psi'}_{2+} \in L^2(\mathbb{R}_+\times\mathbb{S}^2, dK\wedge d\mathbb{S}^2)\cap L^2(\mathbb{R}_+\times\mathbb{S}^2, KdK\wedge d\mathbb{S}^2)$. Finally, in the considered case, directly by the definition, $\mathsf{K}_\mathcal{H}\psi'_2 = \widehat{\psi'}_{2+}$ and $\mathsf{K}_\mathcal{H}\psi_2 = \widehat{\psi}_{2+}$. Using the fact that $\widehat{\psi}_2(-K,\theta,\phi) = \overline{\widehat{\psi}_2(K,\theta,\phi)}$ because ψ_2 and ψ'_2 are real, one straightforwardly achieves the first identity in the following:

$$\begin{aligned}\sigma(\psi_2,\psi'_2) &= -2Im\langle \widehat{\psi}_{2+}, \widehat{\psi'}_{2+}\rangle_{L^2(\mathbb{R}_+\times\mathbb{S}^2, 2KdK\wedge d\mathbb{S}^2)} = -2Im\langle F_{(U)}\psi_{2+}, F_{(U)}\psi'_{2+}\rangle_{L^2(\mathbb{R}_+\times\mathbb{S}^2, 2KdK\wedge d\mathbb{S}^2)}\\ &= -2Im\langle \mathsf{K}_\mathcal{H}\psi_2, \mathsf{K}_\mathcal{H}\psi'_2\rangle_{\mathsf{H}_\mathcal{H}}\,, \end{aligned} \tag{113}$$

The remaining identities follows from the definition of $F_{(U)}$ and $\mathsf{K}_\mathcal{H}$. As another step we notice that

$$\sigma(\psi_1,\psi'_3) = 0 = -2Im\langle \mathsf{K}_\mathcal{H}\psi_1, \mathsf{K}_\mathcal{H}\psi'_3\rangle_{\mathsf{H}_\mathcal{H}} \quad \sigma(\psi_3,\psi'_1) = 0 = -2Im\langle \mathsf{K}_\mathcal{H}\psi_3, \mathsf{K}_\mathcal{H}\psi'_1\rangle_{\mathsf{H}_\mathcal{H}}\,. \tag{114}$$

Indeed, focusing on the first identity, the second being analogous, $\sigma(\psi_1,\psi'_3) = 0$ because the functions have disjoint supports, whereas $\langle \mathsf{K}_\mathcal{H}\psi_3, \mathsf{K}_\mathcal{H}\psi'_1\rangle_{\mathsf{H}_\mathcal{H}} = 0$ since (by straightforward application of (b) in Proposition 3.3) $\psi_1 \in \mathsf{S}(\mathcal{H}^+)$ is the limit of a sequence of real smooth functions $f^{(1)}_n$ with support in $\mathcal{H}^+$ whereas $\psi'_3 \in \mathsf{S}(\mathcal{H}^+)$ is the limit of a sequence of real smooth functions $f^{(3)}_n$ with support in $\mathcal{H}^-$ and

$$\begin{aligned}&Im\langle \mathsf{K}_\mathcal{H} f^{(1)}_n, \mathsf{K}_\mathcal{H} f^{(2)}_m\rangle_{\mathsf{H}_\mathcal{H}} = Im\lambda_{KW}(f^{(1)}_m, f^{(2)}_n)\\ &= -\frac{r_S^2}{\pi} Im\int_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2} \frac{f^{(1)}_n(U_1,\theta,\phi) f^{(2)}_m(U_2,\theta,\phi)}{(U_1-U_2-i0^+)^2} dU_1\wedge dU_2\wedge d\mathbb{S}^2(\theta,\phi)\\ &= -\frac{r_S^2}{\pi} Im\int_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2} \frac{\partial_{U_1} f^{(1)}_n(U_1,\theta,\phi) f^{(2)}_m(U_2,\theta,\phi)}{U_1-U_2-i0^+} dU_1\wedge dU_2\wedge d\mathbb{S}^2(\theta,\phi)\\ &= -r_S^2\int_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2} \partial_{U_1} f^{(1)}_n(U_1,\theta,\phi) f^{(2)}_m(U_2,\theta,\phi)\delta(U_1-U_2) dU_1\wedge dU_2\wedge d\mathbb{S}^2(\theta,\phi) = 0 \end{aligned}$$

as $f_n^{(1)}$ and $f_m^{(2)}$ have disjoint support. Let us pass to examine the term $\sigma(\psi_1, \psi_2')$: in this case we decompose $\psi_1 = f_1 + g_1$ where $f_1 \in C_0^\infty(\mathcal{H}^+; \mathbb{R})$ and $g_1 \in \mathsf{S}(\mathcal{H}^+)$, but supp $(g_1) \cap$ supp $(\psi_2') = \emptyset$. We have:

$$\sigma(\psi_1, \psi_2') = \sigma(f_1, \psi_2') + \sigma(g_1, \psi_2') \,.$$

At the end of this proof we shall prove that:

$$\sigma(g_1, \psi_2') = 0 = -2Im\langle \mathsf{K}_{\mathcal{H}} g_1, \mathsf{K}_{\mathcal{H}} \psi_2' \rangle_{\mathsf{H}_{\mathcal{H}}} \,. \tag{115}$$

Conversely $\sigma(f_1, \psi_2') = -2Im\langle \mathsf{K}_{\mathcal{H}} f_1, \mathsf{K}_{\mathcal{H}} \psi_2' \rangle_{\mathsf{H}_{\mathcal{H}}}$, exactly as in the case $\sigma(\psi_2, \psi_2')$ examined above. Summing up, by $\mathbb{R}$ linearity:

$$\sigma(\psi_1, \psi_2') = -2Im\langle \mathsf{K}_{\mathcal{H}} \psi_1, \mathsf{K}_{\mathcal{H}} \psi_2' \rangle_{\mathsf{H}_{\mathcal{H}}} \,. \tag{116}$$

With an analogous procedure we also achieve:

$$\sigma(\psi_2, \psi_1') = -2Im\langle \mathsf{K}_{\mathcal{H}} \psi_1, \mathsf{K}_{\mathcal{H}} \psi_2' \rangle_{\mathsf{H}_{\mathcal{H}}} \quad \sigma(\psi_2, \psi_3') = -2Im\langle \mathsf{K}_{\mathcal{H}} \psi_2, \mathsf{K}_{\mathcal{H}} \psi_3' \rangle_{\mathsf{H}_{\mathcal{H}}} \tag{117}$$

and

$$\sigma(\psi_3, \psi_2') = -2Im\langle \mathsf{K}_{\mathcal{H}} \psi_3, \mathsf{K}_{\mathcal{H}} \psi_2' \rangle_{\mathsf{H}_{\mathcal{H}}} \,. \tag{118}$$

The identities (111)-(118), by $\mathbb{R}$ linearity, yield the thesis:

$$\sigma(\psi, \psi') = -2Im\langle \mathsf{K}_{\mathcal{H}} \psi, \mathsf{K}_{\mathcal{H}} \psi' \rangle_{\mathsf{H}_{\mathcal{H}}} \,.$$

The proof ends provided we demonstrate (115). We only sketch the argument leaving the details to the reader. The proof is based on the following result. If $\psi \in \mathsf{S}(\mathcal{H})$ and $T \in \mathbb{R}$, let us denote by $\psi_T \in \mathsf{S}(\mathcal{H})$ the function such that $\psi_T(U, \theta, \phi) \doteq \psi(U - T, \theta, \phi)$. It is possible to prove that

$$\left(\mathsf{K}_{\mathcal{H}}(\psi_T)\right)(K, \theta, \phi) = e^{-iKT} \left(\mathsf{K}_{\mathcal{H}}(\psi)\right)(K, \theta, \phi) \,. \tag{119}$$

The proof of (119) is obvious when $\psi \in C_0^\infty(\mathcal{H}; \mathbb{R})$, since, in that case $\mathsf{K}_{\mathcal{H}}$ is the (positive frequency part of) the U-Fourier transform of ψ. If $\psi \notin C_0^\infty(\mathcal{H}; \mathbb{R})$, we can decompose it as $\psi_- + \psi_0 + \psi_+$, as in the definition of $\mathsf{K}_{\mathcal{H}}$, fixing ψ_- and ψ_+ in order that $(\psi_\pm)_T$ are still supported in $(-\infty, 0)$ and $(0, +\infty)$ respectively if $|T'| \leq T$. Then, using the fact which can be proved by inspection, that – up to a redefinition of the initially taken ψ_n – $C_0(\mathcal{H}^+; \mathbb{R}) \ni (\psi_n)_T \to (\psi_+)_T$ in $H^1(\mathcal{H}^+)_u$ if $C_0(\mathcal{H}^+; \mathbb{R}) \ni \psi_n \to \psi_+$, one gets that (119) is valid for ψ_+, and the same argument shows that it is valid for ψ_-, too. The very definition of $\mathsf{K}_{\mathcal{H}}$ entails the validity of (119) for every $\psi \in \mathsf{S}(\mathcal{H})$. (119) yields (115) immediately, because, in the examined case,

$$\sigma(g_1, \psi_2') = 0 = -2Im\langle \mathsf{K}_{\mathcal{H}} g_1, \mathsf{K}_{\mathcal{H}} \psi_2' \rangle_{\mathsf{H}_{\mathcal{H}}}$$

because the left hand side vanishes as g_1, ψ_2' have disjoint supports, whereas the right-hand side can be re-written as:

$$-2Im \int_{\mathbb{R}\times\mathbb{S}^2} \overline{e^{-iTK} \left(\mathsf{K}_{\mathcal{H}} g_1\right)(K, \theta, \phi)} e^{-iTK} \left(\mathsf{K}_{\mathcal{H}} \psi_2'\right)(K, \theta, \phi) 2K dK \wedge d\mathbb{S}^2(\theta, \phi) = -2Im \left\langle \mathsf{K}_{\mathcal{H}}((g_1)_T), \mathsf{K}_{\mathcal{H}}((\psi_2')_T) \right\rangle .$$

Such term is also vanishing, because we can fix T so that supp $((g_1)_T) \subset \mathcal{H}^-$ and supp $((\psi_2')_T) \subset \mathcal{H}^+$, reducing to the case $\sigma(\psi_1, \psi_3') = 0 = -2Im\langle \mathsf{K}_{\mathcal{H}} \psi_1, \mathsf{K}_{\mathcal{H}} \psi_3' \rangle_{\mathsf{H}_{\mathcal{H}}}$ examined beforehand.
(d) is a trivial consequence of (c) : if $\mathsf{K}_{\mathcal{H}} \psi = 0$, then $Im\langle \mathsf{K}_{\mathcal{H}} \psi, \mathsf{K}_{\mathcal{H}} \psi' \rangle = 0$ and thus $\sigma_{\mathcal{H}}(\psi, \psi') = 0$ for

every $\psi' \in \mathsf{S}(\mathcal{H})$. Since $\sigma_{\mathcal{H}}$ is nondegenerate, it implies $\psi = 0$. Let us continue by proving (e). As $C_0^\infty(\mathcal{H};\mathbb{R})) \subset \mathsf{S}(\mathcal{H})$,

$$\mathsf{H}_{\mathcal{H}} = \overline{F_{(U)}(C_0^\infty(\mathcal{H};\mathbb{R}))} = \overline{\mathsf{K}_{\mathcal{H}}(C_0^\infty(\mathcal{H};\mathbb{R}))} \subset \overline{\mathsf{K}_{\mathcal{H}}(\mathsf{S}(\mathcal{H}))} \subset \mathsf{H}_{\mathcal{H}} \quad \text{and thus } \overline{\mathsf{K}_{\mathcal{H}}(\mathsf{S}(\mathcal{H}))} = \mathsf{H}_{\mathcal{H}}.$$

The first identity arises by (a) in Proposition 3.2, the second by (b) in Proposition 3.4.
We can now conclude proving (f). The continuity of $\mathsf{K}_{\mathcal{H}}$ with respect to the considered norm holds for the following reason. If $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathsf{S}(\mathcal{H})$ and $||\psi_n||^{\chi}_{\mathcal{H}} \to 0$, then decomposing $\psi_n = \psi_{0n} + \psi_{+n} + \psi_{-n}$, separately ψ_{0n} and $\psi_{\pm n} \to 0$ in the respective Sobolev topologies. In turn $\mathsf{K}_{\mathcal{H}}(\psi_{0n}) = F_U(\psi_{0n}) \to 0$ because the Sobolev topology is stronger than that of $L^2(\mathbb{R}\times\mathbb{S}^2; dU \wedge d\mathbb{S}^2)$ (noticing that F_U is continuous with respect to the latter it being, up to a restriction, the Fourier-Plancherel transform) and $\mathsf{K}_{\mathcal{H}}(\psi_{\pm n}) \to 0$ for (b) in Proposition 3.3. By definition of $\mathsf{K}_{\mathcal{H}}$, it hence holds $\mathsf{K}_{\mathcal{H}}(\psi_n) \to 0$. Thus the linear map $\mathsf{K}_{\mathcal{H}} : \mathsf{S}(\mathcal{H}) \to \mathsf{H}_{\mathcal{H}}$ is continuous it being continuous in 0. We conclude, in particular that there is $C_\chi > 0$ (the value 0 is not allowed since $\mathsf{K}_{\mathcal{H}}$ is not the null function) with $||\mathsf{K}_{\mathcal{H}}(\psi)||_{\mathsf{H}_{\mathcal{H}}} \leq C_\chi ||\psi||^{\chi}_{\mathcal{H}}$ for every $\psi \in \mathsf{S}(\mathcal{H})$. Then, Cauchy-Schwartz inequality implies the inequality displayed in (f). □

Proof of Proposition 3.7. Define $v = x^2$ if $x \geq 0$ and $v = -x^2$ if $v < 0$, so that, by direct inspection one sees that, if $\psi, \psi' \in C_0^\infty(\mathbb{R}^*_- \times \mathbb{S}^2;\mathbb{R})$, where v is the coordinate over $\mathbb{R}$,

$$\begin{aligned}\int_{\mathbb{R}^2\times\mathbb{S}^2} \frac{\psi(v,\theta,\phi)\psi'(v',\theta,\phi)}{(v-v'-i0^+)^2} dv\wedge dv' \wedge d\mathbb{S}^2(\theta,\phi) &= \int_{\mathbb{R}^2\times\mathbb{S}^2} \frac{\psi(v(x),\theta,\phi)\psi'(v(x'),\theta,\phi)}{(x'-x-i0^+)^2} dx\wedge dx' \wedge d\mathbb{S}^2(\theta,\phi) \\ &+ \int_{\mathbb{R}^2\times\mathbb{S}^2} \frac{\psi(v(x),\theta,\phi)\psi'(v(-x'),\theta,\phi)}{(x'-x-i0^+)^2} dx\wedge dx' \wedge d\mathbb{S}^2(\theta,\phi)\,.\end{aligned} \tag{120}$$

Passing to the x-Fourier transform, and denoting by $\dot\psi = \dot\psi(h,\theta,\phi)$, with $h \in \mathbb{R}$ and $(\theta,\phi) \in \mathbb{S}^2$ the x-Fourier transform of $\psi(v(x))$, and finally, defining $\dot\psi_+ \doteq \dot\psi\restriction_{\mathbb{R}_+\times\mathbb{S}^2}$, this identity can be re-written (using the fact that if ϕ is real valued, as it happens for ψ and ψ', then $\overline{\dot\phi_+(h,\theta,\phi)}$ is the x Fourier transform of $x \mapsto \phi(-x,\theta,\phi)$)

$$\begin{aligned}\lambda_{\Im^+}(\psi,\psi') = \int_{\mathbb{R}_+\times\mathbb{S}^2} \overline{\dot\psi'_+(h,\theta,\phi)}\dot\psi_+(h,\theta,\phi) 2h dh\wedge d\mathbb{S}^2 + \\ \int_{\mathbb{R}_+\times\mathbb{S}^2} \overline{\dot\psi'_+(h,\theta,\phi)} \left(C\dot\psi_+\right)(h,\theta,\phi) 2h dh \wedge d\mathbb{S}^2\,,\end{aligned} \tag{121}$$

where the operator $C : L^2(\mathbb{R}_+\times\mathbb{S}^2, 2hdh) \to L^2(\mathbb{R}_+\times\mathbb{S}^2, 2hdh)$ is anti unitary and is nothing but the complex conjugation. Now take $\psi \in \mathsf{S}(\Im^-)$ which is completely supported in $\mathbb{R}_+\times\mathbb{S}^2$. By definition of $\mathsf{S}(\Im^-)$, the function $\psi = \psi(v(x))$ and its x-derivative belong to $L^2(\mathbb{R}\times\mathbb{S}^2, dx\wedge d\mathbb{S}^2)$ and thus ψ belongs to $H^1(\Im^+)_x$. A sequence of functions $\psi_n \in C^\infty(\mathbb{R}^*_-\times\mathbb{S}^2;\mathbb{R})$ which converges to ψ in $H^1(\Im^-)_x$ can be constructed as $\psi_n = \chi_n \cdot \psi$, where $\chi_n(x) \doteq \chi(x/n) \geq 0$ with $\chi(x) = 1$ if $x \in (-1,+\infty)$ and $\chi(x) = 0$ for $x \leq -2$. By direct inspection (using Lebesgue's dominated convergence theorem) one achieves that $C^\infty(\mathbb{R}^*_-\times\mathbb{S}^2;\mathbb{R}) \ni \psi_n \to \psi$ in $H^1(\Im^-)_x$ as $n \to +\infty$. Consequently $\dot\psi_n \to \dot\psi$ both in $L^2(\mathbb{R}\times\mathbb{S}^2, dh)$ and in $L^2(\mathbb{R}_+\times\mathbb{S}^2, h^2dh)$. Therefore $\dot\psi_{n+} \to \dot\psi_+$ in the topology of $L^2(\mathbb{R}_+\times\mathbb{S}^2, 2hdh)$. Finally, in view of (121) and taking into account that C is continuous, the sequence $\{\psi_n\}_{n\in\mathbb{N}}$ is of Cauchy type with respect to $\lambda_{\Im^-}$. The same argument shows that, if $C^\infty(\mathbb{R}^*_-\times\mathbb{S}^2;\mathbb{R}) \ni \psi'_n \to \psi$ in $H^1(\Im^-)$ as $n \to +\infty$, then $\lambda_{\Im^-}(\psi_n - \psi'_n, \psi_n - \psi'_n) \to 0$ as $n \to +\infty$. Now (61) and (62) are trivial consequences of what proved. We have proved both (a) and (b). □

Proof of Proposition 3.8. The proofs of the items (a),(b),(d),(e) and (f) are very similar tho the the proofs of the corresponding items in Proposition 3.4, so they will be omitted. We instead focus attention of the item (c), whose proof is similar to (c) of the Proposition 3.4, but with some relevant differences. Take $\psi, \psi' \in \mathsf{S}(\Im^-)$ and decompose them as $\psi = \psi_0 + \psi_1$, $\psi' = \psi'_0 + \psi'_1$ where $\psi_0, \psi_1 \in C_0^\infty(\Im^-; \mathbb{R})$ while ψ'_0, ψ'_1 are supported in $(0, +\infty) \times \mathbb{S}^2$. We have, where $\sigma \doteq \sigma_{\Im^-}$ and $\langle , \rangle = \langle , \rangle_{\Im^+}$,

$$\sigma(\psi, \psi') = \sigma(\psi_0, \psi'_0) + \sigma(\psi_0, \psi'_1) + \sigma(\psi_1, \psi'_0) + \sigma(\psi_1, \psi'_1) \, .$$

Exactly as in (c) of the Proposition 3.4, we conclude that

$$\sigma(\psi_0, \psi'_0) = -2Im\langle \mathsf{K}_{\Im^-}\psi_0, \mathsf{K}_{\Im^-}\psi'_0\rangle \, . \tag{122}$$

Concerning the term $\sigma(\psi_1, \psi'_1)$, we have instead:

$$-2Im\langle \mathsf{K}_{\Im^-}\psi_1, \mathsf{K}_{\Im^-}\psi'_1\rangle = -2Im\langle F_{(v)}(\psi_1), F_{(v)}(\psi'_1)\rangle = -2Im\lambda_{\Im^-}(\psi_1, \psi'_1) \, .$$

Making use of (62), using the fact that identity can be used for ψ_1, ψ'_1 as established in (b) of Proposition 3.6, we have (where $\dot{\psi}(h, \theta, \phi)$ is the x-Fourier-Plancherel transform of $\psi = \psi(v(x), \theta, \phi)$):

$$-2Im\langle \mathsf{K}_{\Im^-}\psi_1, \mathsf{K}_{\Im^-}\psi'_1\rangle = -2Im \int_{\mathbb{R}_+\times\mathbb{S}^2} \overline{\dot{\psi}_1}\dot{\psi}_1 2hdh \wedge d\mathbb{S}^2 - 2Im \int_{\mathbb{R}_+\times\mathbb{S}^2} \overline{\dot{\psi}_1\dot{\psi}_1} 2hdh \wedge d\mathbb{S}^2 \, .$$

The last term in the right-hand can be omitted for the following reason. Looking at (120), we see that $-i0^+$ can be replaced by $+i0^+$ without affecting the result, since the functions in the numerator have disjoint supports. This is equivalent to say that, in the right-hand side of (121), the last term can be replaced with its complex conjugation without affecting the final result. Finally, this means that the identity written above can be equivalently re-written:

$$-2Im\langle \mathsf{K}_{\Im^+}\psi_1, \mathsf{K}_{\Im^-}\psi'_1\rangle = -2Im \int_{\mathbb{R}_+\times\mathbb{S}^2} \overline{\dot{\psi}_1}\dot{\psi}_1 2hdh \wedge d\mathbb{S}^2 - 2Im \overline{\int_{\mathbb{R}_+\times\mathbb{S}^2} \overline{\dot{\psi}_1\dot{\psi}_1} 2hdh \wedge d\mathbb{S}^2} \, .$$

As a consequence the last term can be dropped, so that:

$$-2Im\langle \mathsf{K}_{\Im^+}\psi_1, \mathsf{K}_{\Im^-}\psi'_1\rangle = -4Im \int_{\mathbb{R}_+\times\mathbb{S}^2} \overline{\dot{\psi}_1} h\dot{\psi}'_1 dh \wedge d\mathbb{S}^2 = 2i \int_{\mathbb{R}\times\mathbb{S}^2} \overline{\dot{\psi}_1} h\dot{\psi}'_1 dh \wedge d\mathbb{S}^2 = \sigma(\psi_1, \psi'_1) \, . \tag{123}$$

In the last passage we have used that fact that $-ih\dot{\psi}'_1$ is the x-Fourier transform of $\partial_x\psi'_1$, that the integration in $\sigma(\psi_1, \psi'_1)$ can be performed in the variable x (the singularity of the coordinates at $x = 0$ is irrelevant since the supports of ψ_1 and ψ'_1 are away from there) and the fact that these functions are real so that $\overline{\dot{\psi}_i(h, \theta, \phi)} = \dot{\psi}_i(-h, \theta, \phi)$. Let us pass to consider the term $\sigma(\psi_0, \psi'_1)$ the other, $\sigma(\psi_1, \psi'_0)$ can be treated similarly. To this end decompose $\psi'_1 = \phi'_0 + \phi'_1$ in order that $\phi'_0 \in C_0^\infty(\Im^-; \mathbb{R})$ and the support of ϕ'_1 is disjoint from that of ψ_0. Therefore: $\sigma(\psi_0, \psi'_1) = \sigma(\psi_0, \phi'_0) + \sigma(\psi_0, \phi'_1) = -2Im\langle \mathsf{K}_{\Im^+}\psi_1, \mathsf{K}_{\Im^-}\phi'_1\rangle + \sigma(\psi_0, \phi'_1)$. Since we shall prove that:

$$\sigma(\psi_0, \phi'_1) = 0 = -2Im\langle \mathsf{K}_{\Im^-}\psi_0, \mathsf{K}_{\Im^+}\phi'_1\rangle \, , \tag{124}$$

we also have that

$$\sigma(\psi_0, \psi'_1) = -2Im\langle \mathsf{K}_{\Im^-}\psi_0, \mathsf{K}_{\Im^-}\psi'_1\rangle \, , \quad \text{and similarly,} \quad \sigma(\psi_1, \psi'_0) = -2Im\langle \mathsf{K}_{\Im^-}\psi_1, \mathsf{K}_{\Im^-}\psi'_0\rangle \, ,$$

which, together (122) and (123) implies the validity of (c) by bi-linearity:

$$\sigma(\psi,\psi') = -2Im\langle \mathsf{K}_{\Im^-}\psi, \mathsf{K}_{\Im^-}\psi'\rangle \ .$$

To conclude, it is enough proving (124). The left-hand side vanishes since the supports of the functions ψ_0, ϕ_1' are disjoint by construction. Hence it remains to prove that $Im\langle \mathsf{K}_{\Im^+}\psi_0, \mathsf{K}_{\Im^-}\phi_1'\rangle = 0$. If it were supp $(\psi_0) \subset (-\infty,0)\times\mathbb{S}^2$ and supp $(\phi_1') \subset (0,+\infty)\times\mathbb{S}^2$ one would achieve $Im\langle \mathsf{K}_{\Im^-}\psi_0, \mathsf{K}_{\Im^-}\phi_1'\rangle = 0$ through the same argument used in the corresponding case (that of $\sigma(\psi_1,\psi_3')$) in the proof of (c) of the Proposition 3.4 employing a sequence of real smooth functions tending to ϕ_1' in the topology of $\lambda_{\Im^-}$ and with compact supports all enclosed in $(0,+\infty)\times\mathbb{S}^2$ (Such a sequence does exist in view of Proposition 3.6.) As a matter of fact, we can reduce to the case supp $(\psi_0) \subset (-\infty,0)\times\mathbb{S}^2$ and supp $(\phi_1') \subset (0,+\infty)\times\mathbb{S}^2$, in view of the following lemma (which will be useful in the proof of (b) of Theorem 3.2).

Lemma C.3. *For $\psi \in \mathsf{S}(\Im^-)$ and $L \in \mathbb{R}$, let $\psi_L \in \mathsf{S}(\Im^-)$ denote the function with $\psi_L(v,\theta,\phi) \doteq \psi(v-L,\theta,\phi)$ for all $v \in \mathbb{R}$ and $\theta,\phi \in \mathbb{S}^2$. With the given definition for $\mathsf{K}_{\Im^-} : \mathsf{S}(\Im^-) \to \mathsf{H}_{\Im^-}$, it holds:*

$$\left(\mathsf{K}_{\Im^-}\psi_L\right)(k,\theta,\phi) = e^{-iLk}\left(\mathsf{K}_{\Im^-}\psi\right)(k,\theta,\phi)\ , \quad \forall (k,\theta,\phi) \in \mathbb{R}_+\times\mathbb{S}^2\ . \tag{125}$$

Proof of Lemma C.3. By definition, if $\psi \in \mathsf{S}(\Im^-)$ is fixed. $\mathsf{K}_{\Im^-}\psi = F_{(v)}\psi_0 + F_{(v)}\psi_-$, where $\psi = \psi_0 + \psi_-$ with $\psi_0 \in C_0^\infty(\Im^-;\mathbb{R})$ and $\psi_- \in \mathsf{S}(\Im^-)$ with supp $(\psi_-) \subset (-\infty,0)\times\mathbb{S}^2$. Fix $L \in \mathbb{R}$ and notice that, by definition of $F_{(v)}$ when acting on $C_0^\infty(\Im^-;\mathbb{R})$, it trivially holds

$$F_{(v)}(\psi_0)_L = e^{-iLk}F_{(v)}\psi_0\ .$$

To conclude it is sufficient to establish that it also holds:

$$F_{(v)}(\psi_-)_L = e^{-iLk}F_{(v)}\psi_-\ . \tag{126}$$

Let us prove it. Since the definition of $\mathsf{K}_{\Im^-}\psi$ does not depend on the chosen decomposition $\psi = \psi_0 + \psi_-$, we can fix ψ_0 and ψ_- such that the support of $(\psi_-)_L$ is still included in $(-\infty,0)$ (this is obviously true for every ψ_- if $L \leq 0$, but it is not for $L > 0$ and, in this case the support of ψ_- has to be fixed sufficiently far from 0). To establish (126), pass to use the coordinate $x = -\sqrt{-v}$ for $v < 0$. The singularity at $v = 0$ does not affect the procedure since the supports of all the involved functions do not include it. We know, by the proof of Proposition 3.6, that there is a sequence $C_0^\infty(\Im^-;\mathbb{R}) \ni \psi_n \to \psi_-$, all supported in supp $(\psi_-) \subset (-\infty,0)\times\mathbb{S}^2$, and where the convergence is both in in the topology of $H^1(\Im^-)$ and in that of $\lambda_{\Im^-}$. By direct inspection – but it is not completely trivial as it could seem at first glance because the L-displacement is implemented in the variable v and not x – one sees that, for the above-mentioned sequence it holds supp $(\psi_n)_L \subset$ supp $(\psi_L) \subset (0,+\infty)\times\mathbb{S}^2$ and $\psi_n \to \psi_-$ entails $(\psi_n)_L \to (\psi_-)_L$ for $n \to +\infty$ in the topology of $H^1(\Im^-)_x$. By (b) in Proposition 3.6, this implies that the convergence holds also in the topology of $\lambda_{\Im^-}$. Since $F_{(v)}$ is continuous with respect to that topology we get, as $n \to +\infty$:

$$e^{-iLk}F_{(v)}\psi_n = F_{(v)}(\psi_n)_L \to F_{(v)}(\psi_-)_L\ .$$

On the other hand, since $F_{(v)}\psi_n \to F_{(v)}\psi_-$ in $L^2(\mathbb{R}_+\times\mathbb{S}^2, kdk\wedge d\mathbb{S}^2)$, it trivially holds as $n \to +\infty$:

$$e^{-iLk}F_{(v)}\psi_n \to e^{-iLk}F_{(v)}\psi_-\ ,$$

and thus

$$e^{-iLk}F_{(v)}\psi = F_{(v)}(\psi_-)_L\ ,$$

which implies (126), concluding the proof. □

To conclude the proof of (c), we note that, in view of (125), it has to hold $-2Im\langle \mathsf{K}_{\Im^-}\psi_0, \mathsf{K}_{\Im^+}\phi_1'\rangle = -2Im\langle \mathsf{K}_{\Im^-}(\psi_0)_L, \mathsf{K}_{\Im^-}(\phi_1')_L\rangle$ for every $L \in \mathbb{R}$. Therefore, we can fix L so that $\mathrm{supp}\,(\psi_0)_L \subset (-\infty, 0) \times \mathbb{S}^2$ and $\mathrm{supp}\,((\phi_1'))_L) \subset (0, +\infty) \times \mathbb{S}^2$, obtaining, as said before

$$-2Im\langle \mathsf{K}_{\Im^-}\psi_0, \mathsf{K}_{\Im^-}\phi_1'\rangle = -2Im\langle \mathsf{K}_{\Im^-}(\psi_0)_L, \mathsf{K}_{\Im^-}(\phi_1')_L\rangle = 0\,.$$

This implies (124) and concludes the proof of (c). □

Proof of Proposition 4.2. The first assertion arises per direct inspection of definition 4.1 and, thus, we need only to prove that the (complexified) functionals on $C_0^\infty(\mathscr{M};\mathbb{C}) \times C_0^\infty(\mathscr{M};\mathbb{C})$, $\Lambda_{\mathcal{H}}$ and $\Lambda_{\Im^-}$ are separately distributions in $\mathscr{D}'(\mathscr{M} \times \mathscr{M})$. To this end, it suffices to show that the maps $f \mapsto \Lambda_i(f,\cdot)$ and $g \mapsto \Lambda_i(\cdot, g)$ are weakly continuous, *i.e.* such that they tend to 0 when tested with any sequence of functions $h_j \in C_0^\infty(\mathscr{M};\mathbb{C})$ which converges to 0 in the topology of ttest functions (that is, in the sense that there is a compact set K with $supp(h_j) \subset K$ and for all $j \in \mathbb{N}$, and $\sup|\partial^\alpha h_j| \to 0$ as $j \to +\infty$ for all but fixed multindices α referred to a global coordinate frame over $\mathscr{M}$). Here and hereafter the subscript i stands either for $\mathcal{H}$ or for $\Im^-$. According to theorem 2.1.4 of [Hö89], such statement entails that both $\Lambda_i(f,\cdot)$ and $\Lambda_i(\cdot, g)$ are distributions in $\mathscr{D}'(\mathscr{M})$, hence they are sequentially continuous. Once established it, one can, therefore, invoke the Schwartz' integral kernel theorem to conclude that $\Lambda_i \in \mathscr{D}'(\mathscr{M} \times \mathscr{M})$. In view of the complexification procedure, it is sufficient to consider the case of real valued test functions only.

Let us start with $\Im^-$; in this case $\lambda_{\Im^-}$ has the explicit form (62) in proposition 3.7 and, thus, taking into account a generic decomposition (63) generated by a smooth function η supported on $\mathbb{R}^*_- \times \mathbb{S}^2$ and equal to one for $v < v_0 < 0$, because of the continuity property presented in point (f) of Proposition 3.4

$$|\Lambda_{\Im^-}(f,h)| = |\lambda_{\Im^-}(\varphi^f_{\Im^-}, \varphi^h_{\Im^-})| \leq C\, \|\varphi^f_{\Im^-}\|^\eta_{\Im^-}\, \|\varphi^h_{\Im^-}\|^\eta_{\Im^-}\,.$$

We recall that, for every $\varphi \in \mathsf{S}(\Im^-)$, $\|\varphi\|^\eta_{\Im^-}$ is defined in (65) as the sum of

$$\|\varphi\|^\eta_{\Im^-} = \|\eta\varphi\|_{H^1(\Im^-)_x} + \|(1-\eta)\varphi\|_{H^1(\Im^-)_v}\,.$$

In order to prove the continuity, we shall show that, for $f, g \in C_0^\infty(\mathscr{M};\mathbb{R})$, if $f \to 0$ with a fixed h (or $h \to 0$ with a fixed f) in the topology of $C_0^\infty(\mathscr{M};\mathbb{C})$ and the compact $K \subset \mathscr{M}$ includes all the supports, both the Sobolev norm written above tends to zero. Let us start considering the second one. For the given K, as in the proof of Lemma 2.1, fix a sufficiently large larger globally hyperbolic conformal (*i.e.* equipped with the metric $\widetilde{g}$) spacetime $\mathscr{N} \subset \widetilde{\mathscr{M}}$ which extends $\mathscr{M}$ about a portion of $\Im^-$, includes K, and such that $\mathscr{N} \cap \Im^-$ includes all the points with $v \geq v_0$, reached by the closure in $\mathscr{M} \cup \Im^-$ of $J^-(K;\mathscr{M})$. (Such an $\mathscr{N}$ can be obtained by intrsecting $I^+(q;\widetilde{\mathscr{M}})$ – where q is a point in the past of $\Im^-$ on the right of i^- and sufficiently close to both $\Im^-$ and i^- – and $I^-(S;\widetilde{\mathscr{M}})$, where S is a portion of a smooth spacelike Cauchy surface of $\mathscr{B}$ which lies in the future of K.) Noticing that the causal propagator $E_{P_{\widetilde{g}}}$ is a continuous map from $C_0^\infty(\mathscr{M};\mathbb{R}) \to C^\infty(\mathscr{N};\mathbb{R})$, one has that $(1-\eta)\varphi^f_{\Im^-}$ and all their v-derivatives uniformly vanishes as $f \to 0$ in the topology of $C_0^\infty(\mathscr{M};\mathbb{C})$. Since, by constriction, all the functions $(1-\eta)\varphi^f_{\Im^-}$ have support in a common compact of $\mathbb{R} \times \mathbb{S}^2$ (determined by η and $J^-(K;\mathscr{M})$)), also $\|(1-\eta)\varphi^f_{\Im^-}\|_{H^1(\Im^-)_v}$ tends to zero in view of the integral expression of the Sobolev norm.

To conclude, in order to deal with the contribution $\|\eta\varphi^f_{\Im^-}\|_{H^1(\Im^-)_x}$, notice that according to proposition 2.1 (point (b) in particular), the restriction of a solution of the D'Alembert wave equation on $\Im^-$ decays on null infinity, for $|v|$ greater than a certain $|v_0|$, as $\frac{C_f}{\sqrt{1+|v|}}$, while its v-derivative decay as $\frac{C_f}{1+|v|}$, where

C_f tends to 0 as $f \to 0$ in the topology of $C_0^\infty(\mathscr{M};\mathbb{C})$. Hence per direct inspection, passing to work with the coordinate x, also$\|\eta\varphi^f_{\Im^-}\|_{H^1(\Im^-)_x}$ tends to zero as f tends to zero in the topology of $C_0^\infty(\mathscr{M};\mathbb{C})$. The case of $\mathcal{H}$ can be dealt with in the same way using the continuity presented in point (f) of proposition 3.4 and the appropriate decay estimates of the wave functions presented in proposition 2.1, as before, one can reach the conclusion that both $\Lambda_{\mathcal{H}}(f,\cdot)$ and $\Lambda_{\mathcal{H}}(\cdot,g)$ lie in $\mathscr{D}'(\mathscr{M})$. $\square$

Proof of Proposition 4.4. Let us start considering $\|\varphi^{f_p}_{\Im^-}\|_{\Im^-}$ as defined in (65) (for some generic decomposition based on the choice of the function η) and some $f \in C_0^\infty(\mathscr{M})$ and p in a conic neighbourhood V_{k_x} of k_x we are going to specify. The procedure we shall employ can be similarly used also for $\|\chi'\varphi^{f_p}_{\mathcal{H}}\|_{\mathcal{H}^-}$ to show that it is rapidly decreasing in p. Furthermore we recall that $f_p \doteq fe^{i\langle p,\cdot\rangle}$, while $\varphi^{f_p}_{\Im^-}$ is the smooth limit towards of $\Im^-$ of $E_{P_g}f_p$, where E_{P_g} is the causal propagator of P_g as in (13) (we shall omit the index P_g from now on). $\varphi^{f_p}_{\Im^-}$, together with its derivative along the global null coordinate v are known to decay at minus infinity according to the estimates (28) in Proposition 2.1, in turn based on the work of [DR09], *i.e.*,

$$|\varphi^{f_p}_{\Im^-}| \leq \frac{C_3}{\sqrt{1+|v|}}\,, \qquad |X(\varphi^{f_p}_{\Im^-})| \leq \frac{C_4}{1+|v|}\,,$$

where X still stands for the smooth Killing vector field on the conformally extended Kruskal spacetime, coinciding with ∂_v on $\Im^-$. They yields that the norm $\|\varphi^{f_p}_{\Im^-}\|_{\Im^-}$, defined as in (65), is controlled by the above coefficients C_3 and C_4 which depend on φ^{f_p}, since the norms of the remaining universal functions (smoothed about i^-) are finite. Hence, we shall analyse them explicitly. The fact that f_p is complex valued does not change the result using a straightforward complexfication procedure. All the relevant results in 2.1 can straightforwardly be extended to the complex case. Ou goal is establishing that the coefficients C_3 and C_4 are rapidly decreasing in p when computed for φ^{f_p} in the given hypotheses about x and k_x.

As a starting point, let us consider the case in which $x \in I^+(\mathcal{B};\mathscr{M})$, $\mathcal{B}$ being the bifurcation. In order to study this, as well as all other scenarios, we make use of the results and of the techniques available in [DR09] of which we shall adopt nomenclatures and conventions. In this last cited paper it is manifest, that, up to a term depending on the support of initial data, the dependence on the wave function in C_3 and C_4 is factorised in the square root of the so-called coefficient $\tilde{E}_5$, namely formula (5.4) in [DR09]. After few formal manipulations, the formula can be (re)written as an integral over the constant time surface $\Sigma_1 \subset \mathscr{W}$, unambiguously individuated, in the coordinates (t,r,θ,ϕ), as the locus $t=-1$ – here we consider $t=-1$ because we are interested in the decay property in a neighbourhood of i^- – :

$$\tilde{E}_5(\varphi) = \sum_{i=1..2}\int_{\Sigma_1} T_{\mu\nu}(\Omega^i\varphi)\; n^\mu n^\nu d\mu(\Sigma_1) + \sum_{i=1..4}\int_{\Sigma_1} T_{\mu\nu}(\Omega^i\varphi)\; K^\mu n^\nu d\mu(\Sigma_1) + \\ + \sum_{i=1..5}\int_{\Sigma_1} T_{\mu\nu}(\Omega^i\varphi)\; X^\mu n^\nu d\mu(\Sigma_1), \tag{127}$$

where n is the vector orthogonal to Σ_1, pointing towards the past, and normalised as $g^{\mu\nu}n_\mu n_\nu = -1$, $K \doteq v^2\frac{\partial}{\partial v} + u^2\frac{\partial}{\partial u}$ is the so-called *Morawetz vector field*, X is the timelike Killing vector field $\frac{\partial}{\partial t}$, whereas $d\mu(\Sigma_1)$ is the metric induced measure on Σ_1. Furthermore,

$$T_{\mu\nu}(\varphi) = \frac{1}{2}\left(\partial_\mu\overline{\varphi}\partial_\nu\varphi + \partial_\nu\overline{\varphi}\partial_\mu\varphi\right) - \frac{1}{2}g_{\mu\nu}\left(\partial_\lambda\overline{\varphi}\partial^\lambda\varphi\right),$$

stands for the stress-energy tensor computed with respect of the solution φ, while $\Omega^2 \doteq r^2\nabla\!\!\!/\nabla\!\!\!/$ is the squared angular momentum operator, $\nabla\!\!\!/$, being the covariant derivative induced by the metric (11),

normalized with $r = 1$, on the orbits of $SO(3)$ isomorphic to $\mathbb{S}^2$. We remark both that the above expression can be found in theorem 4.1 in [DR08] and, more important to our purposes, that the integrand is a (hermitean) quadratic combination of a finite number of derivatives of φ^{f_p} on Σ_1. Furthermore, since $J^-(supp(f_p); \mathscr{M}) \cap \Sigma_1$ is compact, the integrand in (127) does not vanish on a compact set at most and, thus, the overall integral can be bounded by a linear combination of products of the sup of the absolute value of derivatives of φ^{f_p} up to a certain order, all evaluated on Σ_1. Notice that, all the remaining functions in the integrand defining $\tilde{E}_5$, barring the said products of derivatives, are continuous and thus bounded on the compact set where φ^{f_p} does not vanish on Σ_1.
Let us thus focus on φ^{f_p} and on the initially chosen $x \in I^+(\mathcal{B}; \mathscr{M})$ and k_x. Using global coordinates, we identify an open relatively compact $\mathcal{O}$ set containing both the support of f and that of the function ρ we go to introduce with $\mathbb{R}^4$ by means of a local coordinate patch, so that every vector $p \in \mathbb{R}^4$ can be viewed as an element of the cotangent space at any point in that set. It is always possible to select an $f \in C_0^\infty(\mathscr{M}; \mathbb{R})$ with $f(x) = 1$ and with a sufficiently small support, such that every inextensible geodesic starting from $supp(f)$, with cotangent vector equal to k_x, intersects $\mathcal{H}$ in a point with coordinate $U > 0$. Hence, we can always fix $\rho \in C_0^\infty(\mathscr{K}; \mathbb{R})$ such that (i) $\rho = 1$ on $J^-(supp(f); \mathscr{M}) \cap \Sigma_1$ and (ii) the null geodesics emanating from $supp(f)$ with k_x as cotangent vector do not meet the support of ρ. Furthermore, on account both of the form of the wave front set of $E(z, z')$ (now thought of in the whole Kruskal spacetime $\mathscr{K}$), whose elements $(z, z', k_z, k_{z'})$ have to always fulfil $(z, k_z) \sim (z', k_{z'})$ we realize that, with (x, k_x) fixed as said above and with the given definitions of f and ρ:

$$\left\{(x_1, x_2, k_1, k_x) \in T^*(\mathscr{M} \times \mathscr{M}) \mid x_1 \in \text{supp}\,(\rho),\ x_2 \in \text{supp}\,(f),\ k_1 \in \mathbb{R}^4\right\} \cap WF(E) = \emptyset\,.$$

Employing this result, remembering the definition of wavefrontset, and making use of Lemma 8.1.1 in [Hö89] working in the coordinate frame initially fixed on the compact $\overline{\mathcal{O}}$, it is possible to further adjust ρ, f (preserving the constraints already stated) in such a way that an open conical neighbourhood V_{k_x} of k_x in $T_x^*\mathscr{M}$ exists such that for all $n, n' = 1, 2, \ldots$, one can find two nonegative constants C_n and C'_n such that

$$|\widehat{\rho E f}(k_1, p)| \leq \frac{C_n}{1 + |k_1|^n} \frac{C'_{n'}}{1 + |p|^{n'}}, \tag{128}$$

uniformly for $(k_1, p) \in (\mathbb{R}^4 \setminus \{0\}) \times V_{k_x}$. The searched bounds on the behaviour at large in $|p|$ for C_3 and C_4 (proving that they are rapidly decreasing) when computed for φ^{f_p} for p in a open conical neighborhood of k_x arise in term of corresponding bounds of the derivatives $|\partial_x^a \overline{\varphi^{f_p}(x)} \partial_x^b \varphi^{f_p}(x)|$ taking into account the explicit expression of C_3 and C_4 as integrals over the relevant portion of Σ_1, which has finite measure because it is compact. Each factor $\partial_x^a \varphi^{f_p}(x)$ coincides with the inverse k_1-Fourier transform of $\widehat{\rho E f}(k_1, p)$ multiplied with powers of the components of k_1 up to a finite power depending on the considered order of derivative. As a last step, to get rid of the k_1 dependence, one needs to integrate the absolute value over k_1, but the right hand side of (128) grants us that the overall procedure yields that the supremum of the integrand in (127) is of rapid decreasing in p for all $p \in V_{k_x}$.
Nonetheless, the result is not yet conclusive since we still need to analyse the case in which the point x lies in $\partial J^+(\mathcal{B}; \mathscr{K}) \cap \mathscr{M}$, that is $x \in \mathcal{H}_{ev}$. In that case, for every open cone $\Gamma \in T_x^*\mathscr{M}$ containing k_x, there is $p \in \Gamma$ such that the inextendible geodesic starting form x and tangent to p meets the closure of Σ_1, it reaching $\mathcal{B}$. Therfore, in order to apply the same argument as before, we need to modify the form of Σ_1 in the computation of $\tilde{E}_5$ (127) in a neighbourhood of $\mathcal{B}$. Therefore we need a slightly more refined estimate of the decay-rate of the solutions of (13) on $\Im^-$. This can be achieved if we adapt the proof of Theorem 1.1 in [DR09] under the assumption that we modify the form of Σ_1, used to compute $\tilde{E}_5$ (127), into that of another spacelike hypersurface, say Σ'_1, contained in $\overline{\mathscr{W}}$ and such that it intersects $\mathcal{H}$ at some negative value of the Kruskal null coordinate U and it differs by Σ_1 only in a neighbourhood of $\mathcal{B}$.

In the next, we shall briefly review the arguments given in [DR09] in order to show that it is really possible to deform the initial surface Σ_1 on which the value of $\tilde{E}_5$ (127) is computed preserving the decay estimates presented above and in (28). To this end we shall follow the discussion and the notation introduced in that paper in order to obtain the decay estimates in the neighbourhood of i^+, eventually the desired estimates towards i^- could be obtained employing the time reversal symmetry. Let us start noticing that a central role in the analysis performed in [DR09] is played by the flux generated by the Morawetz vector field $K = v^2\frac{\partial}{\partial v} + u^2\frac{\partial}{\partial u}$. Moreover, as explained in Section 9 of [DR09], the crucial estimates, needed in this mentioned paper, are obtained out of the divergence (or Stokes-Poincaré) theorem applied to the current $J^K_\mu(\varphi)$:

$$J^K_\mu(\varphi) = K^\nu T_{\mu\nu}(\varphi) + |\varphi|^2\nabla_\mu\psi - \psi\nabla_\mu|\varphi|^2\,, \qquad \psi = \frac{tr^*}{4r}\left(1 - \frac{2m}{r}\right)$$

which is generated by K though with a modification due to total derivatives. Following such way of reasoning, let us compute the mentioned flux between two spacelike smooth surfaces Σ_1 and Σ_2 in $\mathscr{W}$, identified respectively as the loci with fixed time coordinate $\{t = t_1\}$ and $\{t = t_2\}$, though with $t_2 > t_1$. The end point is

$$\hat{E}^K_\varphi(t_2) = \hat{E}^K_\varphi(t_1) + \hat{I}^K_\varphi(\mathcal{P}),$$

where $\hat{E}^K_\varphi(t_2)$ is the boundary term computed on Σ_2 and $\hat{I}^K_\varphi(\mathcal{P})$ is the the volume term computed in the region $\mathcal{P} \doteq J^+(\Sigma_1) \cap J^-(\Sigma_2)$. Notice that the integrand of the boundary terms $\hat{E}^K_\varphi(t_1)$ are everywhere positive, while, as it can be seen from Proposition 10.7 of [DR09], the one of the volume element $\hat{I}^K_\varphi(\mathcal{P})$ is negative everywhere but in the region $\mathcal{P} \cap \{r_0 < r < R\}$ where the constant r_0 and R (with $2m < r_0 < 3m < R$) are defined in section 6 of [DR09]. For our later purposes, since we would like to eventually deform both Σ_1 and Σ_2 in a neighbourhood of $\mathcal{B}$, one should notice that the integrand is negative on such a neighbourhood if chosen in the region $r < r_0$.

Since the pointwise decay estimate towards i^+ on $\Im^+$ can be obtained from $\hat{E}^K_\varphi(t)$, the problem boils down to control the bad positive volume term in $\hat{I}^K_\varphi(\mathcal{P})$. Luckily enough, the positive part of $\hat{I}^K_\varphi(\mathcal{P})$ can be controlled by t_2 times $\hat{I}^{\mathbf{X}}_\varphi(\mathcal{P}) + \hat{I}^{\mathbf{X}}_{\Omega\varphi}(\mathcal{P})$ where $\hat{I}^{\mathbf{X}}_\varphi(\mathcal{P})$ is the sum of the volume terms. These arise out of the divergence theorem applied to the modified current generated by vectors like $X_\ell = f_\ell(r^*)\frac{\partial}{\partial r^*}$ acting separately on an angular mode decomposition[5] (we refer toSsection 7 of [DR09] for further details on the construction of $\hat{I}^{\mathbf{X}}_\varphi(\mathcal{P})$ and to [DR07] for recent results that do not require a decomposition in modes).

Notice that, as discussed in proposition 10.2 of [DR09] the boundary terms $|E^{\mathbf{X}}_\varphi(t)|$ are always smaller then a constant C times the conserved flux of energy $E_\varphi(t)$, with respect to the Killing time $\frac{\partial}{\partial t}$. Hence, if we collect all these results, it is possible to write

$$\hat{E}^K_\varphi(t) \leq \hat{E}^K_\varphi(t_1) + (t - t_1)C\left(E_{\varphi^\chi}(t_1) + E_{\Omega\varphi^\chi}(t_1)\right), \tag{129}$$

where Ω is the square root of the angular momentum while φ^χ is a solution of the equation of motion coinciding with φ on $(t_1, t)\times(r_0, R)\times\mathbb{S}^2$ vanishing in a neighborhood of $\mathcal{B}$, as the one constructed in the proof of Proposition 10.12 in [DR09]. More precisely, for t sufficiently close to t_1, φ^χ can be constructed as the solution generated by the following compactly supported Cauchy data on Σ_{t_1}: $\varphi^\chi(t_1, r^*) = \chi(2r^*/t_1)\varphi(t_1, r^*)$ and $\partial_t\varphi^\chi(t_1 r^*) = \chi(2r^*/t_1)\partial_t\varphi(t_1, r^*)$, where χ is a compactly supported smooth function on $\mathbb{R}$ equal to 1 on $[-1, 1]$ and vanishing outside $[-1.5, 1.5]$.

[5] Here $\ell(\ell + 1)$ is the eigenvalue of the angular momentum operator.

As explained in Section 12.1 of [DR09], it can be shown that, if $t_2 = 1.1t_1$ and t_1 is sufficiently large, then $E_{\varphi^\chi}(t_2) \leq C\, t_2^{-2}\, \hat{E}^K \varphi(t_2)$ and this permits to obtain a better estimate then (129), it yields

$$t_2 \hat{I}^{\mathbf{X}}_{\varphi}(\mathcal{P}) \leq \frac{C}{t_2} \hat{E}^K_{\varphi}(t_1) + C\left(E_{\varphi^\chi}(t_1) + E_{\Omega\varphi^\chi}(t_1)\right), \tag{130}$$

which is valid for $t_2 = 1.1t_1$ in particular. The estimate for a generic interval $t - t_1$ can be obtained, as explained in Section 12.1 of [DR09], dividing $t - t_1$ in sub interval $t_{i+1} = 1.1t_i$ and eventually summing the estimates (130) over i. In such a way it is possible to obtain

$$t\hat{I}^{\mathbf{X}}_{\varphi}(\mathcal{P}) \leq C\hat{E}^K_{\varphi}(t_1) + C\log(t)\left(E_{\varphi^\chi}(t_1) + E_{\Omega\varphi^\chi}(t_1)\right),$$

for a generic interval. As a final step, applying the same reasoning for $t\hat{I}^{\mathbf{X}}_{\Omega\varphi}(\mathcal{P})$ and using both of them to control $\hat{I}^K_{\varphi}$ we obtain a better estimate for $\hat{E}^K$ then the one (129), namely

$$\hat{E}^K_{\varphi}(t) = C\hat{E}^K_{\varphi}(t_1) + C\hat{E}^K_{\Omega\varphi}(t_1) + C\log(t)\left(E_{\varphi^\chi}(t_1) + E_{\Omega\varphi^\chi}(t_1) + E_{\Omega\Omega\varphi^\chi}(t_1)\right), \tag{131}$$

where the t in (129) is substituted by $\log(t)$, paying the price of considering higher derivatives.

The $\log(t)$ can eventually be removed employing once again the same line of reasoning, using (131) in place of (129) to improve (130), obtaining

$$\hat{E}^K_{\varphi}(t_2) \leq C\left(\sum_{n=0..3} \hat{E}_{\Omega^n\varphi}(t_1) + \sum_{n=0..2} \hat{E}^K_{\Omega^n\varphi}(t_1)\right) \leq \tilde{E}_5(\Sigma_1).$$

We would like to stress that, since the integrand $I^K_{\varphi}(\mathcal{O})$ is positive whenever $\mathcal{O}$ is a small neighbourhood of $\mathcal{B}$, the very same results can be obtained out of a modification of the surfaces Σ_1 and Σ_2 in such a way that they are still spacelike while they intersect the horizon $\mathcal{H}_{ev}$ at positive V equal to V_0; in this new framework the form of $\tilde{E}_5(\Sigma_1')$ is left unaltered with respect to (127), though it is computed on a modified surface Σ_1'. The decay estimate towards i^+ on $\Im^+$ can eventually be obtained as in Section 13.2 of [DR09]. At this point, out of time reversal, we can employ a similar argument as before in order to get the rapid decrease in p of $\|\varphi^{f_p}_{\Im^-}\|^{\eta}_{\Im^-}$.

The case of the decay on the horizon can be dealt in a similar way and, in such case, the pointwise decay on $\mathcal{H}^-$ can be shown to be controlled by an integral similar to the one defining $\tilde{E}_5$, though here it is again computed on the modified surface. In order to establish the mentioned decay rate, it is, however, necessary to consider another flux, namely that generated by a vector field Y which approaches $\frac{1}{1-\frac{2m}{r}}\partial_u$ on the horizon $\mathcal{H}$, as described in Section 8 of [DR09]. In this framework, even if the integrand of the volume term $\hat{I}^Y_{\varphi}$, associated with Y, is negative in a region formed by the compact interval $[\hat{r}_0, R]$, it can be controlled in a similar way as previously discussed for $\hat{I}^{\mathbf{X}}_{\varphi}$. $\square$

Proof of Lemma 4.2. As a starting point, let us recall that Λ_U is a weak-bisolution of (13), whose antisymmetric part is nothing but the causal propagator $E = E_{P_g}$ in $\mathscr{M}$. The wave-front set of E is well-known [Ra96a] and contains only pair of non-vanishing light-like covectors, so that:

$$(x, y, k_x, 0) \notin WF(E)\,, \qquad (x, y, 0, k_y) \notin WF(E)\,.$$

Therefore, whenever $(x, y, k_x, 0) \in WF(\Lambda_U)$, also $(y, x, 0, k_x)$ must lie in $WF(\Lambda_U)$ and *vice versa*, otherwise the wavefront set of the antisymmetric part of Λ_U, which is nothing but E, would contain a forbidden

element $(x, y, k_x, 0)$. This allow us to focus only on an arbitrary, but fixed $(x, y, k_x, 0) \in T^*(\mathscr{M} \times \mathscr{M}) \setminus \{0\}$; in order to show that it does not lie in $WF(\Lambda_U)$. Furthermore we know, thanks to *Part 1* of the proof of theorem 4.1, that Λ_U is of Hadamard form in $\mathscr{W}$ and, thus, the statement of this lemma holds if $x, y \in \mathscr{W}$. We shall hence focus on the case of $x \in \mathscr{M} \setminus \mathscr{W}$ and $y \in \mathscr{W}$, the remaining cases will be treated later. In the case $x \in \mathscr{M} \setminus \mathscr{W}$ and $y \in \mathscr{W}$, it suffices to consider only those k_x such that there are no representatives of $B(x, k_x)$ lying in $\mathscr{W}$, otherwise we would be falling in the already discussed scenario using a propagation-of-singularities argument already used in other proofs. This restriction yields, however, that a representative $(q, k_q) \in B(x, k_x)$ exists such that $q \in \mathcal{H}^+ \cup \mathcal{B}$. Summarizing, we are going to prove that $(x, y, k_x, 0)$ is a direction of rapid decreasing for $\Lambda_U(f_{k_x}, h)$, for some functions $f, g \in C_0^\infty(\mathscr{M}; \mathbb{R})$ with $f(x) = h(y) = 1$, provided $x \in \mathscr{M} \setminus \mathscr{W}$, $y \in \mathscr{W}$ and a representative $(q, k_q) \in B(x, k_x)$ exists such that $q \in \mathcal{H}^+ \cup \mathcal{B}$. As before $f_{k_x} \doteq f e^{i\langle k_x, \cdot\rangle}$ and $\varphi^h \doteq Eh$.
In this scenario, let us pick a partition of unit $\chi + \chi' = 1 : \mathcal{H} \to \mathbb{R}$ where $\chi \in C_0^\infty(\mathcal{H}; \mathbb{R})$ and $\chi = 1$ in a neighborhood of q. Hence

$$\Lambda_U(f_{k_x}, h) = \lambda_{\mathcal{H}}(\chi \varphi_{\mathcal{H}}^{f_{k_x}}, \varphi_{\mathcal{H}}^h) + \lambda_{\mathcal{H}}(\chi' \varphi_{\mathcal{H}}^{f_{k_x}}, \varphi_{\mathcal{H}}^h) + \lambda_{\Im^-}(\varphi_{\Im^-}^{f_{k_x}}, \varphi_{\Im^-}^h). \tag{132}$$

The second and third terms are rapidly decreasing in k_x because they are respectively dominated by $C\|\chi' \varphi_{\mathcal{H}}^{f_{k_x}}\|_{\mathcal{H}^-} \cdot \|\varphi_{\mathcal{H}}^h\|_{\mathcal{H}^-}$ and $C'\|\varphi_{\Im^-}^{f_{k_x}}\|_{\Im^-} \cdot \|\varphi_{\Im^-}^h\|_{\Im^-}$, C and C' being positive constants, which, in turn, are rapidly decreasing in k_x due to Proposition 4.4. The norms $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\Im^-}$ are those respectively defined in (52) and in (65). Therefore, we need to establish that k_x is of rapid decreasing for $\lambda_{\mathcal{H}}(\chi \varphi_{\mathcal{H}}^{f_{k_x}}, \varphi_{\mathcal{H}}^h)$ only. This can be done by the same procedure as that used at the end of the **case A** in the proof of Theorem 4.1 leading to (86), to prove the rapid decrease of $k_x \mapsto \lambda_{\mathcal{H}}(\chi \varphi_{\mathcal{H}}^{f_{k_x}}, \varphi_{\mathcal{H}}^{h_{k_y}})$ for a fixed k_y and assuming $k_y = 0$ there (that part of the proof is independent form the lemma we are proving here, while this lemma is used elsewhere therein).
Let us now treat the case $y \in \mathscr{M} \setminus \mathscr{W}$ and $x \in \mathscr{W}$, and let us prove that $(x, y, k_x, 0) \notin WF(\Lambda_U)$. To this end we adopt an overall coordinate frame where a coordinate, indicated by t, is tangent to X and the remaining three coordinates are denoted by $\overline{x}$. In this reference frame, the pull-back action of the one-parameter group generated by X trivially acts as $(\beta_\tau f)(t, \overline{x}) = f(t - \tau, \overline{x})$. First of all notice that, due to the restriction (81), the cases of k_x spacelike or timelike are trivial ruled out, so we consider $k_x \in T_x^*(\mathscr{M} \setminus \mathscr{W}) \equiv \mathbb{R}^4$ of null type and exploit the splitting $k_x = (k_{xt}, \overline{k_x})$, where we have isolated the t-component from the three remaining ones $\overline{k_x}$.
For k_x as before, consider the two non-null and non-vanishing covectors $q = (0, \overline{k_x})$ and $q' = (-k_{xt}, 0)$. In view of (81) $(x, y, q, q') \notin WF(\Lambda_U)$, hence, employing (c) of Proposition 2.1 of [Ve99], there exist a conical open neighborhood V' of (q, q'), a function $\psi' \in C_0^\infty(\mathbb{R}^4 \times \mathbb{R}^4; \mathbb{R})$ with $\psi'(0, 0) = 1$, such that for all $n \geq 1$,

$$\sup_{k, k' \in V'} \left| \int d\tau d\tau' d\overline{x}' d\overline{y}' \; \psi'(x', y') \; e^{i\lambda^{-1}(k_t \tau + \overline{k}\overline{x}')} e^{i\lambda^{-1}(k_t' \tau' + \overline{k}' \overline{y}')} \Lambda_U(\beta_\tau(f_{\overline{x}', \lambda}^{(p)}), \beta_{\tau'}(h_{\overline{y}', \lambda}^{(p)})) \right| \leq C_n \lambda^n, \tag{133}$$

which holds for every $0 < \lambda < \lambda_n$, where $C_n \geq 0$ and $\lambda_n > 0$ are suitable constants. In the preceding expression we have used the notation $x' = (\tau, \overline{x}')$, and $y' = (\tau', \overline{y}')$ where we have highlighted the $t-$component. Moreover $f_{\overline{x}', \lambda}^{(p)}(z)$ (and similarly also $h_{\overline{y}', \lambda}^{(p)}$) is defined as follows

$$f_{\overline{x}', \lambda}^{(p)}(z) \doteq f(x + \lambda^{-p}(z - \overline{x}' - x)) \,, \qquad f \in C_0^\infty(\mathscr{M}; \mathbb{R}) \,, \qquad \hat{f}(0) = 1 \,,$$

$\hat{f}$ being the standard Fourier transform. At this point we can use the translation invariance of Λ_U under the action of $\beta_{-\tau-\tau'}$ which implies that $\Lambda_U(\beta_\tau(f_{\overline{x}', \lambda}^{(p)}), \beta_{\tau'}(h_{\overline{y}', \lambda}^{(p)}))$ is equal to $\Lambda_U(\beta_{-\tau'}(f_{\overline{x}', \lambda}^{(p)}), \beta_{-\tau}(h_{\overline{y}', \lambda}^{(p)}))$

and hence from (133) that for all $p \geq 1$:

$$\sup_{k,k' \in V} \left| \int d\tau d\tau' d\overline{x}' d\overline{y}' \; \psi'(x',y') \; e^{i\lambda^{-1}(k_t\tau + \overline{k}\overline{x}')} e^{i\lambda^{-1}(k'_t\tau' + \overline{k}'\overline{y}')} \Lambda_U(\beta_{-\tau'}(f^{(p)}_{\overline{x}',\lambda}), \beta_{-\tau}(h^{(p)}_{\overline{y}',\lambda})) \right| \leq C_n \lambda^n,$$

if $0 < \lambda < \lambda_n$. The found result implies that (133) holds also replacing (i) V' with $V = V' + (k_x - q, -q')$ – which is an open conical neighborhood of $(k_x, 0)$ – and (ii) ψ' with $\psi(x',y') \doteq \psi((\tau',\overline{x}'),(\tau,\overline{y}'))$. Exploiting proposition 2.1 of [Ve99] once again, this yields that $(x,y,k_x,0) \notin WF(\Lambda_U)$ as wanted.
To conclude the proof, we need to analyse the last possible case, namely both $x, y \in \mathscr{M} \setminus \mathscr{W}$. If a representative of either $B(x,k_x)$ or $B(y,k_y)$ lies in $T^*\mathscr{W}$, then we fall back in the previous analysis. Hence, we need to focus only on the scenario where no representatives of both $B(x,k_x)$ and $B(y,k_y)$ lies in $T^*\mathscr{W}$. In this case, we can make use of an argument substantially identically to the one used in the analysis above: Introducing a partition of unit on $\mathcal{H}$ for both variables. In this way we have a decomposition like (132) with two more terms which can be analysed exactly as the others, thus leading to the wanted statement. $\square$

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