An application of biregularity to quaternionic Lagrange interpolation

A. Perotti

Department of Mathematics, University of Trento, Via Sommarive, 14, I-38050 Povo Trento ITALY

Abstract. We revisit the concept of totally analytic variable of one quaternionic variable introduced by Delanghe [1] and its application to Lagrange interpolation by Güerlebeck and Sprössig [2]. We consider left-regular functions in the kernel of the Cauchy-Riemann operator

\[ \partial = 2 \left( \frac{\partial f}{\partial z_1} + j \frac{\partial f}{\partial z_2} \right) = \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} - k \frac{\partial}{\partial x_3}. \]

For every imaginary unit \( p \in S^2 \), let \( C_p = (1, p) \cong \mathbb{C} \) and let \( J_p = p_1 J_1 + p_2 J_2 + p_3 J_3 \) be the corresponding complex structure on \( \mathbb{H} \). We identify totally regular variables with real–affine holomorphic functions from \( (\mathbb{H}, J_p) \) to \( (C_p, L_p) \), where \( L_p \) is the complex structure defined by left multiplication by \( p \). We then show that every \( J_p \)-biholomorphic map, which is always a biregular function, gives rise to a Lagrange interpolation formula at any set of distinct points in \( \mathbb{H} \).

Keywords: Quaternionic regular function, biregular function, Lagrange interpolation

PACS: 02.30.-f, 02.30.Fn, 02.60.Ed

PRELIMINARIES

We identify the space \( \mathbb{C}^2 \) with the set \( \mathbb{H} \) of quaternions by means of the mapping that associates the pair \((z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)\) with the quaternion \( q = z_1 + z_2 j = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H} \). A quaternionic function \( f = f_1 + f_2 j \in C^1(\Omega) \) is (left) regular (or hyperholomorphic) on \( \Omega \) if

\[ \partial f = 2 \left( \frac{\partial f}{\partial z_1} + j \frac{\partial f}{\partial z_2} \right) = \frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1} + j \frac{\partial f}{\partial x_2} - k \frac{\partial f}{\partial x_3} = 0 \quad \text{on} \ \Omega. \]

We will denote by \( \mathcal{R}(\Omega) \) the space of regular functions on \( \Omega \) (cf. e.g. \([8] \) and \([7]\) for properties of these functions). The space \( \mathcal{R}(\Omega) \) contains the identity mapping and every holomorphic mapping \( (f_1, f_2) \) on \( \Omega \) defines a regular function \( f = f_1 + f_2 j \). The original definition of regularity given by Fueter (cf. \([8]\) or \([3]\)) differs from that adopted here by a real coordinate reflection. Let \( \gamma \) be defined by \( \gamma(z_1, z_2) = (z_1, z_2) \). Then \( f \) is regular on \( \Omega \) if and only if \( f \circ \gamma \) is Fueter–regular on \( \gamma(\Omega) = \gamma^{-1}(\Omega) \).

A regular function \( f \in C^1(\Omega) \) is called biregular if \( f \) is invertible and \( f^{-1} \) is regular.

Holomorphic functions with respect to a complex structure \( J_p \)

Let \( J_p = p_1 J_1 + p_2 J_2 + p_3 J_3 \) be the orthogonal complex structure on \( \mathbb{H} \) defined by a unit imaginary quaternion \( p = p_1 i + p_2 j + p_3 k \) in the sphere \( S^2 = \{ p \in \mathbb{H} \ | \ p^2 = -1 \} \). Let \( C_p = (1, p) \) be the complex plane spanned by 1 and \( p \) and let \( L_p \) be the complex structure defined on \( \mathbb{T} \mathbb{C}_p \cong \mathbb{C} \) by left multiplication by \( p \). If \( f = f^0 + i f^1 : \Omega \rightarrow \mathbb{C} \) is a \( J_p \)-holomorphic function, i.e. \( df^0 = J_p^* (df^1) \) or, equivalently, \( df + iJ_p^* (df) = 0 \), then \( f \) defines a regular function \( \tilde{f} = f^0 + pf^1 \) on \( \Omega \). We can identify \( \tilde{f} \) with a holomorphic function

\[ \tilde{f} : (\Omega, J_p) \rightarrow (C_p, L_p). \]

We have \( L_p = J_{\gamma(p)} \), where \( \gamma(p) = p_1 i + p_2 j + p_3 k \). More generally, we can consider the space of holomorphic maps from \( (\Omega, J_p) \) to \( (\mathbb{H}, L_p) \)

\[ Hol_p(\Omega, \mathbb{H}) = \{ f : \Omega \rightarrow \mathbb{H} \text{ of class } C^1 \ | \ \overline{\partial}_p f = 0 \text{ on } \Omega \} = \text{Ker}(\overline{\partial}_p) \]
where $\partial_p$ is the Cauchy–Riemann operator with respect to the structure $J_p$

$$\partial_p = \frac{1}{2}(d + pJ_p^* \circ d).$$

For any positive orthonormal basis $\{1, p, q, pq\}$ of $\mathbf{H}$ ($p, q \in \mathbb{S}^2$), let $f = f_1 + f_2q$ be the decomposition of $f$ with respect to the orthogonal sum

$$\mathbf{H} = \mathbb{C}_p \oplus (\mathbb{C}_p)q.$$  

Let $f_1 = f^0 + pf^1$, $f_2 = f^2 + pf^3$, with $f^0, f^1, f^2, f^3$ the real components of $f$ w.r.t. the basis $\{1, p, q, pq\}$. Then the equations of regularity can be rewritten in complex form as

$$\partial_p f_1 = J^*_q(\partial_p f)_2,$$

where $\overline{f}_2 = f^2 - pf^3$ and $\partial_p = \frac{1}{2}(d - pJ_p^* \circ d)$. Therefore every $f \in \text{Hol}_p(\Omega, \mathbf{H})$ is a regular function on $\Omega$.

### The energy quadric

In [4] and [6] was introduced the energy quadric of a regular function $f$. It is a family (depending on the point $z \in \Omega$) of positive semi-definite quadrics which contains a lot of information about the (Dirichlet) energy of $f$ and the holomorphic properties of the function. In particular, this concept can be used to show that there are regular functions that are not $J_p$-holomorphic for any $p$, and that an affine biregular function is always $J_p$-biholomorphic for some $p$: there exists $p$ such that $f \in \text{Hol}_p(\mathbf{H}, \mathbf{H})$ and $f^{-1} \in \text{Hol}_p(\mathbf{H}, \mathbf{H})$.

### TOTALLY REGULAR FUNCTIONS

**Definition 1** A regular function $f \in \mathcal{R}(\Omega)$ is called totally regular if the powers $f^k$ are regular on $\Omega$ for every integer $k \geq 0$ and $f^k$ is regular on $\Omega' = \{x \in \Omega \mid f(x) \neq 0\}$ for every integer $k < 0$.

**Theorem 1** Let $f \in \mathcal{R}(\Omega)$ with image $\text{Im}(f)$ contained in a (real) plane $H$. Then there exists $p \in \mathbb{S}^2$ such that $f \in \text{Hol}_p(\Omega, \mathbf{H})$. If $f$ is non–constant, the complex structure $J_p$ is uniquely determined.

If $f = \sum_{\alpha=0}^4 x_\alpha a_\alpha + b \in \mathcal{R}(\mathbf{H})$, $a_\alpha, b \in \mathbf{H}$, is (real) affine and $f$ has Jacobian matrix of maximum rank 2, the same conclusion of Theorem 1 follows.

**Corollary 2** If $f \in \mathcal{R}(\Omega)$ and $\text{Im}(f)$ is contained in $\mathbb{C}_p$ for some $p \in \mathbb{S}^2$, then $f$ is a $J_p$–holomorphic function, and therefore it is totally regular.

**Remark 1** The decomposition $f = f_1 + f_2q$ of a function $f \in \text{Hol}_p(\Omega)$ w.r.t. any orthonormal basis $\{p, q, pq\}$ defines totally regular components $f_1, f_2 \in \text{Hol}_p(\Omega, \mathbb{C}_p)$.

We now prove the converse of Corollary 2 for affine functions. Using the energy quadric of a function, we are able to show that the regularity of $f$ and $f^2$ is sufficient to get that $f \in \text{Hol}_p(\mathbf{H}, \mathbb{C}_p)$ and to obtain the total regularity of $f$.

**Theorem 3** If $f \in \mathcal{R}(\mathbf{H})$ is affine and $f^2$ is regular, then $f$ has maximum rank 2 and there exists $p \in \mathbb{S}^2$ such that $f \in \text{Hol}_p(\mathbf{H}, \mathbb{C}_p)$.

**Corollary 4** If $f \in \mathcal{R}(\mathbf{H})$ is affine and $f^2$ is regular, then $f$ is totally regular.

The condition on the rank of $f$ given in Theorem 3 was proved, in the context of Fueter–regularity, in [2]\$1.2$ (cf. also [3]\$10$). The preceding results tell that the set of affine totally regular functions coincides with the set

$$\{f \text{ affine } \mid f \in \bigcup_{p \in \mathbb{S}^2} \text{Hol}_p(\mathbf{H}, \mathbb{C}_p)\}.$$

Note that every subspace $\text{Hol}_p(\mathbf{H}, \mathbb{C}_p)$ is a commutative algebra w.r.t. the pointwise product.
Remark 2

The biregular function $f$ in [2], as an application of totally analytic variables, a Lagrange's Interpolation Theorem was proved. Given

$$v_p(x) = x_0 + (\gamma(p) \cdot \bar{x})p.$$  

In particular, we get the variables $v_i = x_0 + x_1i = z_1 \in Hol_p(H, C_p)$, $v_j = x_0 + x_2j \in Hol_p(H, C_j)$, $v_k = x_0 - x_3k \in Hol_k(H, C_k)$. We can also consider the totally regular variables $v'_p = v_p q \in Hol_p(H, C_p)$, which satisfy the additive property

$$\frac{1}{|p + q|}(v'_p + v'_q) = v'_{p+q} \in Hol_{p+q}(H, C_{p+q}).$$

For every $a \in H, a \neq 0$, let $\text{rot}_a(q) = aqa^{-1}$ be the three–dimensional rotation of $H$ defined by $a$. In [5] was studied the effect of rotations on regularity and holomorphicity of functions. As an application of those results, we get that $v_p$ can be seen as one component of a biregular function.

**Theorem 5**

a) For every $p \in S^2$, the function $v_p$ is regular on $H$ and belongs to the space $Hol_p(H, C_p)$. Therefore $v_p$ is totally regular.

b) For any $p, q \in S^2, q \perp p$, let $a \in H$ be such that $\text{rot}_{\gamma(a)}(i) = p, \text{rot}_{\gamma(a)}(j) = q$. There exists an affine biregular function $f_a = v_p + w_a q$, with totally regular components $v_p, w_a \in Hol_p(H, C_p)$. The function $f_a \in Hol_p(H, H)$ is $J_p$–biholomorphic, with inverse of the same type as $f_a$:

$$f_a^{-1} = f_a' = v_{\gamma(a)} + w_a' q' \in Hol_{p}(H, H) \quad (a' = \gamma(a)^{-1}, \gamma(p) = \text{rot}_a^{-1}(i), q' = \text{rot}_a^{-1}(j)).$$

**Remark 2** The biregular function $f_a$ is defined by the simple formula $f_a = \text{rot}_{\gamma(a)} a$.

### Quaternionic Lagrange Interpolation

In [2], as an application of totally analytic variables, a Lagrange’s Interpolation Theorem was proved. Given $k$ distinct points $b_1, \ldots, b_k \in H$ and $k$ values $u_1, \ldots, u_k \in H$, one wants to construct a Lagrange polynomial in the module of regular functions, i.e., a polynomial $L \in \mathcal{R}(H)$ such that $L(b_j) = u_j$ for every $j = 1, \ldots, k$.

**Theorem 6** Given a $J_p$–biholomorphic mapping $f = f_1 + f_2 q \in Hol_p(H, H)$ ($q \perp p$), there exist (infinitely many) $\alpha, \beta \in C_p$ such that $g = \alpha f_1 + \beta f_2 \in Hol_p(H, C_p)$ is totally regular and satisfies the conditions

$$g(b_i) \neq g(b_j) \quad \forall \ i \neq j \ (i, j = 1, \ldots, k).$$

The numbers $\alpha, \beta$ can also be found in the real field.

Then every $J_p$–biholomorphic mapping $f$ gives rise to a Lagrange interpolation function (a polynomial if $f$ is a polynomial function), given by the formula

$$L = \sum_{j=1}^{k} l_j u_j, \quad \text{where} \quad l_j(x) = \prod_{i \neq j} (g(x) - g(b_i))(g(b_j) - g(b_i))^{-1} \in Hol_p(H, C_p).$$

The functions $l_j^m$ are regular on $H$ for every integer $m > 0$ and $L \in \mathcal{R}(H)$. The powers of $L$ are regular if also the values $u_j$ belong to the subalgebra $C_p$.

**Example 1** If we take the function $f_a$ of Theorem 5 as $J_p$–biholomorphic mapping, and $\alpha, \beta \in \mathbb{R}$, then $g = \alpha f_1 + \beta f_2$ is the linear function

$$\text{rot}_{\gamma(a)}(\alpha z_1 + \beta z_2) \circ \text{rot}_a.$$
ACKNOWLEDGMENTS

The work was partially supported by MIUR (PRIN Project “Proprietà geometriche delle varietà reali e complesse”) and GNSAGA of INdAM.

REFERENCES

6. A. Perotti, Every biregular function is biholomorphic, Advances in Applied Clifford Algebras, in press.