ON MINIMIZING SYMMETRIC SET FUNCTIONS

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Abstract

Mader proved that every loopless undirected graph contains a pair \((u,v)\) of nodes such that the star of \(v\) is a minimum cut separating \(u\) and \(v\). Nagamochi and Ibaraki showed that the last two nodes of a “max-back order” form such a pair and used this fact to develop an elegant min-cut algorithm. M. Queyranne extended this approach to minimize symmetric submodular functions. With the help of a short and simple proof, here we show that the same algorithm works for an even more general class of set functions.

Key words: max-back order, minimum cut, symmetric submodular functions.

Main Section

Let \(V\) be a finite set. A value \(d(\{S,T\})\) is given for every unordered pair of disjoint subsets \(S,T\) of \(V\). For convenience, function \(d\) is called a map on \(V\), even if it is actually defined on a subset of \(2^V \times 2^V\). We also rely on the shorthand \(d(S, T) = d(\{S, T\})\) and leave the fact that \(d(S,T) = d(T,S)\) as understood. Function \(d\) is called monotone if \(d(S, T') \leq d(S, T)\) for any \(S, T\) disjoint and \(T' \subseteq T\). Finally, \(d\) is consistent if \(d(A, W \cup B) \geq d(B, W \cup A)\) whenever \(A, B, W\) are disjoint sets such that \(d(A, W) \geq d(B, W)\). As an example, when \(G = (V, E)\) is an undirected graph, then \(d(S, T) = |\{st \in E : s \in S, t \in T\}|\) for any disjoint sets \(S, T \subseteq V\), is a monotone and consistent map on \(V\). A subset \(S\) of \(V\) is said nontrivial when \(\emptyset \neq S \neq V\). We give an efficient algorithm to solve the following problem (minimum bipartition problem):

Given a finite set \(V\) and a monotone and consistent map \(d\) on \(V\), find a nontrivial subset \(S\) of \(V\) for which \(d(S, V \setminus S)\) is minimum.

A max-back order for \((V, d)\) is an ordering \(v_1, v_2, \ldots, v_n\) of the elements in \(V\) such that

\[d(v_i, \{v_1, \ldots, v_{i-1}\}) \geq d(v_j, \{v_1, \ldots, v_{i-1}\}) \quad \text{for } 2 \leq i < j \leq n\]

Let \(s\) and \(t\) be two elements of \(V\). An st-set is a subset \(S\) of \(V\) with \(|S \cap \{s,t\}| = 1\).

An ordered pair \((s,t)\) of elements of \(V\) is good if \(d(\{t\}, V \setminus \{t\}) \leq d(S, V \setminus S)\) holds for every st-set \(S\). Before the end of this section we will prove the following lemma:

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Lemma 1 Let $v_1, \ldots, v_n$ be a max-back order for $(V, d)$. Then $(v_{n-1}, v_n)$ is good for $(V, d)$.

Lemma 1 gives an efficient procedure, called a _Good_Pair, to find a good pair. When $(s, t)$ is a good pair two cases are possible: either $\{t\}$ is an optimal solution to our problem, or no optimal solution $S$ to the problem is an $st$-set. This motivates the following definitions: Let $s$ and $t$ be any two elements of $V$. Consider identifying $s$ and $t$ into a single new element $v_{st}$ thus obtaining a new set $V_{st} = V \setminus \{s, t\} \cup \{v_{st}\}$. Now, to reconsider a subset $X$ of $V_{st}$ as a subset of $V$, we define $\langle X \rangle = X$ if $v_{st} \notin X$ and $\langle X \rangle = X \setminus \{v_{st}\} \cup \{s, t\}$ if $v_{st} \in X$. When $S$ and $T$ are disjoint subsets of $V_{st}$ then $\langle S \rangle$ and $\langle T \rangle$ are disjoint subsets of $V$ and we define:

$$d_{st}(S, T) = d(\langle S \rangle, \langle T \rangle)$$

Note that, when $d$ is a monotone and consistent map on $V$, then $d_{st}$ is a monotone and consistent map on $V_{st}$. To conclude, the following algorithm solves the minimum bipartition problem.

**Algorithm 1** MIN\_BIPARTITION $(V,d)$

1. if $|V| = 2$ then return either of the two nontrivial subsets of $V$;
2. $(s, t) \leftarrow a_Good_Pair(V, d);$
3. return the best set among $\{t\}$ and $\langle\text{Min\_Bipartition}(V_{st}, d_{st})\rangle$;

We have now enough motivation to prove Lemma 1.

**Proof of Lemma 1:** The lemma is true for $n = 3$ since $d(v_2, v_1) \geq d(v_3, v_1)$ implies $d(\{v_1, v_3\}, v_2) \geq d(\{v_1, v_2\}, v_3)$ for $d$ is consistent. Let $S$ be any $v_nv_{n-1}$-set. We must show that:

$$d(S, V \setminus S) \geq d(\{v_n\}, V \setminus \{v_n\})$$

(1)

Clearly, $v_1v_2v_3, v_4, \ldots, v_n$ is a max-back order for $(V_{v_1v_2}, d_{v_1v_2})$. Thus, either (1) follows by induction or $S$ is a $v_1v_2$-set. Since $d$ is monotone, $v_1, v_2v_3, v_4, \ldots, v_n$ is max-back for $(V_{v_2v_3}, d_{v_2v_3})$ and either (1) follows or $S$ is a $v_2v_3$-set. Assume therefore that $S$ is both a $v_1v_2$-set and a $v_2v_3$-set. But then $S$ is not a $v_1v_3$-set and to derive (1) it suffices to show that $v_2, v_1v_3, v_4, \ldots, v_n$ is max-back for $(V_{v_1v_3}, d_{v_1v_3})$. Assume on the contrary $d_{v_1v_3}(v_k, v_2) > d_{v_1v_3}(v_{v_1v_3}, v_2)$. However $d(v_2, v_1) \geq d(v_3, v_1)$ and $d(v_3, \{v_1, v_2\}) \geq d(v_k, \{v_1, v_2\})$ since $v_1, \ldots, v_n$ is max-back for $(V, d)$. Since $d$ is monotone and consistent, we get $d(v_3, \{v_1, v_2\}) \geq d(v_k, v_2) = d_{v_1v_3}(v_k, v_2) > d_{v_1v_3}(v_{v_1v_3}, v_2) = d(\{v_1, v_3\}, v_2) \geq d(v_3, \{v_1, v_2\})$, a contradiction. \)

**Some Applications**

A couple of observations and a list of applications will follow. In Application 1, Queyranne’s important result on minimizing symmetric submodular functions is derived as a special case of our framework. The generalization is strict as shown in Applications 2 and 3.
Note that Algorithm 1 can also be used to solve maximization problems when \(-d\) is a monotone and consistent map. In practice it follows that we can maximize \(d'(S, V \setminus S)\) over the nontrivial subsets \(S\) of \(V\) whenever \(d'\) is a map on \(V\) with the following properties:

(i) \(d'(S, T') \geq d'(S, T)\) for any \(S, T\) disjoint and \(T' \subseteq T\) – (reverse monotonicity);

(ii) \(d'(A, W \cup B) \geq d'(B, W \cup A)\) whenever \(A, B, W\) are disjoint sets such that \(d'(A, W) \geq d'(B, W)\) – (consistency);

In contrast, maximizing \(d(S, V \setminus S)\) for a generic monotone and consistent map \(d\) is an \(NP\)-complete problem since it contains as a special case the max-cut problem, which is known to be \(NP\)-complete [4].

Application 1 (symmetric submodular functions [9])

Consider a finite set \(V\) and a real function \(f\) on \(2^V\). We are interested in finding a nontrivial subset of \(V\) which minimizes \(f\). For this reason we consider an ordered pair \((s, t)\) of elements of \(V\) to be good if \(\{t\}\) is an \(st\)-set minimizing \(f\). For any two disjoint subsets \(S, T\) of \(V\) let us define

\[
d_f(S, T) = f(S) + f(T) - f(S \cup T)
\]

If \(f\) is symmetric (that is, \(f(S) = f(V \setminus S)\) for every subset \(S\) of \(V\)), then a pair is good with respect to \(f\) if and only if it is good with respect to \(d_f\).

Note that \(d_f\) is consistent. Assume indeed \(A, B, W\) to be disjoint and such that \(d_f(A, W) \geq d_f(B, W)\). This means \(f(A) + f(W) - f(A \cup W) \geq f(B) + f(W) - f(B \cup W)\). But then \(d_f(A, B \cup W) = f(A) + f(B \cup W) - f(A \cup B \cup W) \geq f(B) + f(A \cup W) - f(A \cup B \cup W) = d_f(B, A \cup W)\).

So we are interested in characterizing those \(f\) for which \(d_f\) is monotone, that is, \(d_f(S, T_1) \leq d_f(S, T_1 \cup T_2)\) for any \(S, T_1, T_2\), all disjoint and non-empty. In terms of \(f\) this means, \(f(S) + f(T_1) - f(S \cup T_1) \leq f(S) + f(T_1 \cup T_2) - f(S \cup T_1 \cup T_2)\), or equivalently, \(f(S \cup T_1 \cup T_2) + f(T_1) \leq f(T_1 \cup T_2) + f(S \cup T_1)\). Hence \(d_f\) is monotone if and only if \(f\) satisfies the submodular inequality \(f(A \cap B) + f(A \cup B) \leq f(A) + f(B)\) for any sets \(A\) and \(B\) such that \(A \setminus B, B \setminus A, A \cap B\) and \(V \setminus A \setminus B\) are all non-empty.

In [8], Nagamochi and Ibaraki called such a function \(f\) crossing submodular and observed that the approach proposed by Queyranne in [9] to minimize symmetric submodular functions (where the submodular inequality has to hold for any sets \(A\) and \(B\)), was also valid for symmetric crossing submodular functions.

Algorithm 1 was first employed by Nagamochi and Ibaraki [7] to find minimum cuts in undirected graphs. A simple proof of the validity of Nagamochi and Ibaraki’s min-cut algorithm had been obtained by Frank [2] and Stoer and Wagner [10], while Queyranne was deriving his important, but less simple, extension. Recently, in [3], Fujishige gave another short proof of the validity of Nagamochi and Ibaraki’s min-cut algorithm and indicated how to employ his arguments to obtain a compact proof of Queyranne’s result.

In the next application we show that our simple approach actually embraces an even broader class of problems.
Application 2 (short distance partitions)

Let $G$ be a graph. A symmetric distance $\lambda(u,v)$ is given for every two nodes $u,v$. Assume we want to bipartition the node set $V$ of $G$ as to keep the maximum distance among two nodes on different sides of the partition as small as possible. Even if this problem can easily be solved directly, define $d(S,T) = \max\{\lambda(s, t) : s \in S, t \in T\}$. Note that $d$ is a monotone and consistent map in general. Consider the graph $(V, E) = (\{a, b, c, d\}, \{ab, bc, cd, da\})$ and for every $u, v \in V$ define the distance $\lambda(u, v)$ as the length of a shortest path between $u$ and $v$. (Hence $\lambda(a,c) = \lambda(b,d) = 2$, and $\lambda(a,b) = \lambda(b,c) = \lambda(c,d) = \lambda(a,d) = 1$). The sets $S = \{a, c\}$ and $T = \{a, d\}$ show that the function $f$ on $2^V$ defined by $f(S) = d(S, V \setminus S)$ for every $S \subseteq V$, is not crossing submodular in this special case.

Application 3 (critical cuts)

Let $(G, w)$ be a weighted graph. Assume to be interested in those spanning trees $T$ of $G$ such that $\max\{w(e) : e \in T\}$ is as small as possible. Then it is natural to define the cost of a cut $\delta(S)$ as the minimum of $w(e)$ for $e \in \delta(S)$ and to search for a cut of maximum cost. This is clearly a bottleneck problem and admits a direct and simple solution.

Define $d(S,T) = \min\{w(e) : e \text{ has an endpoint in } S \text{ and the other in } T\}$. Note that $-d$ is a monotone and consistent map. This is indeed a reformulation of the above problem on short distance partitions (see Application 2). Hence we also have that the function $f$ on $2^V$ defined by $f(S) = d(S, V \setminus S)$ for every $S \subseteq V$, is not crossing supermodular in general. (A function $f$ is called crossing supermodular if $f(A \cap B) + f(A \cup B) \geq f(A) + f(B)$ for any sets $A$ and $B$ such that $A \setminus B, B \setminus A, A \cap B$ and $V \setminus A \setminus B$ are all non-empty).

Application 4 (minimum cuts in hypergraphs [5])

Hypergraphs generalize graphs. When $G = (V, H)$ is an hypergraph, then the hyperedges in $H$ are arbitrary subsets of the node set $V$, Thus a graph is an hypergraph in which every hyperedge has cardinality 2. Klimmek and Wagner [5] proposed a Nagamochi-Ibaraki type algorithm to find a minimum cut in an hypergraph. Indeed, the cut function of an hypergraph is symmetric and submodular [5, 9]. Consider the bottleneck version of this problem, that is, finding a cut which minimizes the maximum weight of an hyperedge belonging to it. Submodularity is lost but still we would have to deal with a monotone and consistent map.

Application 5 (partitions minimizing ambivalence)

Let $G$ be a graph. Partition the node set $V$ as $S \cup (V \setminus S)$ in such a way as to minimize the number of nodes with neighbors in both sides of the partition. This problem can be formulated as an hypergraph min cut problem (for every node $v$, we have an hyperedge $h_v$ made of the neighbors of $v$ in $G$). The problem hence falls in the framework of Stoer and Wagner [10], but also in that of Queyranne [9], or finally in our framework.
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