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SIR epidemic models with age structure and immigration.

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Contents

Introduction.

The aim of the present work is to discuss some aspects about a problem of epidemiological modeling, the evolution of a childhood infectious disease in a human population subject to immigration and in which age-strucure is taken into account.
The first attempt to put age-sructure in an epidemiological model dates back to McKendrick (1926). Since then many authors put age-structure in their models, having recognized that it is then possible to gain a better insight into the contact patterns between individuals in the population. Such models are usually compartmental, that is population is subdivided into classes of individuals relevant from the epidemilogical point of view; this is done in the SIR model, where individuals are classified as susceptibles or infectives or removed. Since the 70^s age-structured SIR models were used to develop vaccination policies against diseases like measles, rubella and others in order to achieve eradication. In 1975 Dietz presented the following SIR age-structured model:

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) X &= -(\lambda(t) + \mu)X \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) Y &= \lambda(t)X - (\mu + \gamma)Y \,, \quad +\infty > a > 0 \,, \, t > 0 \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) Z &= \gamma Y - \mu Z \end{cases} \tag{1}$$

and:

$$\lambda(t) = \beta \int_0^{+\infty} Y(a,t)\, da \tag{2}$$

with initial and boundary conditions:

$$X(a,0) = X_0(a) \,, \; Y(a,0) = Y_0(a) \,, \; Z(a,0) = Z_0(a)$$

$$X(0,t) = \mu\, N \,, \; Y(0,t) = 0 \,, \; Z(0,t) = 0$$

where $X(a,t), Y(a,t), Z(a,t)$ are respectively the density of the susceptibles, the infectives and removeds aged a at time t, N is the total (and constant) population size, μ is the birth and death rate, $\lambda(t)$ is the force of infection, here supposed proportional to total amount of infectives in the population, and γ is the removal rate (the force of infection $\lambda(t)$ is the per capita rate of acquisition of infection, that is $\lambda(t)\Delta t$ is the probability that a susceptible individual will become infected in the small time interval $(t, t + \Delta t)$).
A similar model was developed by Hoppensteadt(1975); and not age-structured models but with the same (2) force of infection were developed by Bayley(1975), Waltman(1974). But assuming (2) as force of infection means to suppose that individuals interact homogeneously, age or social habits related differences are not taken into account; interactions between children at school are more likely to affect the transmission process of diseases like measles than child-adult or adult-adult interactions. This was indeed noticed from Anderson and May(1982d), Grenfell and Anderson(1985). A different functional form for the force of infection, taking into account the ages of both the infector and the infected, was proposed by Schenzle(1984), Anderson and May(1985a). In their works they put the functional form:

$$\lambda(a,t) = \int_0^{\omega} \beta(a,a')Y(a',t)\, da' \tag{3}$$

where $\beta(a,a')$ is a contact coefficient describing the interaction between susceptibles aged a and infectives aged a' (then $\lambda(a,t)\,\Delta a\,\Delta t$ is the probability that a susceptible of age in the small interval $(a, a + \Delta a)$ becomes infective during the small time interval $(t, t + \Delta t)$) and ω is the maximal age of individuals. Greenhalgh(1988b) and Inaba(1990) studied SIR age-structured models in case the transmission rate is like in (3). In their works they performed an accurate investigation of properties of the equilibria and obtained threshold results. Indeed they caracterized the dualism extinction/endemicity of the disease in terms of the spectral radius of a linear operator T: if $r(T) < 1$ then there is only the disease free equilibrium, if $r(T) > 1$ then there exists a positive equilibrium (that may be unique under suitable assumptions).
When studing epidemiological models, an important feature to be considered is the underlying demography: is the population stable or growing or experiencing decay, has a young or an old age profile, are all facts that affect the disease transmission. A demographical assumption nowadays commonly made in doing models for an infectious disease in western world is that of "stable population with immigration" (SPI for short), as defined in Manfredi and Valentini(2000). There, focusing on the actual italian situation, they analyze the behaviour of solutions of a system of Lotka-VonFoerster population equations in which a constant immigration inflow has been inserted with the aim of forecasting the demographical situation of a population which is experiencing fertility below replacement (BRF for short) and immigration. They lay stress on the stationarity gained by such population on the long term; this is indeed the more relevant forecast of SPI model, that is a population subject to BRF and to a

constant immigration stream, as age profile, will become stationary and reach an equilibrium age profile. Below replacement fertility and immigration is a situation now shared by almost all western countries; hence to add demographical realism in their models and avoiding at the same time to set unnecessry complications in the problem, some epidemiological modelers as for example Iannelli-Manfredi(2006) are beginnig to build models whose underlying demography is that furnished by the SPI model. A SIS and a SIR model without age-structure and with immigration, a fraction of which is infective, were analyzed from Brauer and Van der Driessche(2001). They showed that under a constant immigration flow on infectives, there is only a positive endemic equilibrium, the disease free equilibrium is lost. And the threshold result is now substituted by a threshold-like result, that is the limiting behaviour evinced from the endemic equilibrium as the infective fraction immigrant approaches zero.

The SIR model considered in this thesis is conceived for analyzing the spread of an infectious disease in an age-structured population, with the infective-susceptible interaction being as that given by the (3) contact rate; and with the SPI assumption as the demographical background.

In chapter one some basic definitions and the system of PDE^s governing the spread of the infection are given. In chapter two, by resorting to semigroup theory, is proved a result about existence and uniqueness of the solution of the system of PDE^s. In chapter three the steady states of the system are studied: following the analysis done by [10] and resorting to the teory of positive monotone operators on a cone in a Banach space, we prove existence and uniqueness for the steady state on a subset of $L^1_+(0, \omega)$, the cone of the positive functions of $L^1(0, \omega)$; then, by mean of the Sawashima theorem (a generalization of the Krein-Rutman theorem), a threshold-like result is proved. Then follows discussion about the posibility of obtaining a uniqueness result without resorting to the theory of monotone operators in case of using as a contact coefficient a 2×2 WAIFW matrix.

In chapter four we firstly rewrite force of infection by subdividing the life span into a finite number of discrete age classes and using a suitable WAIFW matrix as contact coefficient; then we numerically solve the equations of the system of PDE^s describing the model by mean of a discretization scheme along the characteristics. The algorithm is then used to perform some simulations aimed at having a glance in the transient phase between the initial time and the long term situation, with initial conditions given by actual italian age profile and under various assumptions on the immigrants (data about immigrants are less observable); and to compare these results with the simulations with the stationary population as initial condition, that is with the forecasts of the SPI model.

Chapter 1

The setting of the problem.

In SIR compartmental models what is firstly done is to classify individuals of the population, as regard as to the disease, as members of one of three classes:

the susceptible class, which consists of those individuals who are not sick, but can contract the infection by mean of a suitable contact with infective individuals;
the infective class, which consists of those individuals who have contracted the infection and can transmit it to others;
the removed class, which consists of those individuals who are immune from the disease because they contracted it and then they healed or by vaccination.
The number of individuals of the population without outer influences is not constant over time: individuals are born, they can generate new individuals and then they die before reach a maximal age. But we assume that the population can vary also by mean of a stream of individuals coming from the outside. We name these individuals "immigrants"; individuals natives of the population under examination, we name "natives".
We do the following fudamental demographical assumption: native population is in demographic decay but is kept in an equilibrium condition by mean of the immigration stream, and that this stream is constant as number of individuals coming in the time unit and as age profile. Hence we study the spread process of the disease in the SPI demographical context. Furthemore, we assume that immigrants acquire the same demographic parameters of the native population (mortality and fertility rates) after their arrival: immigrants suddenly becomes natives;
that a fraction of the immigrants is infective;
that there is no vertical transmission of the disease, that is newborn people are healty
that the disease doesn't affects the mortality rate;
that age affects the transmission rate of the disease.
We write:

- $X(a,t)$, the density of susceptible individuals of age a at time t;

- $Y(a,t)$, the density of infective individuals of age a at time t;

- $Z(a,t)$, the density of removed individuals of age a at time t;

- $I_X(a)$, the stream of susceptible immigrants that enter the population in the time unit;

- $I_Y(a)$, the stream of infective immigrants that enter the population in the time unit;

- $I_Z(a)$, the stream of susceptible immigrants that enter the population in the time unit.

Then we write ω as the maximum life length of individuals;
$I(a) = I_X(a) + I_Y(a) + I_Z(a)$ as the total immigration stream in the time unit;
$\mu \in L^1_{loc}\big([0,\omega)\big)$ as the mortality rate, which satisfies $\mu(a) \geq 0$ for a.e. $a \in (0,\omega)$, $\int_0^\omega \mu(s)ds = +\infty$ and $e^{-\int_0^a \mu(s)ds}$ is the fraction of individuals who are still living at age a;
γ as the recovery rate (hence $1/\gamma$ is the average time length of the disease).
Under these assumptions the system of partiale differential equations which governs the disease's diffusion is:

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) X &= -(\lambda(a,t) + \mu(a))X + I_X(a) \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) Y &= \lambda(a,t)X - (\mu(a) + \gamma)Y + I_Y(a) \qquad \omega > a > 0\,,\ t > 0 \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) Z &= \gamma Y - \mu(a)Z + I_Z(a) \end{cases} \tag{1.1}$$

where $\lambda(a,t)$ is the force of infection:

$$\lambda(a,t) = \int_0^\omega \beta(a,s)Y(s,t)\,ds\ . \tag{1.2}$$

as introduced from Schenzle(1984), Anderson and May(1985a); $\lambda(a,t)$ is the rate with whom susceptible individuals of age a fall ill becoming infective, so $\lambda(a,t)X(a,t)da\,dt$ is the number of susceptible individuals whose age is in the interval $(a, a+da)$, that pass in the infective class in a time in $(t, t+dt)$;
$\beta(\cdot,\cdot) \in L^\infty\big((0,\omega) \times (0,\omega)\big)$, $\beta(a,b)$ represents a contact coefficient between

individuals of age a and individuals of age b.
To make the problem handable with analytical tools it is then necessary to give boundary conditions:

$$X(0,t)\ ,\ Y(0,t)\ ,\ Z(0,t) \quad t \geq 0 \tag{1.1a}$$

and initial conditions:

$$X(a,0)\ ,\ Y(a,0)\ ,\ Z(a,0) \quad a \geq 0 \tag{1.1b}$$

(1.1a)'s give respectively the susceptible, infective and removed newborns at time t. Let us remember that we did the hypothesis: the population is in demographic decay but is substained by a constant immigration stream. If we call $n(a,t)$ the density of individuals of age a at time t, for $n(a,t)$ is valid the McKendrick-Von Foerster non-homogeneous equation (which it obtains by summing up the three equations and the pertinent boundary and initial conditions in (1.1)):

$$\left\{ \begin{array}{rcl} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) n & = & -\mu(a)\, n + I(a) \qquad a,t > 0 \\ n(0,t) & = & B(t) = \int_0^\omega \beta(a) n(a,t) da \quad t \geq 0 \\ n(a,0) & = & n_0(a) \qquad a \geq 0 \end{array} \right. \tag{1.3}$$

which has solution given by:

$$n(a,t) = \left\{ \begin{array}{ll} n_0(a-t)\frac{\Pi(a)}{\Pi(a-t)} + \int_0^t I(s)\frac{\Pi(a)}{\Pi(s+a-t)} ds & 0 < t < a < \omega \\ B(t-a)\Pi(a) + \int_0^a I(s)\frac{\Pi(a)}{\Pi(s)} ds & 0 < a < t \end{array} \right. \tag{1.4}$$

As in [8], we write $\beta(a)$ for the age specific fertility, that gives the number of newborn generated from a single inividual whose age is in $(a, a+da)$; and $\Pi(a) = e^{-\int_0^a \mu(s) ds}$ is the survival function. The equilibrium solution of (1.3) is given by the solution of the ordinary differential equation:

$$\left\{ \begin{array}{rcl} \frac{d}{da} n(a) & = & -\mu(a) n(a) + I(a) \quad a > 0 \\ n(0) & = & \int_0^\omega \beta(a) n(a) da \end{array} \right. \tag{1.5}$$

and then:

$$n(a) = n(0)\Pi(a) + \int_0^\omega I(s)\frac{\Pi(a)}{\Pi(s)} ds \ \ , \ \ 0 \leq a \leq \omega \tag{1.6}$$

with $n(0)$ given by:

$$n(0) = \frac{1}{1-R}\int_0^\omega \beta(a)\Pi(a)\int_0^a \frac{I(s)}{\Pi(s)}ds\,da \tag{1.7}$$

where $R = \int_0^\omega \beta(a)\Pi(a)\,da$ is the net reproduction rate ([8]) , and it represents the average number of newborn individuals produced by an individual during his reproductive life. It is possible to show in the population model with age structure and without immigration of McKendrick-Von Foerster, that population decay is equivalent to $R < 1$; and in the model with immigration it has been shown ([5]) that, if the native population is assumed decaing, the equilibrium solution (1.6) of equation (1.3) is globally asymptotically stable. Then it makes sense to consider as initial and boundary data of system (1.1) respectively the equilibrium solution of equation (1.3) of McKendrick-Von Foerster and the constant number of newborn individuals which we have in such equilibrium situation, given by (1.7); that is to say, let us consider the evolution of the disease in the population stabilized by the immigration stream. Then the system with boundary and initial conditions to study is:

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) X &= -(\lambda(a,t) + \mu(a))X + I_X(a) \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) Y &= \lambda(a,t)X - (\mu(a) + \gamma)Y + I_Y(a) \qquad 0 < a < \omega\,,\, t > 0 \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) Z &= \gamma Y - \mu(a)Z + I_Z(a) \\ X(0,t) = n(0)\;,\; Y(0,t) = 0\;,\; Z(0,t) = 0 \quad t \geq 0 \\ X(a,0) = X_0(a)\;,\; Y(a,0) = Y_0(a)\;,\; Z(a,0) = Z_0(a) \quad a \geq 0 \end{cases} \tag{1.8}$$

with $X_0(\cdot)\,,\, Y_0(\cdot)\,,\, Z_0(\cdot)$ such that $n(a) = X_0(a) + Y_0(a) + Z_0(a) \quad 0 \leq a \leq \omega$.

Chapter 2

Existence and uniqueness of solutions.

To establish existence and uniqueness of the solution of system (1.8) in a suitable functional class, let us normalize the variables X, Y, Z:

$$x(a,t) = \frac{X(a,t)}{n(a)}\ ,\ y(a,t) = \frac{Y(a,t)}{n(a)}\ ,\ z(a,t) = \frac{Z(a,t)}{n(a)}$$

and the variables pertinent to the migratory stream:

$$i_x(a) = \frac{I_X(a)}{n(a)}\ ,\ i_y(a) = \frac{I_Y(a)}{n(a)}\ ,\ i_z(a) = \frac{I_Z(a)}{n(a)}$$

that is we consider the susceptible, infective and removed fractions of age a at time t and the analogous fractions of susceptibles, infective and removed immigrants who arrive in the unit of time, over the total native population of age a.
By assumption, our analysis starts with initial condition given by the demographic equilibrium of system (1.3), so we have:

$$n(a,t) = n(a) = X(a,t) + Y(a,t) + Z(a,t) \quad 0 \leq a \leq \omega\ ,\ t \geq 0$$

and hence:

$$1 = x(a,t) + y(a,t) + z(a,t) \quad 0 \leq a \leq \omega\ \ t \leq 0 \tag{2.1}$$

Then we rewrite system (1.8) in this way:

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) x &= -(\lambda(a,t) + i(a))x + i_x(a) \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) y &= \lambda(a,t)x - (\gamma + i(a))y + i_y(a) \qquad 0 < a < \omega\,,\, t > 0 \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) z &= \gamma y - i(a)z + i_z(a) \\ x(0,t) = 1\ ,\ y(0,t) = 0\ ,\ z(0,t) = 0 \quad t \geq 0 \\ x(a,0) = x_0(a)\ ,\ y(a,0) = y_0(a)\ ,\ z(a,0) = z_0(a) \quad a \geq 0 \end{cases} \tag{2.2}$$

(the condition x(0,t)=1 means that there is no vertical transmission of the disease: the newborns are all susceptibles) where $i(a) = i_x(a) + i_y(a) + i_z(a)$ and:

$$\lambda(a,t) = \int_0^\omega \beta(a,s)n(s)y(s,t)\,ds\ .$$

In the first two equations of (2.2) the $z(\cdot,\cdot)$ variable doesn't appear, then by mean of (2.1), it suffices to consider the system:

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) x &= -(\lambda(a,t) + i(a))x + i_x(a) \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) y &= \lambda(a,t)x - (\gamma + i(a))y + i_y(a) \qquad 0 < a < \omega\,,\, t > 0 \\ x(0,t) = 1\ ,\ y(0,t) = 0 \quad t \geq 0 \\ x(a,0) = x_0(a)\ ,\ y(a,0) = y_0(a) \quad a \geq 0 \end{cases} \tag{2.3}$$

Let us now write system (2.3) as an abstract Cauchy problem. To make the problem treatable with the aid of semigroup theory, let us set $\tilde{x}(a,t) = 1 - x(a,t)$. The system 2.3 then become:

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) \tilde{x} &= (\lambda(a,t) + i(a))(1 - \tilde{x}) - i_x(a) \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) y &= \lambda(a,t)(1 - \tilde{x}) - (\gamma + i(a))y + i_y(a) \qquad 0 < a < \omega\,,\, t > 0 \\ \tilde{x}(0,t) = 0\ ,\ y(0,t) = 0 \quad t \geq 0 \\ \tilde{x}(a,0) = \tilde{x}_0(a)\ ,\ y(a,0) = y_0(a) \quad a \geq 0 \end{cases} \tag{2.4}$$

Let us suppose $i_x, i_y, i_z \in L^\infty(0,\omega)$; let us set $X = L^1\big((0,\omega);\mathbb{R}^2\big)$ and let $A : D(A) \subseteq X \longrightarrow X$ be the linear operator defined by:

$$A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(a) = \begin{pmatrix} -\frac{d}{da}u_1(a) - i(a)u_1(a) \\ -\frac{d}{da}u_2(a) - (\gamma + i(a))u_2(a) \end{pmatrix}$$

with domain:

$$D(A) = \{u \in X \,:\, u_1, u_2 \in AC([0,\omega]),\ u_1(0) = u_2(0) = 0\}\,.$$

Let $G : X \longrightarrow X$ be the nonlinear operator defined by:

$$G\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(a) = \begin{pmatrix} (Fu_2)(a)\cdot(1 - u_1(a)) + i_y(a) + i_z(a) \\ (Fu_2)(a)\cdot(1 - u_1(a)) + i_y(a) \end{pmatrix}$$

where $F : L^1(0,\omega) \longrightarrow L^1(0,\omega)$ is the linear operator defined by:

$$(Fv)(a) = \int_0^{\omega} \beta(a,s)n(s)v(s)ds\ ,\ v \in L^1(0,\omega).$$

Then we can write (2.4) as the abstract Cauchy problem:

$$\left\{ \begin{array}{rcl} \frac{d}{dt}u(t) & = & Au(t) + Gu(t)\ \ ,\ \ t > 0 \\ u(0) & = & u_0 \in X\ \ \ ,\ \ \ u_0 = \begin{pmatrix} \tilde{x}_0 \\ y_0 \end{pmatrix}. \end{array} \right. \tag{2.5}$$

A is the infinitesimal generator of the strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on X given by:

$$T(t)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(a) = \left\{ \begin{array}{ll} \begin{pmatrix} e^{-\int_0^t i(a-t+\tau)d\tau}\, u_1(a-t) \\ e^{-\gamma t - \int_0^t i(a-t+\tau)d\tau}\, u_2(a-t) \end{pmatrix} & \omega > a > t \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & a < t \end{array} \right.$$

with $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in X$. (We can write $T(t)$ in a more compact way as:

$$T(t) = \begin{pmatrix} e^{-\int_0^t T_r(t-\tau)i(\cdot)d\tau}\, T_r(t) \\ e^{-\gamma t - \int_0^t T_r(t-\tau)i(\cdot)d\tau}\, T_r(t) \end{pmatrix}$$

with $\{T_r(t)\}_{t\geq 0}$ the right-translation semigroup on $L^1(0,\omega)$ given by:

$$\big(T_r(t)f\big)(a) = \begin{cases} f(a-t) & a-t>0\,,\ a<\omega \\ 0 & a-t\leq 0 \end{cases}\quad , \qquad f\in L^1(0,\omega)\ \ ).$$

The nonlinear operator G is Frechet differentiable on X, with derivative given by:

$$\left(G'(u)\varphi\right)(a) = \left(\begin{array}{c} -(Fu_2)(a)\cdot\varphi_1(a) + (F\varphi_2)(a)\cdot(1-u_1(a)) \\ " \end{array} \right)$$

$u,\varphi \in X$, $a\in(0,\omega)$. Hence it is continuously Frechet differentiable on X; then:

1. for each $u_0 \in X$ there exists a maximal interval of existence $[\,0,t_0)$ and a unique continuous function $t \longrightarrow u(t;u_0)$ from $[\,0,t_0)$ to X that is mild solution of (2.5) defined by:

$$u(t;u_0) = T(t)u_0 + \int_0^\omega T(t-s)G(u(s;u_0))\,ds \qquad 0\leq t<t_0 \tag{2.6}$$

   and either $t_0=+\infty$ or $\limsup_{t\longrightarrow {t_0}^-}\|u(t;u_0)\|_X = +\infty$ (blow-up);

2. if $u_0\in D(A)$ then $u(t;u_0)\in D(A)\ \forall\, 0\leq t<t_0$ and the function (2.6) is continuosly differentiable and satisfies (2.5) on $[0,t_0)$. ([20],teor. 4.16, pag.194).

We now show that blow-up doesn't happen, that is $t_0=+\infty$; to do this let us assume $x_0\in L^\infty(0,\omega)$ (and it is reasonable to do it).

Lemma 1 *Let* $u(t;u_0) = \begin{pmatrix} \tilde{x}(t) \\ y(t) \end{pmatrix}$ *be the mild solution of (2.5). Then:*

(i) $u(t;u_0)\in[0,1]\times[0,1]$ *a.e. on* $(0,\omega)$, $\forall\, t\geq\omega$ *if* $\tilde{x}_0\leq 1$, $y_0\geq 0$ *a.e. on* $(0,\omega)$*;*

(ii) $u(t;u_0)\in[0,1]\times[0,1]$ *a.e. on* $(0,\omega)$, $\forall\, t\geq 0$ *if* $\tilde{x}_0$, $y_0\in[0,1]$ *a.e. on* $(0,\omega)$.

Proof. From (2.3) by integration along characteristic curves we obtain:

$$x(a,t) = \begin{cases} x_0(a-t)\, e^{-\int_0^t \bar\lambda(\tau+a-t,\tau)d\tau} + \int_0^t i_x(\tau+a-t) e^{-\int_\tau^t \bar\lambda(\tau'+a-t,\tau')d\tau'} d\tau & a>t \\ e^{-\int_0^a \bar\lambda(\tau,\tau+t-a)d\tau} + \int_0^a i_x(\tau) e^{-\int_\tau^a \bar\lambda(\tau',\tau'+t-a)d\tau'} d\tau & a<t \end{cases}$$

with $\bar\lambda(a,t) = \lambda(a,t) + i(a)$. If $a > t$ then from $0 \le i_x(a) \le i(a) \le \bar\lambda(a,t)$ we have:

$$x_0(a-t)\, e^{-\int_0^t \bar\lambda(\tau+a-t,\tau)d\tau} + \int_0^t i_x(\tau+a-t) e^{-\int_\tau^t \bar\lambda(\tau'+a-t,\tau')d\tau'} d\tau \le$$

$$\le x_0(a-t)\, e^{-\int_0^t \bar\lambda(\tau+a-t,\tau)d\tau} + \int_0^t \bar\lambda(\tau+a-t,\tau) e^{-\int_\tau^t \bar\lambda(\tau'+a-t,\tau')d\tau'} d\tau =$$

$$= x_0(a-t)\, e^{-\int_0^t \bar\lambda(\tau+a-t,\tau)d\tau} + 1 - e^{-\int_0^t \bar\lambda(\tau+a-t,\tau)d\tau} =$$

$$= 1 - \big(1 - x_0(a-t)\big) e^{-\int_0^t \bar\lambda(\tau+a-t,\tau)d\tau} \le 1\,;$$

and in the same way if $a < t$. Then $\tilde x(a,t) \le 1$ a.e. on $(0,\omega)$, for all $t \ge 0$ if $\tilde x_0(a) = 1 - x_0(a) \le 1$ a.e. on $(0,\omega)$; and if $0 \le \tilde x_0(a) \le 1$ a.e. on $(0,\omega)$, then $0 \le \tilde x(a,t) \le 1$ a.e. on $(0,\omega)$, for all $t \ge 0$.

Let us now show that if $y_0 \ge 0$ a.e. on $(0,\omega)$, then $y(t) \ge 0$ a.e. on $(0,\omega)$, for all $t \ge 0$. To establish this, let us write the second equation in (2.3) as the abstract Cauchy problem:

$$\begin{cases} \frac{d}{dt} y(t) &=& By(t) + \big(Fy(t)\big)(1-\tilde x(t)) + i_y \;\;,\;\; t>0 \\ y(0) &=& y_0 \end{cases} \tag{2.7}$$

with $B : L^1(0,\omega) \longrightarrow L^1(0,\omega)$ the linear operator defined by:

$$(Bf)(a) = -\frac{df}{da}(a) - \big(\gamma + i(a)\big) f(a)\;,\; 0<a<\omega$$

with domain:

$$D(B) = \{ f \in L^1(0,\omega) \;:\; f \in AC([0,\omega])\;,\; f(0)=0 \}$$

B is the generator of the positive strongly continuous semigroup of linear operators $\{S(t)\}_{t\ge0}$ given by:

$$S(t) = e^{-\gamma t - \int_0^t T_r(t-\tau) i(\cdot) d\tau} T_r(t) \, .$$

Then by the variation of parameters formula, $y(t)$ is given by:

$$y(t) = S(t) y_0 + \int_0^t S(t-s) \big((F y(s))(1 - \tilde{x}(s)) + i_y \big) \, ds.$$

Given $T > 0$, consider the space $Y_T = C\big([0,T]; L^1(0,\omega)\big)$ which is Banach with

the norm $\|\varphi\|_{Y_T} = \sup_{0 \le t \le T} \|\varphi(t)\|_1$; consider the closed subsets:

$$W_T = \big\{ u(\cdot) \in Y_T \, : \, u(0) = y_0 \big\} \, , \; Y_{T,+} = \big\{ u(\cdot) \in Y_T \, : \, u(t) \in L^1_+(0,\omega) \; \forall t \in [0,T] \big\}$$

and the positive operator:

$$V : Y_T \longrightarrow Y_T \; : \; (V\varphi)(t) = S(t)\varphi(0) + \int_0^t S(t-s) \big((F\varphi(s))(1 - \tilde{x}(s)) + i_y \big) \, ds.$$

We have $V\big(W_T \cap Y_{T,+}\big) \subseteq W_T \cap Y_{T,+}$ and V has in $W_T \cap Y_{T,+}$ 1! fixed point $y(\cdot)$ which is the mild solution of (2.7). Thus if we define the sequence:

$$\begin{cases} y_1(t) &= S(t) y_0 \\ y_{n+1}(t) &= (V y_n)(t) \, , \; n \ge 1 \end{cases}$$

with $y_0 \in L^1_+(0,\omega)$, we have $\{y_n\} \subset W_T \cap Y_{T,+}$ and $y = \lim_{n \to +\infty} y_n$ in Y_T for

all $T > 0$. Then $y(t) \ge 0$ a.e. on $(0,\omega)$, for all $t \ge 0$.

To show that eventually $u(t; u_0) \in [0,1] \times [0,1]$, let us define $v(t) = \tilde{x}(t) - y(t)$ and consider the abstract Cauchy problem:

$$\begin{cases} \frac{d}{dt} v(t) &= C y(t) + \gamma y(t) + i_z \; , \quad t > 0 \\ v(0) &= \tilde{x}_0 - y_0 \end{cases} \tag{2.8}$$

with $C : L^1(0,\omega) \longrightarrow L^1(0,\omega)$ the linear operator defined by:

$$(Cf)(a) = -\frac{df}{da}(a) - i(a) f(a) \; , \; 0 < a < \omega \;\; , \; D(C) = D(B).$$

C is the generator of the positive strongly continuous semigroup of linear operators $\{U(t)\}_{t \ge 0}$ given by:

$$U(t) = e^{-\int_0^t T_r(t-\tau) i(\cdot) d\tau} T_r(t) \,.$$

So we have:

$$v(t) = U(t)v(0) + \int_0^t U(t-s)\big(\gamma y(s) + i_z\big)\, ds \geq U(t)v(0) \;\;, \;\; \forall\, t \geq 0$$

and then:

$$v(t)(a) \geq U(t)v(0)(a) = \begin{cases} e^{-\int_0^t i(\tau+a-t)d\tau}\big(\tilde{x}_0(a-t) - y_0(a-t)\big) \;\;, \;\; a > t \\ 0 \qquad\qquad\qquad t > a \end{cases} \tag{2.9}$$

From (2.9) it follows $0 \leq \tilde{x}(t) - y(t)$ for all $t \geq \omega$; then if $\tilde{x}_0 \leq 1$, $0 \leq y_0$ a.e. on $(0, \omega)$, from $\tilde{x}(t) \leq 1$, $y(t) \geq 0$ a.e. on $(0, \omega)$ for all $t \geq 0$, it follows $0 \leq y(t) \leq \tilde{x}(t) \leq 1$ a.e. on $(0, \omega)$ for all $t \geq \omega$. This establishes (i).

If furthermore $\tilde{x}_0$, $y_0 \in [0, 1]$ a.e. on $(0, \omega)$ we get for $0 \leq t < a < \omega$:

$$0 \leq \tilde{x}_0(a-t) - y_0(a-t) = 1 - \big(x_0(a-t) + y_0(a-t)\big) \leq 1$$

and hence from (2.9) we obtain $\tilde{x}(t)(a) - y(t)(a) \geq 0$ a.e. on $(0, \omega)$ for all $t \geq 0$ and then $1 \geq \tilde{x}(t)(a) \geq y(t)(a) \geq 0$ a.e. on $(0, \omega)$ for all $t \geq 0$. And this establishes (ii).

q.e.d.

If we now define:

$$D = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in X \;:\; u_1(a) \leq 1 \,,\; u_2(a) \geq 0 \; for\; a.e.\; a \in (0, \omega) \right\}$$

then from the lemma it follows:

Proposition 1 *There exists one and only one classical solution of (2.3) for every initial data $u_0 \in D \cap D(A)$; this solution is maximal and defined for all $t \geq 0$.*

Chapter 3

Steady states.

A problem which makes sense to face in tring to understand the evolution of the infection in the population, is the analysis of the equilibrium solutions of system (2.3). Are there reasonable conditions under whose we have existence and uniqueness of such solution?
The question of uniqueness for this kind of problems is often studied in the setting of of the teory of positive monotone operators in partially ordered Banach spaces. Let us remark some basic theory.
Let X be a real Banach space. A subset C of X is called a cone if C is closed, convex, invariant under multiplication by elements of $\mathbb{R}_+ = [0, +\infty)$ and if $C \cap (-C) = \{0\}$. Each cone induces a partial ordering in X by definig $u \geq v$ if and only if $u - v \in C$. This ordering is antisymmetric, reflexive, transitive, compatible with the linear structure, that is $\alpha \in \mathbb{R}_+$ and $u \geq 0$ imply $\alpha u \geq 0$ and for every $w \in X$, $u \geq v$ implies $u + w \geq v + w$; and compatible with the topology, that is if $\{u_n\}_{n\in\mathbb{N}} \subset X$ is such that $u_n \geq 0$, $u_n \underset{n}{\longrightarrow} u$ then $u \geq 0$. On the other hand, let X be a real Banach space with an ordering $\leq$ which is compatible with the linear structure and the topology. Then the set $C = \{u \in X : u \geq 0\}$ is a cone on X. We shall write $u > 0$ if $u \in C \setminus \{0\} = \dot{C}$, and hence $u > v$ if $u - v \in \dot{C}$ (here $\dot{C}$ does not mean "the set of the inner points of C") and [u,v]= $\{w \in X : u \leq w \leq v\} = (u + C) \cap (v - C)\ \forall\, u, v \in X$ s.t. $u \leq v$ (order interval). Let X_1, X_2 be ordered Banach spaces with positive cones C_1, C_2. A linear operator $T : X_1 \longrightarrow X_2$ is called positive if $T \neq 0$ and $T(C_1) \subseteq C_2$; it is called strictly positive if $T(\dot{C_1}) \subseteq \dot{C_2}$; it is called strongly positive if $T(\dot{C_1}) \subseteq Int(C_2)$ (where we now suppose $Int(C_2) \neq \emptyset$).
A nonlinear mapping $A : D(A) \subseteq X_1 \longrightarrow X_2$ is called increasing if, for all $u, v \in D(A)$ with $u < v$, we have $Au \leq Av$; it is called strictly increasing if $Au < Av$; it is called e-increasing if there exists an $e \in \dot{C_2}$ such that for every $u, v \in D(A)$ with $u < v$ there exist constants $\alpha = \alpha(u, v)$, $\beta = \beta(u, v) > 0$ with $\alpha\, e \leq Av - Au \leq \beta\, e$; and, supposing that $Int(C_2) \neq \emptyset$, it is called strongly increasing if for every $u, v \in D(A)$ we have that $u < v$ implies $Av - Au \in Int(C_2)$.

3.1 Formulation of the problem.

We now introduce the problem of determinig and studing the equilibrium solutions of (2.3) and do some assumptions that will make us able to give some answer.
Equilibrium solutions of (2.3) are solution of the system of ordinary differential equations:

$$\begin{cases} \dfrac{d}{da}x^*(a) = -\big(\lambda^*(a)+i(a)\big)x^*(a)+i_x(a) & 0<a\le\omega \\ \dfrac{d}{da}y^*(a) = \lambda^*(a)\,x^*(a)-\big(\gamma+i(a)\big)\,y^*(a)+i_y(a) & \\ x^*(0)=1 & \\ y^*(0)=0 & \end{cases} \tag{3.1}$$

where $\lambda^*(a)=\int_0^\omega \beta(a,s)n(s)y^*(s)\,ds$; by solving (3.1) directly we obtain:

$$x^*(a)=e^{-\int_0^a(\lambda^*(s)+i(s))ds}+\int_0^a i_x(s)e^{-\int_\sigma^a(\lambda^*(s)+i(s))ds}d\sigma \tag{3.2}$$

$$y^*(a)=\int_0^a e^{-\gamma(a-\sigma)-\int_\sigma^a i(s)ds}\left(\lambda^*(\sigma)x^*(\sigma)+i_y(\sigma)\right)d\sigma \tag{3.2$'$}$$

Then we obtain for the force of infection $\lambda^*(\cdot)$:

$$\lambda^*(a)=\int_0^\omega \beta(a,\xi)n(\xi)y^*(\xi)\,d\xi=$$

$$=\int_0^\omega \beta(a,\xi)n(\xi)\int_0^\xi e^{-\gamma(\xi-\sigma)-\int_\sigma^\xi i(s)ds}\left(\lambda^*(\sigma)x^*(\sigma)+i_y(\sigma)\right)d\sigma\,d\xi=$$

$$=\int_0^\omega \left(\lambda^*(\sigma)x^*(\sigma)+i_y(\sigma)\right)\int_\sigma^\omega \beta(a,\xi)n(\xi)e^{-\gamma(\xi-\sigma)-\int_\sigma^\xi i(s)ds}d\xi\,d\sigma=$$

$$=\int_0^\omega \Big(\lambda^*(\sigma)e^{-\int_0^\sigma(\lambda^*(s)+i(s))ds}+\lambda^*(\sigma)\int_0^\sigma i_x(s)e^{-\int_s^\sigma(\lambda^*(\tau)+i(\tau))d\tau}\,ds+$$
$$+\,i_y(\sigma)\Big)\phi(a,\sigma)d\sigma$$

where $\phi(\cdot,\cdot)$ is given by:

$$\phi(a,\sigma) = \int_\sigma^\omega \beta(a,\xi)n(\xi)e^{-\gamma(\xi-\sigma)-\int_\sigma^\xi i(s)ds}d\xi \tag{3.3}$$

We now set:

$$L^1_+(0,\omega) = \left\{ f \in L^1(0,\omega) \; s.t. \; f \geq 0 \; q.o. \; on \; (0,\omega) \right\} , \tag{3.4}$$

as the cone of the nonnegative functions in the Banach space $L^1(0,\omega)$, and define the nonlinear operator $\Phi : L^1_+(0,\omega) \longrightarrow L^1_+(0,\omega)$ by setting:

$$(\Phi\psi)(a) \; = \; \int_0^\omega \Big(\psi(\sigma)e^{-\int_0^\sigma(\psi(s)+i(s))ds} + \psi(\sigma)\int_0^\sigma i_x(s)e^{-\int_s^\sigma(\psi(\tau)+i(\tau))d\tau}\,ds + \\ + i_y(\sigma)\Big)\,\phi(a,\sigma)d\sigma \; , \; a \in (0,\omega) \tag{3.5}$$

To each fixed point of Φ in the positive cone (3.4) there corresponds a force of infection $\lambda^*(\cdot)$ and hence an equilibrium solution of (2.3) and viceversa. That is, we transfer the problem of finding equilibruium solutions of (2.3) into that of finding solutions of a nonlinear operator equation.
To succeed in doing an analysis of the existence and uniqueness of a fixed point for Φ, we do now some assumptions on the contact coefficient between individuals $\beta(\cdot,\cdot)$ ([10]):

Assumption 1 *$\beta(\cdot,\cdot) \in L^\infty\big((0,\omega)\times(0,\omega)\big)$, $\beta(a,s) \geq 0$ for a.e. $a,s \in (0,\omega)$ and it is such that:*

$$\lim_{h\to 0}\int_0^\omega |\beta(a+h,s)-\beta(a,s)|da \; = \; 0$$

uniformly for $s \in \mathrm{R}$ (and $\beta(\cdot,\cdot)$ is extended by setting $\beta(a,s)=0$ for $a,s \in (-\infty,0)\cup(\omega,+\infty)$);

Assumption 2 *there exists $m>0$, $0<\alpha<\omega$ such that $\beta(a,s)\geq m$ for a.e. $(a,s) \in (0,\omega)\times(\omega-\alpha,\omega)$.*

Furthermore let us suppose that:

Assumption 3 *There exist $0 \leq a_1 < a_2 \leq \omega$ such that $i_y(a) > 0 \;\; \forall a \in (a_1,a_2)$.*

(this means we assume a continuous entry of infective immigrants in the population).

3.2 Existence of steady states.

We now show that under these assumptions, we have a positive (endemic) equilibrium. Let us recall the following definitions of compact and completely continuous operator:

Definition 1 *An operator $A : D(A) \subseteq X \longrightarrow X$ is said compact if it maps bounded sets into relatively compact sets; it is said completely continuous if it is compact and continuous.*

We now show that the operator defined by (3.5) is completely continuous.

Proposition 2 *Let assumptions 1-3, be valid then:*

(a) *there exists $D \subset L^1_+(0,\omega)$ such that $\Phi(D) \subseteq D$, with D closed, bounded and convex;*

(b) *Φ is completely continuous.*

It then follows that $\exists\, \psi \in D$ such that $\psi = \Phi\psi$.

Proof. a) Given $\psi \in L^1_+(0,\omega)$ we have:

$$\|\Phi\psi\|_{L^1} = \int_0^\omega \int_0^\omega \Big(\psi(\sigma) e^{-\int_0^\sigma (\psi(s)+i(s))ds} + \\ + \psi(\sigma) \int_0^\sigma i_x(s) e^{-\int_s^\sigma (\psi(\tau)+i(\tau))d\tau}\, ds \,+\, i_y(\sigma) \Big) \phi(a,\sigma) d\sigma\, da =$$

$$= \int_0^\omega \Big(\dots \Big) \int_0^\omega \phi(a,\sigma) da\, d\sigma < \omega^2 \|\beta\|_\infty \|n\|_\infty \int_0^\omega \Big(\dots \Big) d\sigma <$$

$\Big($last inequality follows from the definition (3.3) of $\phi(\cdot,\cdot)$:

$$\int_0^\omega \phi(a,\sigma)\, da = \int_0^\omega \int_\sigma^\omega \beta(a,s) n(s) e^{-\gamma(s-\sigma) - \int_\sigma^s i(\tau)d\tau} ds\, da \leq \\ \leq \omega^2 \|\beta\|_\infty \|n\|_\infty \;\; for\ a.e.\ \sigma \in (0,\omega) \;\Big)$$

$$< \omega^2 \|\beta\|_\infty \|n\|_\infty \int_0^\omega \Big(\psi(\sigma) e^{-\int_0^\sigma \psi(s)ds} + \psi(\sigma) \int_0^\sigma i_x(s) e^{-\int_s^\sigma \psi(\tau)d\tau}\, ds \,+\, i_y(\sigma) \Big) d\sigma <$$

$$< \omega^2 \|\beta\|_\infty \|n\|_\infty \left(1 + \omega \left(\|i_x\|_\infty + \|i_y\|_\infty\right)\right) \;\; for\ a.e.\ \ a \in (0,\omega),\ \forall \psi \in L^1_+(0,\omega)$$

$\Big($for we have:

$$\int_0^\omega \psi(\sigma) e^{-\int_0^\sigma \psi(s)ds}\, d\sigma = 1 - e^{-\int_0^\omega \psi(s)ds} = 1 - e^{-\|\psi\|_1} < 1 \quad \forall \psi \in L^1_+(0,\omega)$$

$$\begin{aligned} \int_0^\omega \psi(\sigma) \int_0^\sigma e^{-\int_s^\sigma \psi(\tau)d\tau}\, ds\, d\sigma &= \int_0^\omega \int_s^\omega \psi(\sigma) e^{-\int_s^\sigma \psi(\tau)d\tau}\, d\sigma ds = \\ &= \int_0^\omega \Big(1 - e^{-\int_s^\omega \psi(\tau)d\tau}\Big)\, ds < \omega \quad \forall \psi \in L^1_+(0,\omega) \quad \Big). \end{aligned}$$

Hence if we set:

$$R = \omega^2 \|\beta\|_\infty \|n\|_\infty \; (1 + \omega\,(\|i_x\|_\infty + \|i_y\|_\infty))$$

it obtains:

$$\Phi(L^1_+(0,\omega)) \subset B_R(0) \cap L^1_+(0,\omega)$$

(we obviously have $\Phi(L^1_+(0,\omega)) \subset L^1_+(0,\omega)$ too), where $B_R(0) = \big\{\psi \in L^1(0,\omega) \;:\; \|\psi\|_1 < 1\big\}$ is the open ball of $L^1(0,\omega)$ of radius R and centered in zero.

$\Big($let us observe that indeed we have $\Phi(L^1_+(0,\omega)) \subset L^\infty_+(0,\omega)$, the cone of nonnegative functions of $L^\infty(0,\omega)$:

$$|\Phi\psi(a)| \le \|\phi\|_\infty \int_0^\omega \big(\psi(\sigma) e^{-\int_0^\sigma \psi(s)ds} + \int_0^\sigma i_x(s) e^{-\int_s^\sigma \psi(\tau)d\tau}\, ds + i_y(\sigma)\big) d\sigma \le$$

$$\le \omega \|\beta\|_\infty \|n\|_\infty \,(1 + \omega\,(\|i_x\|_\infty + \|i_y\|_\infty)) \quad for \; a.e. \; a \in (0,\omega), \; \forall \psi \in L^1_+(0,\omega)\Big)$$

Furthermore if we set:

$$u_0(a) = \big(\Phi\,(0)\big)(a) = \int_0^\omega i_y(\sigma) \phi(a,\sigma) d\sigma \tag{3.6}$$

it obtains:

$$\Phi\psi \ge u_0 \quad \forall \psi \in L^1_+(0,\omega)$$

because of the definition of Φ.
From assumptions 2 and 3 it follows that $u_0 \in L^1_+(0,\omega) \setminus \{0\}$, or more, $u_0 > 0$ in $(0,\omega)$. In fact:

$$\phi(a,\sigma) = \int_\sigma^\omega \beta(a,\xi) n(\xi) e^{-\gamma(\xi-\sigma)-\int_\sigma^\xi i(s)ds} d\xi \geq$$

$$\geq m \int_{max\{\sigma,\,\omega-\alpha\}}^\omega n(\xi) e^{-\gamma(\xi-\sigma)-\int_\sigma^\xi i(s)ds} d\xi > 0 \quad for\ a.e.\ (a,\sigma) \in (0,\omega)\times(0,\omega)$$

and hence:

$$\begin{aligned} u_0(a) = &\int_0^\omega i_y(\sigma)\phi(a,\sigma)d\sigma \geq \\ &\geq m \int_{a_1}^{a_2} i_y(\sigma) \int_{max\{\sigma,\,\omega-\alpha\}}^\omega \dots d\xi\, d\sigma > 0 \quad for\ a.e.\ a \in (0,\omega). \end{aligned}$$

Then if we set:

$$D = \big\{\psi \in L^1_+(0,\omega)\ :\ \|\psi\|_1 \leq R\big\} \cap \big\{\psi \in L^1_+(0,\omega)\ :\ \psi \geq u_0\big\} \tag{3.7}$$

we have that D is a bounded, convex and closed subset of $L^1_+(0,\omega)$ and that $\Phi(D) \subset D$.

b) *Φ is continuous*; in fact we have:

$$\begin{aligned} \|\Phi\psi_2 - \Phi\psi_1\|_{L^1} \leq &\int_0^\omega \int_0^\omega \left|\psi_2(\sigma) e^{-\int_0^\sigma (\psi_2+i)d\tau} - \psi_1(\sigma) e^{-\int_0^\sigma (\psi_1+i)d\tau}\right| \phi(a,\sigma)\, d\sigma\, da + \\ &+ \int_0^\omega \int_0^\omega \Big|\psi_2(\sigma) \int_0^\sigma i_x(s) e^{-\int_s^\sigma (\psi_2+i)d\tau} ds - \psi_1(\sigma) \int_0^\sigma i_x(s) \cdot \\ &\qquad \cdot e^{-\int_s^\sigma (\psi_1+i)d\tau} ds\Big|\, \phi(a,\sigma)\, d\sigma\, da \leq \end{aligned}$$

$$\begin{aligned} \leq C\Big\{ &\int_0^\omega \left|\psi_2(\sigma) e^{-\int_0^\sigma \psi_2 d\tau} - \psi_1(\sigma) e^{-\int_0^\sigma \psi_1 d\tau}\right| d\sigma + \\ &+ \int_0^\omega \left|\psi_2(\sigma) \int_0^\sigma i_x(s) e^{-\int_s^\sigma \psi_2 d\tau} ds - \psi_1(\sigma) \int_0^\sigma i_x(s) e^{-\int_s^\sigma \psi_1 d\tau} ds\right| d\sigma \Big\} \end{aligned}$$

where $C = \omega\,\|\phi\|_\infty$. For the first term between the brackets we have:

$$\int_0^\omega \big|\psi_2(\sigma)e^{-\int_0^\sigma \psi_2 d\tau} - \psi_1(\sigma)e^{-\int_0^\sigma \psi_1 d\tau}\big|\, d\sigma \leq$$

$$\leq \int_0^\omega \big|\psi_2(\sigma) - \psi_1(\sigma)\big| e^{-\int_0^\sigma \psi_2 d\tau}\, d\sigma + \int_0^\omega \psi_1(\sigma)\big|e^{-\int_0^\sigma \psi_2 d\tau} - e^{-\int_0^\sigma \psi_1 d\tau}\big|\, d\sigma \leq$$

$$\leq \int_0^\omega \big|\psi_2(\sigma) - \psi_1(\sigma)\big|\, d\sigma + \int_0^\omega \psi_1(\sigma) \int_0^\omega \big|\psi_2(\tau) - \psi_1(\tau)\big|\, d\tau\, d\sigma \ =$$

$$= \|\psi_2 - \psi_1\|_{L^1} \ + \int_0^\omega \big|\psi_1(\tau) - \psi_2(\tau)\big| \int_\tau^\omega \psi_1(\sigma)\, d\sigma\, d\tau \ \leq$$

$$\leq \|\psi_2 - \psi_1\|_{L^1} \ + \|\psi_1\|_{L^1}\|\psi_2 - \psi_1\|_{L^1} \ = (1 + \|\psi_1\|_{L^1})\ \|\psi_2 - \psi_1\|_{L^1}$$

$$\forall\, \psi_1, \psi_2 \in L^1_+(0,\omega)$$

$\Big($ in fact we have:

$$\big|e^{-\int_0^\sigma \psi_2 d\tau} - e^{-\int_0^\sigma \psi_1 d\tau}\big| = e^{\xi}\Big|\int_0^\sigma \big(\psi_2(\tau) - \psi_1(\tau)\big) d\tau\Big| \leq \int_0^\sigma \big|\psi_2(\tau) - \psi_1(\tau)\big| d\tau \leq$$

$$\leq \int_0^\omega \big|\psi_2(\tau) - \psi_1(\tau)\big| d\tau = \|\psi_2 - \psi_1\|_1 \ \ \forall\, \psi_1, \psi_2 \in L^1_+(0,\omega) \quad \Big)$$

and in a like manner we find for the second term:

$$\int_0^\omega \big|\psi_2(\sigma)\int_0^\sigma \cdots ds\,\big|\, d\sigma \ \leq \omega \|i_x\|_{L^\infty}\,(1 + \|\psi_1\|_{L^1})\ \|\psi_2 - \psi_1\|_{L^1} \ \ \forall\, \psi_1, \psi_2 \in L^1_+(0,\omega)$$

Then:

$$\|\Phi\psi_2 - \Phi\psi_1\|_1 \leq C\omega \|i_x\|_\infty\,(1 + \|\psi_1\|_1)\ \|\psi_2 - \psi_1\|_1 \quad \forall\, \psi_1, \psi_2 \in L^1_+(0,\omega)$$

hence Φ is continuous (or rather, Φ is locally Lipschitz on $L^1_+(0,\omega)$ and Lipschitz on D).

Φ is compact; that is it maps bounded sets into relatively compact sets.

Let us set $\Phi = \Phi_1 + \Phi_2 + u_0$ with $\Phi_1, \Phi_2 : L^1_+(0,\omega) \longrightarrow L^1_+(0,\omega)$ defined in this way:

$$\begin{aligned}
(\Phi_1\,\psi)\,(a) &= \int_0^\omega \psi(\sigma) e^{-\int_0^\sigma (\psi+i)d\tau}\,\phi(a,\sigma)\,d\sigma \\
(\Phi_2\,\psi)\,(a) &= \int_0^\omega \psi(\sigma)\int_0^\sigma i_x(s) e^{-\int_s^\sigma (\psi+i)d\tau} ds\,\phi(a,\sigma)\,d\sigma
\end{aligned} \qquad \forall a \in (0,\omega)$$

and with u_0 defined as in (3.6). Let us define the linear operator:

$$T : L^1(0,\omega) \longrightarrow L^1(0,\omega) \ \mid\ (T\psi)(a) = \int_0^\omega \psi(\sigma)\,\phi(a,\sigma)\,d\sigma \quad a \in (0,\omega)$$

Remark. We have $T = \Phi_1'(0)$, the Frechet derivative of Φ in the zero of $L^1(0,\omega)$.

T is a linear completely continuous operator.
T is continuous because of the boundedness of $\phi(\cdot,\cdot)$. Compactness of T follows from assumption (1) and from the Riesz-Frechet-Kolmogorov theorem ([4], teor.4.26, p.116), which we now recall:

Theorem 1 *(Riesz-Frechet-Kolmogorov) Let $\mathcal{F} \subset L^p(\mathbb{R}^n)$, $\mathcal{F}$ be bounded, $1 \le p < +\infty$; let us suppose that*

$$\lim_{|h|\to 0} \|\tau_h\, f - f\|_p = 0 \quad uniformly\ for\ f \in \mathcal{F}, \qquad \text{(R-F-K)}$$

where $\big(\tau_h\, f\big)(x) = f(x+h)$, $h \in \mathbb{R}^n$ $\Big($ that is $\forall\,\epsilon > 0\ \ \exists\,\delta > 0$ s.t. $\|\tau_h f - f\|_p < \epsilon \quad \forall h \in \mathbb{R}^n$ with $|h| < \delta$, $\forall f \in \mathcal{F}$ $\Big)$.

Then $\mathcal{F}\,|_\Omega$ has compact closure in $L^p(\Omega)$, for each $\Omega \subset \mathbb{R}^n$, Ω measurable and bounded.

($\mathcal{F}\,|_\Omega$ is the restriction on Ω of the functions of $\mathcal{F}$.)

Given $\psi \in L^1_+(0,\omega)$, we have:

$$\|\tau_h\,(T\psi) - T\psi\|_1 \le \int_0^\omega \psi(\sigma)\int_0^\omega \mid \phi(a+h,\sigma) - \phi(a,\sigma)\mid\ da\,d\sigma$$

and by assumption 1

$$\int_0^\omega |\phi(a+h,\sigma)-\phi(a,\sigma)\,|\,\,da\,\le$$

$$\le \int_0^\omega \int_\sigma^\omega \,|\,\beta(a+h,\xi)-\beta(a,\xi)\,|\,n(\xi)\,e^{-\gamma(\xi-\sigma)-\int_\sigma^\xi i(s)ds} d\xi\,da \le$$

$$\le \int_\sigma^\omega n(\xi) \int_0^\omega \,|\,\beta(a+h,\xi)-\beta(a,\xi)\,|\,da\,d\xi \le$$

$$\le \omega\|n\|_\infty \left(\underset{\xi\in\mathbb{R}}{sup} \int_0^\omega \,|\,\beta(a+h,\xi)-\beta(a,\xi)\,|\,\,da \right)$$

and hence:

$$\begin{aligned} &\|\tau_h\,(T\psi)-T\psi\|_1 \le \\ &\le \omega\|n\|_\infty \left(\underset{\xi\in\mathbb{R}}{sup} \int_0^\omega \,|\,\beta(a+h,\xi)-\beta(a,\xi)\,|\,\,da \right) \int_0^\omega \psi(\sigma)d\sigma = \end{aligned} \tag{3.7}$$

$$= \omega\|n\|_\infty\|\psi\|_1 \left(\underset{\xi\in\mathbb{R}}{sup} \int_0^\omega \,|\,\beta(a+h,\xi)-\beta(a,\xi)\,|\,\,da \right) \quad \forall\psi\in L^1_+(0,\omega)$$

If B is a bounded subset of $L^1_+(0,\omega)$, we can think of it as a subset of $L^1(\mathbb{R})$ by setting f(x)=0 $\forall x\in\mathbb{R}\setminus(0,\omega)$, $\forall f\in B$. Because of (3.7), condition (R-F-K) of Riesz-Frechet-Kolmogorov is satisfied from $T(B)$, for each $B\subset L^1(0,\omega)$, B bounded.

Hence the closure $\overline{T(B)}$ of $T(B)$ is compact in $L^1(0,\omega)$ for each $B\subset L^1(0,\omega)$, B bounded; that is, T is a completely continuous operator.

Given $\psi\in L^1_+(0,\omega)\setminus\{0\}$ we have:

$$(\Phi_1\,\psi)\,(a) = \int_0^\omega \psi(\sigma)e^{-\int_0^\sigma(\psi+i)d\tau}\,\phi(a,\sigma)\,d\sigma = \big(T\varphi\big)(a) \quad a\in(0,\omega)$$

where $\varphi(a)=\psi(a)e^{-\int_0^\omega(\psi+i)d\tau}$, $\|\varphi\|_1=\int_0^\omega\varphi(a)\,da<\int_0^\omega\psi(a)\,da=\|\psi\|_1$. So, given B bounded subset of $L^1_+(0,\omega)$ and $r>0$ such that $B\subseteq B_r(0)\cap L^1_+(0,\omega)$, it follows that:

$$\Phi_1(B)\subseteq\Phi_1\left(B_r(0)\cap L^1_+(0,\omega)\right)\subseteq T\left(B_r(0)\cap L^1_+(0,\omega)\right)$$

and $\overline{\Phi_1(B)} \subseteq \overline{T\big(B_r(0) \cap L^1_+(0,\omega)\big)}$, which is compact because T is compltetely continuous.

Hence $\overline{\Phi_1(B)}$ is compact and Φ_1 is a compact (nonlinear) operator. In the same way we can show that Φ_2 is compact.

Then from $\big(\Phi_1 + \Phi_2\big)(B) \subseteq \Phi_1(B) + \Phi_2(B)$ it follows:

$$\overline{\big(\Phi_1 + \Phi_2\big)(B)} \subseteq \overline{\Phi_1(B) + \Phi_2(B)} \subseteq \overline{\overline{\Phi_1(B)} + \overline{\Phi_2(B)}} = \overline{\Phi_1(B)} + \overline{\Phi_2(B)}$$

which is compact; hence $\overline{\big(\Phi_1 + \Phi_2\big)(B)}$ is compact and $\Phi_1 + \Phi_2$ is a compact operator and then $\Phi_1 + \Phi_2 + u_0$ is a compact operator. Hence $\Phi = \Phi_1 + \Phi_2 + u_0$ is a completely continuous operator.

q.e.d.

Let us now recall the Schauder's principle ([21]):

Theorem 2 *(Schauder) If a completely continuous operator $A : X \longrightarrow X$ acting on a Banach space X, transforms a bounded, convex and closed set $B \subset X$ into itself, then the operator A has at least one fixed point on B.*

From the preceding result and the Schauder's principle it follows:

Theorem 3 *Under assumptions 1-3, the nonlinear operator Φ defined by (3.5) has at least one fixed point on the set*

$D = \big\{\psi \in L^1_+(0,\omega) \,:\, \|\psi\|_1 \le R\big\} \cap \big\{\psi \in L^1_+(0,\omega) \,:\, \psi \ge u_0\big\}$.

This fixed point is strictly positive (that is it is in $L^1_+(0,\omega)\backslash\{0\}$) and it corresponds to an equilibrium solution of (2.3) given by the (3.2), (3.2′). Hence, under assumptions 1-3 between which there is a constant inflow of infective immigrants, we ever have an equilibrium solution for which we have $\int_0^\omega y^*(s)ds > 0$. If the equilibrium is unique and globally asymptotically stable, this enables us to assert that the disease persists in the population.
Then, in order to be able in doing previsions about the evolution of the infection in the population, it raises another question: do we have uniqueness for an equilibrium solution of (2.3)?

3.3 The question of uniqueness.

A way to face the problem that has shown to be effective ([10]) is to recover to the concept of monotonicity for sublinear operators as done by Amann in

[1]. Let X be an ordered Banach space. A mapping $A : [v, w] \longrightarrow X$ is called sublinear with respect to $[v, w]$ if we have:

$$A\big(v + \tau(u - v)\big) - \big(v + \tau(Au - v)\big) \geq 0$$

for every $u \in [v, w]$ and every $\tau \in [0, 1]$.
A is called strictly sublinear if it holds the strict inequality sign for $u \in (v, w] \equiv [v, w] \setminus \{v\}$ and $\tau \in (0, 1)$.
A is called e-sublinear if there exists an $e \in X$ with $e > 0$ such that for every $u \in (v, w]$ and every $\tau \in (0, 1)$, there exists a $\delta = \delta(u, \tau) > 0$ such that

$$A\big(v + \tau(u - v)\big) - \big(v + \tau(Au - v)\big) \geq \delta\, e$$

Let us recall the following result concernig fixed points of sublinear and increasing maps ([1]):

Theorem 4 *Let X be an ordered Banach space. Let us set* $[v, \infty) = \big\{u \in X \,:\, u \geq v\big\}$ *and suppose* $A : [v, \infty) \longrightarrow X$ *is e-sublinear and e-increasing, and suppose thre exists a constant* $\gamma > 0$ *such that* $0 < Av - v < \gamma\, e$. *Then A has at most one fixed point in* $(v, \infty) = \big\{u \in X \,:\, u > v\big\}$

Proof. Let $u_1, u_2 \in (v, \infty)$ be fixed points of A and we may suppose that $u_1 \not\leq u_2$. Hence $u_1 - v \not\leq u_2 - v$ and consider the set $\big\{t > 0 \,:\, t(u_1 - v) \leq u_2 - v\big\}$. We have:

$$u_i - v = Au_i - v = Au_i - Av + Av - v \leq (\beta_i + \gamma)e \quad i = 1, 2$$

and

$$u_i - v = Au_i - Av + Av - v \geq \alpha_i\, e + Av - v \geq \alpha_i e \quad i = 1, 2$$

hold, from which we obtain:

$$u_1 - v \leq (\beta_1 + \gamma)e \leq \frac{\beta_1 + \gamma}{\alpha_2}(u_2 - v)\,.$$

Hence $\big\{t > 0 \,:\, t(u_1 - v) \leq u_2 - v\big\} \neq \emptyset$ and we set

$\tau = sup\big\{t > 0 \,:\, t(u_1 - v) \leq u_2 - v\big\}$. We have $\tau > 0$; and also $\tau < 1$. For otherwise there exists $t \geq 1$ such that $u_2 - v \geq t(u_1 - v) = (t-1)(u_1 - v) + u_1 - v \geq u_1 - v$ but this implies $u_1 \leq u_2$. Then $0 < \tau < 1$.
From the fact that $u_2 - v \geq \tau(u_1 - v)$, A is e-increasing and e-sublinear we obtain:

$$u_2 = Au_2 \geq \overline{\alpha}_2 e + A(v+\tau(u_1-v)) \geq \overline{\alpha}_2 e + \delta_1 e + v + \tau(Au_1 - v) =$$

$$= \big(\overline{\alpha}_2 + \delta_1\big)e + v + \tau(u_1 - v) \geq v + \left(\frac{\overline{\alpha}_2+\delta_1}{\beta_1+\gamma} + \tau\right)(u_1 - v)$$

with $\frac{\overline{\alpha}_2+\delta_1}{\beta_1+\gamma} > 0$, which contradicts the maximality of τ.

Otherwise, if $u_1 < u_2$, we set
$\tau = sup\big\{t > 0 \, : \, \tau(u_2 - v) \leq u_1 - v\big\} < 1$ and arrive at a contradiction in a similar way.

q.e.d.

We now show that in fact $\Phi : [u_0, +\infty) \longrightarrow [u_0, +\infty)$ is an e-sublinear operator and do an assumption under which we will show that Φ is e-increasing.

Proposition 3 *There exists $e \in L^1_+(0,\omega) \setminus \{0\}$ such that $\Phi : [u_0, +\infty) \longrightarrow [u_0, +\infty)$ is e-sublinear.*

Proof. We have to show that there exists $e \in L^1_+(0,\omega) \setminus \{0\}$ such that for every $\psi > u_0$, for every $\tau \in (0,1)$ there exists $\delta = \delta(\psi, \tau) > 0$ such that

$$\Phi\big(u_0 + \tau(\psi - u_0)\big) - \big(u_0 + \tau(\Phi\psi - u_0)\big) \geq \delta\, e.$$

Now given $\psi > u_0$, we have:

$$\Phi\big(u_0 + \tau(\psi - u_0)\big)(a) - \big(u_0 + \tau(\Phi\psi - u_0)\big)(a) =$$

$$\int_0^\omega \bigg[\big(u_0(\sigma) + \tau(\psi(\sigma) - u_0(\sigma))\big)e^{-\int_0^\sigma [u_0+\tau(\psi-u_0)+i]d\eta} + \big(u_0(\sigma) + \tau(\psi(\sigma) - u_0(\sigma))\big)$$

$$\cdot \int_0^\sigma i_x(s)e^{-\int_s^\sigma [u_0+\tau(\psi-u_0)+i]d\eta}ds + i_y(\sigma) - \tau\psi(\sigma)e^{-\int_0^\sigma(\psi+i)d\eta} -$$

$$-\,\tau\psi(\sigma)\int_0^\sigma i_x(s)e^{-\int_s^\sigma(\psi+i)d\eta}ds - \tau i_y(\sigma)\bigg]\phi(a,\sigma)d\sigma - (1-\tau)u_0(a) =$$

$$= (1-\tau)\int_0^\omega \bigg[u_0(\sigma)e^{-\int_0^\sigma [u_0+\tau(\psi-u_0)+i]d\eta} + u_0(\sigma)\int_0^\sigma i_x(s)e^{-\int_s^\sigma [u_0+\tau(\psi-u_0)+i]d\eta}\,ds\bigg]\cdot$$

$$\cdot\,\phi(a,\sigma)d\sigma +$$

$$+\tau\int_0^\omega \bigg[\psi(\sigma)e^{-\int_0^\sigma(\psi+i)d\eta}\Big(e^{(1-\tau)\int_0^\sigma(\psi-u_0)d\eta} - 1\Big) + \psi(\sigma)\int_0^\sigma i_x(s)e^{-\int_s^\sigma(\psi+i)d\eta}\cdot$$

$$\cdot\Big(e^{(1-\tau)\int_0^\sigma(\psi-u_0)d\eta} - 1\Big)\,ds\bigg]\phi(a,\sigma)\,d\sigma \geq$$

for we have $e^{-\int_0^\sigma [u_0+\tau(\psi-u_0)+i]d\eta} - e^{-\int_0^\sigma (\psi+i)d\eta} =$

$$= e^{-\int_0^\sigma (\psi+i)d\eta} \left(e^{(1-\tau)\int_0^\sigma (\psi-u_0)d\eta} - 1 \right) \geq 0 \quad \forall \sigma \in (0,\omega).$$

Because of assumption (2) we have:

$$\phi(a,s) = \int_s^\omega \beta(a,\xi) n(\xi) e^{-\gamma(\xi-s)-\int_s^\xi i(\eta)d\eta} d\xi \geq$$

$$\geq m \int_{max\{s,\omega-\alpha\}}^\omega n(\xi) e^{-\gamma(\xi-s)-\int_s^\xi i(\eta)d\eta} d\xi \overset{def}{=} g(s) > 0$$

for a.e. $(a,s) \in (0,\omega) \times (0,\omega)$ and $g(s) > 0 \ \forall s \in (0,\omega)$.
Therefore:

$$\Phi\big(u_0 + \tau(\psi - u_0)\big)(a) - \big(u_0 + \tau(\Phi\psi - u_0)\big)(a) \geq$$

$$\geq (1-\tau) \int_0^\omega \Big[u_0(\sigma) \dots \Big] g(\sigma) d\sigma + \tau \int_0^\omega \Big[\psi(\sigma) e^{-\int_0^\sigma (\psi+i)d\eta} \dots \Big] g(\sigma) d\sigma = \delta\, e(a)$$

for a.e. $a \in (0,\omega)$ where $e(a) = 1 \quad a \in (0,\omega)$ and $\delta = \delta(\psi,\tau)$ is the constant defined by:

$$\delta(\psi,\tau) = (1-\tau) \int_0^\omega \Big[u_0(\sigma) e^{-\int_0^\sigma [u_0+\tau(\psi-u_0)+i]d\eta} +$$

$$u_0(\sigma) \int_0^\sigma i_x(s) e^{-\int_s^\sigma [u_0+\tau(\psi-u_0)+i]d\eta}\, ds \Big] g(\sigma) d\sigma +$$

$$+\tau \int_0^\omega \Big[\psi(\sigma) e^{-\int_0^\sigma (\psi+i)d\eta} \left(e^{(1-\tau)\int_0^\sigma (\psi-u_0)d\eta} - 1 \right) + \psi(\sigma) \int_0^\sigma i_x(s) e^{-\int_s^\sigma (\psi+i)d\eta} \cdot$$

$$\cdot \left(e^{(1-\tau)\int_0^\sigma (\psi-u_0)d\eta} - 1 \right) ds \Big] g(\sigma)\, d\sigma > 0.$$

That is, Φ is e-sublinear with respect to $[u_0, +\infty)$.

q.e.d.

Following Inaba ([10]), we do the assumption on the kernel $\phi(\cdot,\cdot)$:

Assumption 4 *There exists $\epsilon_1 > 0$ such that it holds the following inequality:*

$$-\frac{d\,\phi}{ds}(a,s) = \beta(a,s)n(s) - \gamma\phi(a,s) \geq \epsilon_1 \quad for\ a.e.\ (a,s) \in [0,\omega]\times[0,\omega] \quad (3.8)$$

Proposition 4 *Under assumption 4 we have that the operator Φ is e-increasing.*

Proof. Let $\psi_1, \psi_2 \in L^1_+(0,\omega)$ with $\psi_1 < \psi_2$; then we have:

$$(\Phi\psi_1)(a) - (\Phi\psi_2)(a) = \int_0^\omega \Big(\psi_2(\sigma)e^{-\int_0^\sigma(\psi_2+i)d\eta} - \psi_1(\sigma)e^{-\int_0^\sigma(\psi_1+i)d\eta} +$$

$$+\psi_2(\sigma)\int_0^\sigma i_x(s)e^{-\int_s^\sigma(\psi_2+i)d\eta}ds - \psi_1(\sigma)\int_0^\sigma i_x(s)e^{-\int_s^\sigma(\psi_1+i)d\eta}ds\Big)\phi(a,\sigma)d\sigma =$$

$$= \int_0^\omega \Big(\psi_2(\sigma)e^{-\int_0^\sigma \psi_2(\eta)d\eta} - \psi_1(\sigma)e^{-\int_0^\sigma \psi_1(\eta)d\eta}\Big)e^{-\int_0^\sigma i(\eta)d\eta}\phi(a,\sigma)d\sigma +$$

$$+\int_0^\omega i_x(s)\int_s^\omega \Big(\psi_2(\sigma)e^{-\int_s^\sigma \psi_2(\eta)d\eta} - \psi_1(\sigma)e^{-\int_s^\sigma \psi_1(\eta)d\eta}\Big)e^{-\int_s^\sigma i(\eta)d\eta}\phi(a,\sigma)d\sigma\,ds =$$

$$= -\int_0^\omega \frac{d}{d\sigma}\big(e^{-\int_0^\sigma \psi_2(\eta)d\eta} - e^{-\int_0^\sigma \psi_1(\eta)d\eta}\big)\tilde{\phi}(a,\sigma)\,d\sigma -$$

$$-\int_0^\omega i_x(s)\int_s^\omega \frac{d}{d\sigma}\big(e^{-\int_s^\sigma \psi_2(\eta)d\eta} - e^{-\int_s^\sigma \psi_1(\eta)d\eta}\big)\tilde{\phi}_s(a,\sigma)d\sigma\,ds$$

where we have set:

$$\tilde{\phi}(a,\sigma) = e^{-\int_0^\sigma i(\eta)d\eta}\,\phi(a,\sigma) = e^{-\int_0^\sigma i(\eta)d\eta}\int_\sigma^\omega \beta(a,\xi)n(\xi)e^{-\gamma(\xi-\sigma)-\int_\sigma^\xi i(\eta)d\eta}d\xi =$$

$$= \int_\sigma^\omega \beta(a,\xi)n(\xi)e^{-\gamma(\xi-\sigma)-\int_0^\xi i(\eta)d\eta}d\xi$$

$$\tilde{\phi}_s(a,\sigma) = e^{-\int_s^\sigma i(\eta)d\eta}\,\phi(a,\sigma) = \int_\sigma^\omega \beta(a,\xi)n(\xi)e^{-\gamma(\xi-\sigma)-\int_s^\xi i(\eta)d\eta}d\xi$$

Hence we have:

$$(\Phi\psi_1)(a) - (\Phi\psi_2)(a) = -\left(e^{-\int_0^\sigma \psi_2(\eta)d\eta} - e^{-\int_0^\sigma \psi_1(\eta)d\eta}\right)\tilde{\phi}(a,\sigma)\Big|_0^\omega +$$

$$+\int_0^\omega \left(e^{-\int_0^\sigma \psi_2(\eta)d\eta} - e^{-\int_0^\sigma \psi_1(\eta)d\eta}\right)\frac{d\,\tilde{\phi}}{d\sigma}(a,\sigma)\,d\sigma +$$

$$+\int_0^\omega i_x(s)\Big\{\Big((e^{-\int_0^\sigma \psi_2(\eta)d\eta} - e^{-\int_0^\sigma \psi_1(\eta)d\eta}\Big)\tilde{\phi}_s(a,\sigma)\Big|_s^\omega +$$

$$+\int_s^\omega \left(e^{-\int_s^\sigma \psi_2(\eta)d\eta} - e^{-\int_s^\sigma \psi_1(\eta)d\eta}\right)\frac{d\,\tilde{\phi}_s}{d\sigma}(a,\sigma)d\sigma\Big\}ds =$$

$$= \int_0^\omega \left(e^{-\int_0^\sigma \psi_2(\eta)d\eta} - e^{-\int_0^\sigma \psi_1(\eta)d\eta}\right)\frac{d\,\tilde{\phi}}{d\sigma}(a,\sigma)\,d\sigma +$$

$$+\int_0^\omega i_x(s)\int_s^\omega \left(e^{-\int_s^\sigma \psi_2(\eta)d\eta} - e^{-\int_s^\sigma \psi_1(\eta)d\eta}\right)\frac{d\,\tilde{\phi}_s}{d\sigma}(a,\sigma)d\sigma\,ds > 0$$

for a.e. $a \in (0,\omega)$ (because by assumption 4 we have

$$\frac{d\,\tilde{\phi}}{d\sigma}(a,\sigma) = e^{-\int_0^\sigma i(\eta)d\eta}\frac{d\,\phi}{d\sigma}(a,\sigma) < 0 \ \ , \ \ \frac{d\,\tilde{\phi}_s}{d\sigma}(a,\sigma) = e^{-\int_s^\sigma i(\eta)d\eta}\frac{d\,\phi}{d\sigma}(a,\sigma) < 0$$

for a.e. $(a,\sigma) \in (0,\omega)\times(0,\omega)$).

Then sure we have $\Phi\psi_2 - \Phi\psi_1 \in L^1_+(0,\omega)\setminus\{0\}$; hence the definition of e-increasing operator is satisfied once we set $e(\cdot) \equiv 1$ and

$$\alpha(\psi_1,\psi_2) = \epsilon_1\Big\{\int_0^\omega \left(e^{-\int_0^\sigma \psi_1(\eta)d\eta} - e^{-\int_0^\sigma \psi_2(\eta)d\eta}\right)d\sigma +$$

$$+\int_0^\omega i_x(s)\int_s^\omega \left(e^{-\int_s^\sigma \psi_2(\eta)d\eta} - e^{-\int_s^\sigma \psi_1(\eta)d\eta}\right)d\sigma\,ds\Big\}$$

$$\beta(\psi_1,\psi_2) = (1+\omega)\|\beta\|_\infty\|n\|_\infty\Big\{\int_0^\omega \left(e^{-\int_0^\sigma \psi_1(\eta)d\eta} - e^{-\int_0^\sigma \psi_2(\eta)d\eta}\right)d\sigma +$$

$$+\int_0^\omega i_x(s)\int_s^\omega \left(e^{-\int_s^\sigma \psi_1(\eta)d\eta} - e^{-\int_s^\sigma \psi_2(\eta)d\eta}\right)d\sigma\,ds\Big\}$$

$\big($ we have indeed $\|\frac{d\tilde{\phi}_s}{d\sigma}\|_\infty \leq (1+\omega)\|\beta\|_\infty\,\|n\|_\infty \;\; \forall\, s \in (0,\omega)\,\big)$

q.e.d.

From theorem (4) and proposition (4) it then follows:

Theorem 5 *Under ausumptions (1)-(4) there exists a unique fixed point of* Φ *in D; that is, a unique equilibrium solution for (2.3).*

We now drop assumption (4) to show that Φ is an e-increasing operator. We replace assumption (2) with the more restrictive:

Assumption 2'. There exists $m > 0$ such that

$$\tilde{\phi}(a,s;\sigma) = e^{-\int_s^\sigma i(\tau)d\tau}\,\phi(a,s) \geq m \;\; for\ a.e.\ a,s \in (0,\omega)\ ,\ \sigma \in (s,\omega).$$

Then under assumption (2') we have:

$$m \leq \tilde{\phi}(a,s;\sigma) \leq M \;\; for\ a.e.\;\ a,s \in (0,\omega)$$

where $M = \|\phi\|_\infty$ (the sup norm in $L^\infty\big((0,\omega)\times(0,\omega)\big)$).

We replace assumption (4) with the:

Assumption 4'. Let $(I-T)^{-1}$ be defined and continuous; we assume that u_0 satisfies:

$$\frac{e^{-\|(I-T)^{-1}u_0\|}}{\|(I-T)^{-1}u_0\|} \geq \frac{M}{m}\ .$$

Then for small enough $\|u_0\|_1$, we can prove uniqueness in the case $r(T) < 1$. We prove before the following lemma concerning an upper bound for a fixed point of Φ in terms of $\|u_0\|_1$:

Lemma 2 *If* $r(T) < 1$, $\psi \in L^1_+(0,\omega)$ *is a fixed point of* Φ *then we have:*

$$(0 \leq)\,\psi \leq (I-T)^{-1}u_0\ .$$

Proof. We have $0 \leq T\psi + u_0 - \Phi\psi = T\psi + u_0 - \psi$.

$r(T) < 1$, then there exists $(I-T)^{-1}$ and $(I-T)^{-1} \in \mathcal{L}_+(L^1(0,\omega))$, because $T \in \mathcal{L}_+(L^1(0,\omega))$. So we have:

$$0 \leq (I-T)^{-1}\big(T\psi + u_0 - \psi\big) = (I-T)^{-1}\big(u_0 - (I-T)\psi\big) = (I-T)^{-1}u_0 - \psi\ ,$$

hence:

$$\psi \leq (I-T)^{-1}u_0$$

q.e.d.

This lemma establishes, in the case $r(T) < 1$, that a fixed point of Φ belongs to the order interval $[u_0, (I-T)^{-1}u_0]$.

Proposition 5 *Let assumptions (2'),(4') hold and let $r(T) < 1$. Then the operator Φ is e-increasing on the order interval $[0, (I-T)^{-1}u_0]$.*

Proof. Given $\psi_1, \psi_2 \in L^1_+(0,\omega)$ such that $\psi_1 \leq \psi_2 \leq (I-T)^{-1}u_0$ we have:

$$(\Phi\psi_2)(a) - (\Phi\psi_1)(a) =$$

$$= \int_0^\omega \left(\psi_2(\sigma)e^{-\int_0^\sigma \psi_2(\tau)d\tau} - \psi_1(\sigma)e^{-\int_0^\sigma \psi_1(\tau)d\tau}\right)e^{-\int_0^\sigma i(\tau)d\tau}\phi(a,\sigma)d\sigma +$$

$$+ \int_0^\omega i_x(s)\int_s^\omega \left(\psi_2(\sigma)e^{-\int_s^\sigma \psi_2(\tau)d\tau} - \psi_1(\sigma)e^{-\int_s^\sigma \psi_1(\tau)d\tau}\right)e^{-\int_s^\sigma i(\tau)d\tau}\phi(a,\sigma)d\sigma\, ds =$$

$$= \int_0^\omega \big(\psi_2(\sigma) - \psi_1(\sigma)\big)e^{-\int_0^\sigma \psi_2(\tau)d\tau}\,\tilde\phi(a,\sigma)\,d\sigma -$$

$$- \int_0^\omega \psi_1(\sigma)\left(e^{-\int_0^\sigma \psi_1(\tau)d\tau} - e^{-\int_0^\sigma \psi_2(\tau)d\tau}\right)\tilde\phi(a,\sigma)\,d\sigma +$$

$$+ \int_0^\omega i_x(s)\int_s^\omega \big(\psi_2(\sigma) - \psi_1(\sigma)\big)e^{-\int_s^\sigma \psi_2(\tau)d\tau}\,\tilde\phi_s(a,\sigma)\,d\sigma\, ds -$$

$$- \int_0^\omega i_x(s)\int_s^\omega \psi_1(\sigma)\left(e^{-\int_s^\sigma \psi_1(\tau)d\tau} - e^{-\int_s^\sigma \psi_2(\tau)d\tau}\right)\tilde\phi_s(a,\sigma)\,d\sigma\, ds \geq$$

$\Big($where we have set $\tilde\phi_s(a,\sigma) = e^{-\int_s^\sigma i(\tau)d\tau}\phi(a,\sigma)$ $\Big)$

$$\geq \int_0^\omega \big(\psi_2(\sigma) - \psi_1(\sigma)\big)e^{-\int_0^\sigma \psi_2(\tau)d\tau}\,\tilde\phi(a,\sigma)\,d\sigma -$$

$$- \int_0^\omega \psi_1(\sigma)\int_0^\sigma \big(\psi_2(\tau) - \psi_1(\tau)\big)d\tau\,\tilde\phi(a,\sigma)\,d\sigma +$$

$$+\int_0^\omega i_x(s)\int_s^\omega \big(\psi_2(\sigma)-\psi_1(\sigma)\big)e^{-\int_s^\sigma \psi_2(\tau)d\tau}\,\tilde\phi_s(a,\sigma)\,d\sigma\,ds-$$

$$-\int_0^\omega i_x(s)\int_s^\omega \psi_1(\sigma)\int_s^\sigma \big(\psi_2(\tau)-\psi_1(\tau)\big)d\tau\,\tilde\phi_s(a,\sigma)\,d\sigma\,ds=$$

(because we have:

$$0\geq e^{-\int_0^\sigma \psi_2(\tau)d\tau}-e^{-\int_0^\sigma \psi_1(\tau)d\tau}\geq\int_0^\sigma \big(\psi_2(\tau)-\psi_1(\tau)\big)d\tau\ \ )$$

$$=\int_0^\omega \big(\psi_2(\tau)-\psi_1(\tau)\big)\Big[e^{-\int_0^\tau \psi_2(\tau')d\tau'}\,\tilde\phi(a,\tau)-\int_\tau^\omega \psi_1(\sigma)\tilde\phi(a,\sigma)d\sigma\Big]\,d\tau\,+$$

$$+\int_0^\omega i_x(s)\int_s^\omega \big(\psi_2(\tau)-\psi_1(\tau)\big)\Big[e^{-\int_s^\tau \psi_2(\tau')d\tau'}\,\tilde\phi_s(a,\tau)-$$

$$-\int_\tau^\omega \psi_1(\sigma)\tilde\phi_s(a,\sigma)d\sigma\Big]\,d\tau\,ds\ \geq$$

$$\geq\ \Big(e^{-\|\psi_2\|_1}m-M\|\psi_1\|_1\Big)\int_0^\omega \big(\psi_2(\tau)-\psi_1(\tau)\big)d\tau+$$

$$+\Big(e^{-\|\psi_2\|_1}m-M\|\psi_1\|_1\Big)\int_0^\omega i_x(s)\int_s^\omega \big(\psi_2(\tau)-\psi_1(\tau)\big)\,d\tau\,ds=$$

$$=\ \Big(e^{-\|\psi_2\|_1}m-M\|\psi_1\|_1\Big)\int_0^\omega \big(\psi_2(\tau)-\psi_1(\tau)\big)\big(\,1+\int_0^\tau i_x(s)\,ds\,\big)\,d\tau\ >\ 0$$

$$\forall\,\psi_1,\psi_2\in D\ \textit{such that}\ \psi_1\leq\psi_2.$$

for if $\psi_1\leq\psi_2\leq(I-T)^{-1}u_0$, then:

$$e^{-\|\psi_2\|_1}m-M\|\psi_1\|_1\geq e^{-\|\psi_2\|_1}m-M\|\psi_2\|_1\ >$$

$$>e^{-\|(I-T)^{-1}u_0\|}m-M\,\|(I-T)^{-1}u_0\|\geq 0$$

because $\|(I-T)^{-1}u_0\|$ satisfies assumption (4'). Hence the definition of e-increasing operator is satisfied once we set:

$$e(a) = 1 \;\; , \;\; \forall\, a \in (0,\omega)$$

$$\alpha(\psi_1,\psi_2) = \Big(e^{-\|\psi_2\|_1} m - M\|\psi_1\|_1\Big) \int_0^\omega \big(\psi_2(\sigma) - \psi_1(\sigma)\big)\big(1 + \int_0^\sigma i_x(s)\, ds\,\big)\, d\sigma$$

$$\beta(\psi_1,\psi_2) = 2\, max\big\{\|\Phi\psi_1\|_\infty, \|\Phi\psi_2\|_\infty\big\}\;.$$

q.e.d.

From theorem (4) and proposition (5) it follows again:

Theorem 6 *If $r(T) < 1$ then under assumptions (1),(2'), (3),(4') there exists a unique fixed point of Φ in $[u_0, (I-T)^{-1}u_0]$.*

Proof. Existence follows from theorem (3). Uniqueness follows from theorem (4) because for the previous lemma a fixed point of Φ lies in the order interval $[u_0, (I-T)^{-1}u_0]$ upon which Φ is e-increasing because of proposition (5).

q.e.d.

Under a quite close assumption on Φ it is possible to obtain that Φ is an e-increasing operator in both cases $r(T) < 1$ and $r(T) > 1$. We replace assumption (4) with the:

Assumption 4". We set $R = sup_{\psi \in L^1_+(0,\omega)} \|\Phi\psi\|_1 < \infty$ and assume that R satisfies $\frac{e^{-R}}{R} \geq \frac{M}{m}$

In the same way as for proposition (5) we prove the following:

Proposition 6 *Let assumptions (2'),(4") hold. Then the operator Φ is e-increasing on $L^1_+(0,\omega) \cap B_R(0)$.*

From theorem (4) and proposition (6) it follows again:

Theorem 7 *Under assumptions (1),(2'),(3),(4") there exists a unique fixed point of Φ in $D = [u_0, +\infty) \cap B_R(0)$.*

3.4 Threshold-like results.

In the stable population model without immigration, Inaba ([10]) proved that the spectral radius $r(T)$ of $T = \Phi'(0)$, is a threshold value for the infection. Indeed he proved that:

Theorem 8 *(Inaba)*

1) *If $r(T) \leq 1$, the only fixed point of the operator Φ is the null vector $\psi \equiv 0$;*

2) *if $r(T) > 1$ there is at least a non-zero fixed point of Φ.*

Now, in our model with immigration, in the same way it is possible to prove that if the proportion $i_y(\cdot)$ of immigrant individuals is zero, then we have the same behaviour; that is we have $\psi \equiv 0$ (disease free equilibrium) as the only equilibrium if $r(T) < 1$ and that a positive (endemic) equilibrium is ever present if $r(T) > 1$. If the proportion of infective immigrants is not zero, i.e. $u_0 \neq 0$, then we know (Theor. 3) that a positive equilibrium is present in both cases; and that we have no more the disease free equilibrium. But it is still possible to distinguish between two different situations for the steady states characterized in terms of the spectral radius of a positive linear operator; we now look at the limiting behaviour of a fixed point of Φ as $\|u_0\|_1$ goes to zero.
Now, having a mind to let $i_y(\cdot)$ going to zero, for fixed $i_x(\cdot)$, $i_z(\cdot)$, we consider a sequence $i_{y,n} \in L^1_+(0,\omega)$ such that $\|i_{y,n}\|_1 \longrightarrow 0$ and Φ_n, u_n as the analogous of (3.5), (3.6) in which $i_{y,n}$ has been inserted in place of i_y ; and $\Phi_{0,n} = \Phi_n - u_n$. We consider the positive linear operators T_n, T on $L^1(0,\omega)$:

$$(T_n\psi)(a) \;=\; \int_0^\omega \psi(\sigma)\Big(e^{-\int_0^\sigma (i_x(\tau)+i_{y,n}(\tau)+i_z(\tau))d\tau} + \\ + \int_0^\sigma i_x(s)e^{-\int_s^\sigma ((i_x(\tau)+i_{y,n}(\tau)+i_z(\tau)))d\tau}\,ds\Big)\,\phi_n(a,\sigma)d\sigma\ ,\ a\in(0,\omega) \tag{3.9}$$

$$\phi_n(a,\sigma) = \int_\sigma^\omega \beta(a,\xi)n(\xi)e^{-\gamma(\xi-\sigma)-\int_\sigma^\xi (i_x(\tau)+i_{y,n}(\tau)+i_z(\tau))d\tau}d\xi$$

$$(T\psi)(a) \;=\; \int_0^\omega \psi(\sigma)\Big(e^{-\int_0^\sigma (i_x(\tau)+i_z(\tau))d\tau} + \\ + \int_0^\sigma i_x(s)e^{-\int_s^\sigma ((i_x(\tau)+i_z(\tau)))d\tau}\,ds\Big)\,\phi(a,\sigma)d\sigma\ ,\ a\in(0,\omega) \tag{3.10}$$

$$\phi(a,\sigma) = \int_{\sigma}^{\omega} \beta(a,\xi) n(\xi) e^{-\gamma(\xi-\sigma) - \int_{\sigma}^{\xi}(i_x(\tau)+i_z(\tau))d\tau} d\xi \ ,$$

$$u_n(a) = \int_0^{\omega} i_{y,n}(s)\phi(a,s)ds\,.$$

T_n , T are the Frechet derivative in zero of Φ_n, Φ where now we think about Φ as the (3.5) in which $i_y \equiv 0$. Let ψ_n be a fixed point for Φ_n. Then we have:

Theorem 9 **i)** *If* $r(T) < 1$ *then* $\psi_n \underset{n\to+\infty}{\longrightarrow} 0$*;*

ii) *if* $r(T) > 1$ *then there exists* $\delta > 0$ *such that* $\psi_n \geq \delta \ \ \forall n \in \mathbb{N}$ *.*

Remark. This kind of limiting behaviour has been observed from Brauer and Van der Driessche ([3]); in their SIS and SIR models without age structure, they assume that there is a constant flow of new members into a population of which a fraction p is infective and that the disease is transmitted under a law of mass action. If the positive parameter $\mathcal{R}_0$ (the so called "basic reproduction number") is greater than 1, then the equilibrium number of infectives individuals goes to a positive number as p goes to zero; otherwise, if $\mathcal{R}_0 < 1$, the equilibrium number of infectives goes to zero as p goes to zero.

Before proving the theorem we need to recall some definitions and results about positive operator theory, extending the Perron-Frobenius theorem in the context of infinite dimensional Banach spaces. A first extension of this kind was carried over from Krein and Rutman ([12]).

Definition 2 *Let E be a Banach space,* $K \subset E$ *a cone. The cone K is called total if we have*

$$\overline{K-K} = \overline{\{\psi - \varphi \, : \, \psi, \varphi \in K\}} = E.$$

Theorem 10 *(Krein, Rutman (1948))(1) Let* $A : E \longrightarrow E$ *be a completely continuous linear operator such that* $A(K) \subseteq K$*,* $K \subseteq E$ *a total cone and* $r(A) > 0$*. Then r(A) is an eigenvalue of A (of* A^**) and there exists* $v \in K \setminus \{0\}$ *(*$f \in K^* \setminus \{0\}$*) such that* $A\,v = r(A)\,v$ *(*$A^* f = r(A)\,f$*).*

If A is a strongly positive operator, Krein and Rutman ("Linear operators leaving invariant a cone in a Banach space") said more:

Definition 3 *A linear operator $A : E \longrightarrow E$ is called strongly positive with respect to the cone K with non-empty interior, if for each $v \in Fr(K) \setminus \{0\}$ ($Fr(K)$ the frontier of K), there is $n=n(v) \in \mathbb{N}$ such that $A^n(v) \in Int(K)$.*

Theorem 11 *(Krein, Rutman (1948))(2) If the assumptions of the previous theorem are fulfilled and, moreover, A is strongly positive with respect to K, then:*

i) *A has one and only one (except for a constant) eigenvector $v \in Int(K)$, $A\,v = r(A)\,v$;*

ii) *A^* has one and only one (except for a constant) eigenvector $f \in K^* \setminus \{0\}$, f strictly positive (that is $f(v) > 0$ for each $v \in K \setminus \{0\}$), $A^* f = r(A)\,f$;*

iii) *$|\lambda| < r(A)$ $\forall \lambda$ eigenvalue of A, $\lambda \neq r(A)$.*

Hence if A is a strongly positive operator with respect to the cone with interior K, not only we know that r(A) is an eigevalue which possesses a positive (i.e. in K) eigenvector, but we know also that r(A) has geometric multiplicity one and it has an eigenvector in Int(K).
But we cannot apply this result to our case because $L^1_+(0, \omega)$ is a total cone in $L^1(0, \omega)$ but has empty interior. A different class of positive linear operators that permits to obtain the same results was introduced by Sawashima ([18]) and looks to fit our case:

Definition 4 *(Sawashima (1964)) Given $A \in \mathcal{L}(E)$, A positive with respect to the cone K, A is called non-supporting with respect to K, if for all $v \in K \setminus \{0\}$, for all $f \in K^* \setminus \{0\}$ there exists $p = p(v, f) \in \mathbb{N}$ such that $< f, A^n v >> 0$ $\forall n \geq p$.*

For a non-supporting operator it holds the following result:

Theorem 12 *(Sawashima (1964)) Let the cone K be total, let $A : E \longrightarrow E$ be non-supporting with respect to K, and suppose that $r(A)$ is a pole of the resolvent of A, then:*

i) *$r(A) > 0$ and it is an algebraically simple pole of the resolvent;*

ii) *the eigenspace corresponding to $r(T)$ is one-dimensional and there is an eigenvector $v \in K \setminus \{0\}$ and it satisfies $< f, v >> 0$ $\forall\, f \in K^* \setminus \{0\}$; the relation $T\varphi = \mu\varphi$ with $\varphi \in K$ implies that $\varphi = c\,v$ for some constant $c > 0$;*

iii) *the eigenspace of T^* corresponding to $r(T)$ is also a one-dimensional subspace of E^* spanned by a strictly positive functional $f \in K^*$.*

With regard to our case, we have that $T = \Phi'(0)$, $T_n = \Phi'_n(0)$ are non-supporting operators:

Lemma 3 *Under assumption (2), T, T_n are non-supporting operators with respect to $L^1_+(0,\omega)$.*

Proof. Given $\psi \in L^1_+(0,\omega)\setminus\{0\}$, we have:

$$\big(T\psi\big)(a) = \int_0^\omega \Big[\psi(\sigma)e^{-\int_0^\sigma i(\tau)d\tau} + \psi(\sigma)\int_0^\sigma i_x(s)e^{-\int_s^\sigma i(\tau)d\tau}\,ds\Big]\cdot$$

$$\cdot\int_\sigma^\omega \beta(a,\xi)n(\xi)e^{-\gamma(\xi-\sigma)-\int_\sigma^\xi i(\tau)d\tau}\,d\xi\,d\sigma \geq$$

$$\geq \int_0^\omega \psi(\sigma)\,e^{-\gamma\omega-\|i\|_1}\,m\int_{max\{\sigma,\omega-\alpha\}}^\omega n(\xi)\,d\xi\,d\sigma > 0 \quad for\; a.e.\; a\in(0,\omega).$$

That is we have $T\psi \in L^1_+(0,\omega)\setminus\{0\}$ $\forall\,\psi\in L^1_+(0,\omega)\setminus\{0\}$ and hence $< f, T\psi >> 0$ $\forall f \in (L^1_+(0,\omega))^*\setminus\{0\}$. In the same way, given $n>1$, if $T^n\psi \in L^1_+(0,\omega)\setminus\{0\}$, it follows that $(T^{n+1}\psi)(a) > 0$ for a.e. $a\in(0,\omega)$ and then
$< f, T^{n+1}\psi >> 0\,\forall f\in(L^1_+(0,\omega))^*\setminus\{0\}$. Then the definition of non-supporting operator is satisfied once we set $p = p(\psi,f) = 1\ \forall\,\psi \in L^1_+(0,\omega)\setminus\{0\}$, $\forall\, f\in (L^1_+(0,\omega))^*\setminus\{0\}$. And the same calculations are to be performed for T_n.

q.e.d.

Lemma 4 *We have that $T_n\,, n\in\mathbb{N}\,,\ T$ are completely continuous operators such that $\|T_n - T\|_{\mathcal{L}(L^1)} \longrightarrow 0$;*
if we assume there exists $m>0$ such that $\phi(a,s)\geq m\ \forall\, a,s\in(0,\omega)$, then we have also $r(T_n)\longrightarrow r(T)$.

Proof. That T_n, T are completely continuous operators follows as in proof of proposition (2); that $\|T_n - T\|_{\mathcal{L}(L^1)} \longrightarrow 0$ follows by applying the Lebesgue dominated convergence theorem.
From the theory of linear completely continuous operators, we have that every eigenvalue $\lambda\neq 0$ of T, T_n is a pole of the resolvent operators
$R(\lambda,T) = (\lambda I - T)^{-1}$, $R(\lambda,T_n)$. We know from theorem (10) of Krein-Rutman that $r(T)$, $r(T_n)$ are eigenvalues of T, T_n hence are also poles of $R(\lambda,T)$,$R(\lambda,T_n)$ and from lemma (3) we know that T, T_n are non-supporting with respect to the total cone $L^1_+(0,\omega)$. Hence it is valid theorem (12) from Sawashima.
Let $r_n = r(T_n)$, $r = r(T)$ and let φ_n, φ be the eigenvectors of T, T_n with respect to $r(T)$, $r(T_n)$ of norm one, $\|\varphi_n\|_1 = 1$ and $\|\varphi\|_1 = 1$.

From $r_n\varphi_n = T_n\varphi_n = T\varphi_n + (T_n - T)\varphi_n$ we have $r_n\varphi_n - T\varphi_n \underset{n}{\longrightarrow} 0$.

T is compact, $\{\varphi_n\}$ is bounded then there exists a converging subsequence $\{T\varphi_{n_k}\}$ of $\{T\varphi_n\}$; hence there exists $r \geq 0$ such that $r_{n_k} \underset{k}{\longrightarrow} r$.

We have $r > 0$, because of

$$(T\varphi)(a) \geq m\, e^{-\|i_x+i_y\|_1} \, \|\varphi\|_1 \quad \forall a \in (0,\omega) \ \ , \ \forall \varphi \in L^1_+(0,\omega)$$

and then

$$r_{n_k} \approx \|T\varphi_{n_k}\|_1 \geq m\, e^{-\|i_x+i_y\|_1} \, \|\varphi_{n_k}\|_1 = m\, e^{-\|i_x+i_y\|_1} \quad \forall k \in \mathbb{N}\,.$$

From $r_{n_k}\varphi_{n_k} - T\varphi_{n_k} = r_{n_k}\left(\varphi_{n_k} - \frac{T\varphi_{n_k}}{r_{n_k}}\right) \underset{k}{\longrightarrow} 0$ it follows there exists the limit:

$$\lim_k \varphi_{n_k} = \lim_k \frac{1}{r_{n_k}} T\varphi_{n_k} = \varphi \ \ , \ \ \varphi \in L^1_+(0,\omega) \ \ , \ \ \|\varphi\|_1 = 1\,.$$

We have also $T\varphi_{n_k} \underset{k}{\longrightarrow} T\varphi$, therefore $\frac{1}{r}T\varphi = \varphi$, that is $r\varphi = T\varphi$; r is an eigenvalue of T which has positive eigenvectors. From Sawashima's Theorem it follows that $r = r(T)$.

Let now $\{r_{n'_k}\}$ be another subsequence of $\{r_n\}$; in the same way it is proved that it is possible to extract a converging subsequence $\overline{r}_k$, $\overline{r}_k \underset{k}{\longrightarrow} \overline{r}$, and that $\overline{r} = r(T)$. Then from every subsequence of $\{r_n\}$ it is possible to extract a subsequence converging to $r(T)$. Therefore we have $r(T) = \lim_{n\to+\infty} r(T_n)$.

q.e.d.

In the following lemma we prove an inequality regarding the operators Φ_0, T:

Lemma 5 *It holds the following inequality:*

$$e^{-\sum_{k=0}^{n-1}\|T^k\psi\|_1} \, T^n\psi \leq \Phi_0^n \, \psi \leq T^n\psi \quad \forall\, \psi \in L^1_+(0,\omega),\ \forall\, n \in \mathbb{N} \tag{3.11}$$

Proof. We proceed by induction; given $\psi \in L^1_+(0,\omega)$ we have:

$$\big(\Phi_0\big)(a) = \int_0^\omega \left[\psi(\sigma)e^{-\int_0^\sigma(\psi+i)d\tau} + \psi(\sigma)\int_0^\sigma i_x(s)e^{-\int_s^\sigma(\psi+i)d\tau}\, ds\right] \phi(a,\sigma)\, d\sigma \leq$$

$$\leq \int_0^\omega \left[\psi(\sigma)e^{-\int_0^\sigma i(\tau)d\tau} + \psi(\sigma)\int_0^\sigma i_x(s)e^{-\int_s^\sigma i(\tau)d\tau}\, ds\right] \phi(a,\sigma) d\sigma = \big(T\psi\big)(a)$$

and:

$$\big(\Phi_0\big)(a) \geq e^{-\int_0^\omega \psi d\tau} \int_0^\omega \big[\psi(\sigma)e^{-\int_0^\sigma i(\tau)d\tau} +$$
$$\psi(\sigma) \int_0^\sigma i_x(s) e^{-\int_s^\sigma i(\tau)d\tau}\, ds\big]\, \phi(a,\sigma)d\sigma = e^{-\|\psi\|_1}\, \big(T\psi\big)(a)$$

for a.e. a$\in (0,\omega)$. Let now the assert be valid for $n > 1$, then we have:

$$\big(\Phi_0^{n+1}\big)(a) = \Phi_0\big(\Phi_0^n\psi\big)(a) \leq T\big(\Phi_0^n\psi\big)(a) \leq T\big(T^n\psi\big)(a) = \big(T^{n+1}\psi\big)(a)$$

for a.e. a$\in (0,\omega)$, because of the first step and the inductive hypothesis; and further:

$$\big(\Phi_0^{n+1}\psi\big)(a) = \Phi_0^n\, \Phi_0\psi\,(a) \geq e^{-\sum_{k=0}^{n-1} \|T^k\Phi_0\psi\|_1} \big(T^n\Phi_0\psi\big)(a) \geq$$
$$\geq e^{-\sum_{k=0}^{n-1} \|T^k T\psi\|_1}\, T^n\big(e^{-\|\psi\|_1}\, T\psi\big)(a) = e^{-\sum_{k=0}^{n} \|T^k\psi\|_1} \big(T^{n+1}\psi\big)(a)$$

for a.e. a$\in (0,\omega)$, again because of the first step and the inductive hypothesis and because of the monotoneity of the L^1-norm.

q.e.d.

We can now prove theorem (9):
Proof.

i) If $r(T) < 1$ then we have eventually $r(T_n) < 1$ as $n \to +\infty$: there exists $\nu \in \mathbb{N}$ such that $r(T_n) < 1\ \forall\, n > \nu$. Further, there exist $(I-T)^{-1}$, $(I-T)^{-1} \in \mathcal{L}_+\big(L^1(0,\omega)\big)$ and there exists $(I-T_n)^{-1}\ \forall\, n > \nu$ with $(I-T_n)^{-1} \in \mathcal{L}_+\big(L^1(0,\omega)\big)$. Let $\psi_n \in L^1_+(0,\omega)$ be a fixed point of Φ_n , we have:

$$0 \leq \psi_n = \Phi_n\psi_n = \Phi_{0,n}\psi_n + u_n \leq T_n\psi_n + u_n$$

because of the previous lemma and then:

$$\psi_n - T_n\psi_n = (I-T_n)\psi_n \leq u_n\ ,\ \ (0 \leq)\psi_n \leq (I-T_n)^{-1}u_n\ \ \forall\, n > \nu$$

for $(I-T_n)^{-1}$ is a positive operator. Then from:

$$(I-T_n)^{-1}u_n = (I-T)^{-1}u_n + \big((I-T_n)^{-1} - (I-T)^{-1}\big)u_n$$

the assert follows for the monotoneity of the L^1-norm, the boundedness of $(I-T)^{-1}$ and the fact that $\|T_n - T\|_{\mathcal{L}(L^1)} \underset{n}{\longrightarrow} 0$.

ii) From $r(T) > 1$ it follows there exists a $\nu \in \mathbb{N}$ such that $r(T_n) > 1 \ \ \forall n > \nu$. Let $\psi_n \in L^1_+(0,\omega)$ be such that $\psi_n = \Phi_n\, \psi_n$.

Let us suppose that $\|\psi_n\|_1 \underset{n\to+\infty}{\longrightarrow} 0$. Let $f_n \in (L^1_+(0,\omega))^* \setminus \{0\}$ be the strictly positive eigenvector of T_n^* with respect to the eigenvalue $r(T_n)$. Then we have:

$$< f_n, \psi_n > = < f_n, \Phi_n\, \psi_n > = < f_n, \Phi_{0,n}\, \psi_n + u_n > \geq \tag{3.12}$$

$$\geq < f_n, e^{-\|\psi_n\|_1}\, T_n \psi_n + u_n > > < f_n, e^{-\|\psi_n\|_1}\, T_n \psi_n > = e^{-\|\psi_n\|_1} < T_n^* f_n, \psi_n > =$$

$$= e^{-\|\psi_n\|_1} r(T_n) < f_n, \psi_n > \quad \forall\, n \in \mathbb{N},$$

where the first inequality follows from lemma (5). For $\psi_n \geq u_n > 0$ and f_n is a strictly positive functional, we have that:

$$1 > e^{-\|\psi_n\|_1}\, r(T_n) \quad \forall\, n \in \mathbb{N}$$

which is an absurd because we have eventually

$e^{-\|\psi_n\|_1}\, r(T_n) \geq 1$ as $n \longrightarrow +\infty$. It then follows that there exists $\delta > 0$ such that $\|\psi_n\|_1 \geq \delta \ \ \forall n$.

q.e.d.

Remark. From (3.12) we can say something about the location of a fixed point ψ of Φ in $L^1_+(0,\omega)$ (in the case $r(T) > 1$ and with u_0 not necessarily zero); indeed from:

$$< f, \psi > \geq e^{-\|\psi\|_1}\, r(T) < f, \psi > + < f, u_0 >$$

it follows:

$$\left(1 - \frac{< f, u_0 >}{< f, \psi >}\right) \frac{1}{r(T)} \geq e^{-\|\psi\|_1}$$

and hence:

$$\|\psi\|_1 \geq \ln\left(\frac{1}{1 - \frac{< f, u_0 >}{< f, \psi >}}\right) + \ln\big(r(T)\big) > \ln\big(r(T)\big).$$

3.5 Two remarks on the issue of uniqueness.

H. Inaba in his work [10] about a SIR model without immigration did an accurate analyis of conditions under which it is possible to prove uniqueness for solutions of the nonlinear operator equation one obtains when studing the steady states of (2.3) (without immigration). In the case $r(T) > 1$, he wed the theory of positive monotone operators on cones in Banach spaces under the technical hypothsis (4). When proving the uniqueness of the disesease free equilibrium, for $r(T) < 1$, he resorted to the Sawashima's theorem.
In either case, he did not employ the fact $n(a)$ is a non-increasing function, since it is the given by the form:

$$n(a) = N_0\, e^{-\int_0^a \mu(\tau)d\tau}$$

(N_0 the yearly number of newborns).
A relevant question is if it is possible to drop assumption (4) in proving uniqueness in the case without immigration or in our case with immigration (let us remember that now the disease free equilibrium has disappeared).

Here we show that the assumption that $n(a)$ is not decreasing is necessary for proving the uniqueness. Indeed we exhibit an example with an increasing $n(a)$, for which the necessary condition of a tangent bifurcation of the positive fixed point is satisfied. Adding some technical conditions, this will yeld parameter values with multiple stationary solutions. Since, with immigration, the function $n(a)$ may be increasing, this show that it is possible to find immigration functions $I(a)$ and infection kernels $\beta(a, a')$ with multiple stationary solutions. Now to the construction of the example.

Given $0 < \alpha < \omega$ we set $I_1 = (0, \alpha)$, $I_2 = (\alpha, \omega)$, and consider the contact coefficient $\beta(\cdot, \cdot)$ in the form:

$$\beta(a, s) = \beta_{ij} \iff (a, s) \in I_i \times I_j \;\; i, j = 1, 2\,, \tag{3.13}$$

with $\beta_{ij} \geq 0$ and not all zero (this choice of subdividing the interval $(0, \omega)$ in a finite number of intervals and taking a $\beta(\cdot, \cdot)$ of this kind is what is usually done in practice to perform simulations of real situations [2], and we will do in the next chapter).

Then with this choice of $\beta(\cdot, \cdot)$, we rewrite Φ in this way:

$$(\Phi\psi)(a) = \int_0^\omega (F\psi)(s)\phi(a, s)ds = \int_0^\omega (F\psi)(s) \int_s^\omega \beta(a, \xi)n(\xi)e^{-\gamma(\xi - s)}d\xi\, ds =$$

(with $F : L^1_+(0,\omega) \longrightarrow L^1_+(0,\omega)$ such that $(F\psi)(a) = \psi(a)e^{-\int_0^a \psi(\tau)d\tau}$)

$$= \int_0^\omega \beta(a,\xi)n(\xi)\int_0^\xi (F\psi)(s)\, e^{-\gamma(\xi-s)}ds\, d\xi =$$

$$= \sum_{i=1}^2 \mathbf{1}_{I_i}(a)\sum_{j=1}^2 \beta_{ij}\int_{I_j} n(\xi)\int_0^\xi (F\psi)(s)\, e^{-\gamma(\xi-s)}ds\, d\xi.$$

where $I_i(\cdot)$ is the indicator function of the interval I_i. Therefore $Im\Phi \subseteq <\mathbf{1}_{I_1}, \mathbf{1}_{I_2}, >_+$; if $\psi \in L^1_+(0,\omega)$ is a positive fixed point for Φ we have $\psi = \sum_{i=1}^2 x_i\, \mathbf{1}_{I_i}$, $(x_1, x_2) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$.
(x_1, x_2) is solution of the equation on $\mathbb{R}^2_+$:

$$x_i = \beta_{i1}\int_0^\alpha n(\xi)\int_0^\xi x_1\, e^{-sx_1}e^{-\gamma(\xi-s)}ds\, d\xi \,+\, \beta_{i2}\int_\alpha^\omega n(\xi)\int_0^\alpha x_1\, e^{-sx_1}e^{-\gamma(\xi-s)}ds\, d\xi +$$

$$+\, \beta_{i2}\int_\alpha^\omega n(\xi)\int_\alpha^\xi x_2 e^{-\alpha x_1}e^{-(s-\alpha)x_2}e^{-\gamma(\xi-s)}ds\, d\xi \quad , \; i = 1,2;$$

that is:

$$x_i = \beta_{i1}\int_0^\alpha n(\xi)e^{-\gamma\xi}\, x_1\frac{e^{\xi(\gamma-x_1)}-1}{\gamma-x_1}\, d\xi + \beta_{i2}\, x_1\, \frac{e^{\alpha(\gamma-x_1)}-1}{\gamma-x_1}\int_\alpha^\omega n(\xi)e^{-\gamma\xi}d\xi +$$

$$+\, \beta_{i2}\, x_2\, e^{-\alpha x_1}e^{\alpha x_2}\int_\alpha^\omega n(\xi)e^{-\gamma\xi}\frac{e^{\xi(\gamma-x_2)}-e^{\alpha(\gamma-x_2)}}{\gamma-x_2}d\xi \;. \tag{3.14}$$

By writing as $f_i(x)$ the right term of x_i, we have defined an operator $f : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+$. Hence, with the choices we have did, we have transformed our operator equation problem on $L^1_+(0,\omega)$, into a problem in dimension two.

For the derivatives of the various terms containig x_1, x_2 we have:

$$\frac{d}{dx_1}\left(x_1\frac{e^{\xi(\gamma-x_1)}-1}{\gamma-x_1}\right) = \frac{e^{\xi(\gamma-x_1)}-1}{\gamma-x_1} + x_1\, \frac{-\xi e^{\xi(\gamma-x_1)}(\gamma-x_1)+e^{\xi(\gamma-x_1)}-1}{(\gamma-x_1)^2} \quad ,$$

$$\frac{d}{dx_2}\left(x_2 e^{\alpha x_2}\frac{e^{\xi(\gamma-x_2)}-e^{\alpha(\gamma-x_2)}}{\gamma-x_2}\right)=(1+\alpha\, x_2)e^{\alpha x_2}\frac{e^{\xi(\gamma-x_2)}-e^{\alpha(\gamma-x_2)}}{\gamma-x_2}+$$

$$+\ x_2 e^{\alpha x_2}\frac{\left(-\xi e^{\xi(\gamma-x_2)}+\alpha e^{\alpha(\gamma-x_2)}\right)(\gamma-x_2)\ +\ e^{\xi(\gamma-x_2)}-e^{\alpha(\gamma-x_2)}}{(\gamma-x_2)^2}\ .$$

Then by choosing all the parameters $\beta_{ij},\gamma,\alpha,\omega$, in such a way that $x^*=(1,1)$ be a fixed point, we have:

$$1=\beta_{i1}\int_0^\alpha n(\xi)e^{-\gamma\xi}\,\frac{e^{\xi(\gamma-1)}-1}{\gamma-1}\,d\xi\ +\ \beta_{i2}\,\frac{e^{\alpha(\gamma-1)}-1}{\gamma-1}\int_\alpha^\omega n(\xi)e^{-\gamma\xi}d\xi+$$

$$+\ \beta_{i2}\int_\alpha^\omega n(\xi)e^{-\gamma\xi}\frac{e^{\xi(\gamma-1)}-e^{\alpha(\gamma-1)}}{\gamma-1}d\xi\ ,\ i=1,2.$$

We write $\mathbf{I}$ as the unit matrix of two dimension, $J_f(x)$ as the Jacobian of f in $x=(x_1,x_2)$ and with respect to the x_1,x_2 variables. Then we have:

$$\big(\mathbf{I}-J_f(x^*)\big)_{11}=\beta_{11}\int_0^\alpha n(\xi)e^{-\gamma\xi}\,\frac{1-e^{\xi(\gamma-1)}+\xi(\gamma-1)e^{\xi(\gamma-1)}}{(\gamma-1)^2}\,d\xi+$$

$$+\ \beta_{12}\frac{1-e^{\alpha(\gamma-1)}+\alpha(\gamma-1)e^{\alpha(\gamma-1)}}{(\gamma-1)^2}\int_\alpha^\omega n(\xi)e^{-\gamma\xi}d\xi+$$

$$+\ \beta_{12}\int_\alpha^\omega n(\xi)e^{-\gamma\xi}(1+\alpha)\,\frac{e^{\xi(\gamma-1)}-e^{\alpha(\gamma-1)}}{\gamma-1}\,d\xi\ \ ;$$

$$\big(\mathbf{I}-J_f(x^*)\big)_{22}=\beta_{21}\int_0^\alpha n(\xi)\,e^{-\gamma\xi}\,\frac{e^{\xi(\gamma-1)}-1}{\gamma-1}\,d\xi+$$

$$+\ \beta_{22}\frac{e^{\alpha(\gamma-1)}-1}{\gamma-1}\int_\alpha^\omega n(\xi)\,e^{-\gamma\xi}d\xi\ +\ \beta_{22}\int_\alpha^\omega n(\xi)\,e^{-\gamma\xi}\,\Big[-\alpha\frac{e^{\xi(\gamma-1)}-e^{\alpha(\gamma-1)}}{\gamma-1}+$$

$$+\ \frac{\xi(\gamma-1)e^{\xi(\gamma-1)}-e^{\xi(\gamma-1)}-\big(\alpha(\gamma-1)e^{\alpha(\gamma-1)}-e^{\alpha(\gamma-1)}\big)}{(\gamma-1)^2}\Big]\,d\xi\ ;$$

$$\big(\mathbf{I}-J_f(x^*)\big)_{12}=\beta_{12}\int_\alpha^\omega n(\xi)\,e^{-\gamma\xi}\Big[-(1+\alpha)\frac{e^{\xi(\gamma-1)}-e^{\alpha(\gamma-1)}}{\gamma-1}+$$

$$+\frac{\xi(\gamma-1)e^{\xi(\gamma-1)}-e^{\xi(\gamma-1)}-\big(\alpha(\gamma-1)e^{\alpha(\gamma-1)}-e^{\alpha(\gamma-1)}\big)}{(\gamma-1)^2}\Big]\,d\xi\ ;$$

$$\big(\mathbf{I}-J_f(x^*)\big)_{21}=\beta_{21}\int_0^\alpha n(\xi)e^{-\gamma\xi}\Big[-\frac{e^{\xi(\gamma-1)}-1}{\gamma-1}+\frac{1-e^{\xi(\gamma-1)}+\xi(\gamma-1)e^{\xi(\gamma-1)}}{(\gamma-1)^2}\Big]d\xi+$$

$$+\beta_{22}\Big[-\frac{e^{\alpha(\gamma-1)}-1}{\gamma-1}+\frac{1-e^{\alpha(\gamma-1)}+\alpha(\gamma-1)e^{\alpha(\gamma-1)}}{(\gamma-1)^2}\Big]\int_\alpha^\omega n(\xi)\,e^{-\gamma\xi}d\xi+$$

$$+\beta_{22}\,\alpha\int_\alpha^\omega n(\xi)\,e^{-\gamma\xi}\frac{e^{\xi(\gamma-1)}-e^{\alpha(\gamma-1)}}{\gamma-1}\,d\xi\ .$$

By taking a not monotone decreasing density of population, we now let determinant $det\,\big(\big(\mathbf{I}-J_f(x^*)\big)\big)$ making a sign jump. This means that for some values of the parameters, the necessary condition for x^* to be a bifurcation point is satisfied $\big($[21] prop. (8.2)$\big)$. We take: $\beta_{11}=\beta_{22}=0\,,\ \gamma=1\,,\ n(\xi)=e^{\xi}\,$; then:

Proposition 7 *For the upper choice of the parameters, there exists suitabble $0<\alpha^*<\omega^*<+\infty$ such that:*

$$det\,\big(\big(\mathbf{I}-J_f(x^*)\big)\big)=\beta_{12}\beta_{21}\Bigg(\int_{\alpha^*}^{\omega^*}\Big[(1+\alpha)\big(\xi-\alpha\big)\,+\,\alpha^2/2\Big]d\xi\cdot\int_0^{\alpha^*}\xi\,d\xi+$$

$$+\int_{\alpha^*}^{\omega^*}\Big[(1+\alpha)\big(\xi-\alpha\big)\,-\,\frac{1}{2}\big(\xi^2-\alpha^2\big)\Big]d\xi\cdot\int_0^{\alpha^*}\Big[-\xi+\xi^2/2\Big]d\xi\Bigg)=0. \tag{3.15}$$

Proof. We now see how to choose α and ω. We take $\alpha>0$ "little" in such a way that we have:

$$\xi>\left|-\xi+\frac{1}{2}\,\xi^2\right|\quad\forall\,0<\xi<\alpha;$$

and $\omega = \alpha + \epsilon$, $\epsilon > 0$, in such a way that:

$$(1+\alpha)\big(\xi-\alpha\big) + \frac{1}{2}\,\alpha^2 > \left|(1+\alpha)\big(\xi-\alpha\big) - \frac{1}{2}\,\big(\xi^2-\alpha^2\big)\right| \;\; \forall\, \alpha < \xi < \omega\,.$$

Then $det\,\big(\big(\mathbf{I} - J_f(x^*)\big)\big) > 0$.
And now we let α raise in such a way that:

$$\int_0^\alpha \Big[-\xi + \xi^2/2\Big] d\xi \;>\; 0\,. \tag{3.16}$$

Let us observe that we have:

$$\int_\alpha^\omega \Big[(1+\alpha)\big(\xi-\alpha\big) \;+\; \alpha^2/2\Big] d\xi \;=\; O(\omega^2) \;\; per\,\omega \longrightarrow +\infty\,,$$

$$\int_\alpha^\omega \Big[(1+\alpha)\big(\xi-\alpha\big) \;-\; \frac{1}{2}\big(\xi^2-\alpha^2\big)\Big] d\xi = O(\omega^3) \;\; per\,\omega \longrightarrow +\infty\,,$$

$$\int_\alpha^\omega \Big[(1+\alpha)\big(\xi-\alpha\big) \;-\; \frac{1}{2}\big(\xi^2-\alpha^2\big)\Big] d\xi \longrightarrow -\infty \;\; per\,\omega \longrightarrow +\infty\,.$$

Then for α as in (3.16) e ω big enough we have $det\,\big(\big(\mathbf{I} - J_f(x^*)\big)\big) < 0$. Then follows the thesis by continuity reasons.

q.e.d.

Remark. A choice for ω suitable for the proof could be $\omega = \alpha + \alpha^2$.

The second remark is as follows.
If $n(a)$ is a non-increasing function and $\beta(a, a')$ is given by the functional form (3.13), then the sign jump of proposition (7) cannot happen. With some additional considerations this show that if $\beta(a, a')$ is (3.13), the model without immigrationa has a unique stationary solution.
To prove this fact, we begin by observing it is possible to consider the determinant $det\,\big(\big(\mathbf{I} - J_f(x^*)\big)\big)$ as composed of the sum of three pieces:

$$det\,\big(\big(\mathbf{I} - J_f(x^*)\big)\big) = \beta_{11} \int_0^\alpha \cdots\, d\xi \cdot \big(\mathbf{I} - J_f(x^*)\big)_{22} + \beta_{12}\,\beta_{21}\Big(\cdots\Big) + \beta_{12}\,\beta_{22}\Big(\cdots\Big).$$

The first piece is positive for very possible choice of the parameters, indeed we have:

$$\int_0^{\alpha} n(\xi)e^{-\gamma\xi}\,\frac{1-e^{\xi(\gamma-1)}+\xi(\gamma-1)e^{\xi(\gamma-1)}}{(\gamma-1)^2}\,d\xi\;>\;0$$

because the function:

$$v(\xi)=\frac{1-e^{\xi(\gamma-1)}+\xi(\gamma-1)e^{\xi(\gamma-1)}}{(\gamma-1)^2}$$

is strictly monotone increasing on all of $[0,+\infty)$ and with $v(0)=0$;

and the terms contained in $\big(\mathbf{I}-J_f(x^*)\big)_{22}$ are all positive becuse of the monotoneity of the exponential function and because the function:

$$v(\xi)=-\alpha\frac{e^{\xi(\gamma-1)}-e^{\alpha(\gamma-1)}}{\gamma-1}+$$
$$+\;\frac{\xi(\gamma-1)e^{\xi(\gamma-1)}-e^{\xi(\gamma-1)}-\big(\alpha(\gamma-1)e^{\alpha(\gamma-1)}-e^{\alpha(\gamma-1)}\big)}{(\gamma-1)^2}$$

is strictly positive on $(\alpha,+\infty)$.

If we write all of the second piece, $\beta_{12}\,\beta_{21}\Big(\cdots\Big)$, we have:

$$\beta_{12}\beta_{21}\Bigg(\int_{\alpha}^{\omega} n(\xi)e^{-\gamma\xi}\Big[\frac{1-e^{\alpha(\gamma-1)}+\alpha(\gamma-1)e^{\alpha(\gamma-1)}}{(\gamma-1)^2}+(1+\alpha)\,\frac{e^{\xi(\gamma-1)}-e^{\alpha(\gamma-1)}}{\gamma-1}\Big]\,d\xi\;\cdot$$
$$\cdot\int_0^{\alpha} n(\xi)\,e^{-\gamma\xi}\,\frac{e^{\xi(\gamma-1)}-1}{\gamma-1}\,d\xi+$$
$$+\int_{\alpha}^{\omega} n(\xi)e^{-\gamma\xi}\Big[(1+\alpha)\,\frac{e^{\xi(\gamma-1)}-e^{\alpha(\gamma-1)}}{\gamma-1}-\frac{\xi(\gamma-1)e^{\xi(\gamma-1)}-e^{\xi(\gamma-1)}-\big(\alpha(\gamma-1)e^{\alpha(\gamma-1)}-e^{\alpha(\gamma-1)}\big)}{(\gamma-1)^2}\Big]\,d\xi\cdot$$
$$\cdot\int_0^{\alpha} n(\xi)e^{-\gamma\xi}\Big[-\,\frac{e^{\xi(\gamma-1)}-1}{\gamma-1}\;+\;\frac{1-e^{\xi(\gamma-1)}+\xi(\gamma-1)e^{\xi(\gamma-1)}}{(\gamma-1)^2}\Big]d\xi\Bigg)\qquad(3.17)$$

Remark. This is the term of the sign jump of proposition (7).

We have the lemma:

Lemma 6 *The following inequalities hold:*

$$\int_{\alpha}^{\omega} n(\xi)e^{-\gamma\xi}\Big[\frac{1-e^{\alpha(\gamma-1)}+\alpha(\gamma-1)e^{\alpha(\gamma-1)}}{(\gamma-1)^2}+(1+\alpha)\,\frac{e^{\xi(\gamma-1)}-e^{\alpha(\gamma-1)}}{\gamma-1}\Big]\,d\xi >$$

$$> \int_{\alpha}^{\omega} n(\xi)e^{-\gamma\xi}\Big[(1+\alpha)\,\frac{e^{\xi(\gamma-1)}-e^{\alpha(\gamma-1)}}{\gamma-1}-$$

$$-\;\frac{\xi(\gamma-1)e^{\xi(\gamma-1)}-e^{\xi(\gamma-1)}-\big(\alpha(\gamma-1)e^{\alpha(\gamma-1)}-e^{\alpha(\gamma-1)}\big)}{(\gamma-1)^2}\Big]\,d\xi > 0 \;\;;$$

$\forall\gamma>0$, $\forall\, 0<\alpha<\omega$, $\forall\, n(\cdot)\in L^1_+(0,\omega)$, *$n(\cdot)$ monotone decreasing.*

Proof. First inequality suddenly follows from:

$$\frac{1-e^{\xi(\gamma-1)}+\xi(\gamma-1)e^{\xi(\gamma-1)}}{(\gamma-1)^2} > 0 \quad \forall\xi\in\mathbb{R}\setminus\{0\}\,.$$

To prove second inequality, let us consider before the case $n(\cdot)\equiv 1$. We have the equality:

$$\int_{\alpha}^{+\infty} e^{-\gamma\xi}\Big[(1+\alpha)\,\frac{e^{\xi(\gamma-1)}-e^{\alpha(\gamma-1)}}{\gamma-1}-$$

$$-\;\frac{\xi(\gamma-1)e^{\xi(\gamma-1)}-e^{\xi(\gamma-1)}-\big(\alpha(\gamma-1)e^{\alpha(\gamma-1)}-e^{\alpha(\gamma-1)}\big)}{(\gamma-1)^2}\Big]\,d\xi = 0$$

$\forall\,\alpha,\gamma>0$. (you can check this for example with maple). Let us consider the functions $u:(\alpha,+\infty)\longrightarrow\mathbb{R}$ defined as:

$$u(\omega)=\int_{\alpha}^{\omega} e^{-\gamma\xi}\Big[(1+\alpha)\,\frac{e^{\xi(\gamma-1)}-e^{\alpha(\gamma-1)}}{\gamma-1}-$$

$$-\;\frac{\xi(\gamma-1)e^{\xi(\gamma-1)}-e^{\xi(\gamma-1)}-\big(\alpha(\gamma-1)e^{\alpha(\gamma-1)}-e^{\alpha(\gamma-1)}\big)}{(\gamma-1)^2}\Big]\,d\xi$$

and $v:(\alpha,+\infty)\longrightarrow\mathbb{R}$ defined as:

$$v(\xi) = (1+\alpha)\,\frac{e^{\xi(\gamma-1)} - e^{\alpha(\gamma-1)}}{\gamma-1} -$$

$$- \;\frac{\xi(\gamma-1)e^{\xi(\gamma-1)} - e^{\xi(\gamma-1)} - \big(\alpha(\gamma-1)e^{\alpha(\gamma-1)} - e^{\alpha(\gamma-1)}\big)}{(\gamma-1)^2}.$$

We have $v'(\xi) = e^{\xi(\gamma-1)}\big(1+\alpha-\xi\big)$, hence v is monotone increasing on $(\alpha, \alpha+1)$, decreasing on $(\alpha+1, +\infty)$; and we have $v(\alpha) = 0$,

$$lim_{\xi\to+\infty}\, v(\xi) = \begin{cases} -\infty & if \quad \gamma \geq 1 \\ \frac{e^{\alpha(\gamma-1)}}{1-\gamma}\left(1 - \frac{1}{1-\gamma}\right) & if \quad 0 < \gamma < 1 \end{cases}$$

with $1 - \frac{1}{1-\gamma} < 0$ for $\gamma \in (0,1)$.

Then there exists $\omega_0 > \alpha + 1$ such that $v > 0$ on (α, ω_0), $v < 0$ on $(\omega_0, +\infty)$. Hence we have: u is strictly monotone increasing on (α, ω_0), strictly monotone decreasing on $(\omega_0, +\infty)$; and with $u(\alpha) = 0$, $lim_{\omega\to+\infty}\, u(\omega) = 0$.
It is then possible to conclude that $u(\omega) > 0 \;\forall\, \omega > \alpha$.

Let us now consider the case $n(\cdot)$ monotone decreasing and $n(\cdot) \neq 1$.
If $\alpha < \omega < \omega_0$, then:

$$\int_\alpha^\omega n(\xi)e^{-\gamma\xi}v(\xi)\,d\xi > n(\omega)\int_\alpha^\omega e^{-\gamma\xi}v(\xi)\,d\xi = n(\omega)u(\omega) > 0.$$

Otherwise if $\omega > \omega_0$:

$$\int_\alpha^\omega n(\xi)e^{-\gamma\xi}v(\xi)\,d\xi = \int_\alpha^{\omega_0} \cdots d\xi \;+\; \int_{\omega_0}^{\omega} \cdots d\xi \geq$$

$$\geq n(\omega_0)\int_\alpha^{\omega_0} e^{-\gamma\xi}v(\xi)\,d\xi \;+\; n(\omega_0)\int_{\omega_0}^{\omega} e^{-\gamma\xi}v(\xi)\,d\xi = n(\omega_0)u(\omega)\; > 0$$

($v(\cdot)$ is negative on $(\omega_0, +\infty)$).

q.e.d.

We now write as I_1, I_2, I_3, I_4 the integrals in (3.17) with the order they appear, then:

$$\beta_{12}\,\beta_{21}\big(I_1\,I_2 \;+\; I_3\,I_4\big) = \beta_{12}\,\beta_{21}\Big(\big(I_1 - I_3\big)\,I_2 \;+\; I_3\,\big(I_2 + I_4\big)\Big) > 0$$

because we have:

- $I_2 > 0\,,\, I_2 + I_4 > 0$ because they are integrals of positive functions;

  (we have:

$$I_2 + I_4 = \int_0^{\alpha} n(\xi) e^{-\gamma\xi} \frac{e^{\xi(\gamma-1)} - 1}{\gamma - 1} d\xi +$$

$$+ \int_0^{\alpha} n(\xi) e^{-\gamma\xi} \Big[- \frac{e^{\xi(\gamma-1)} - 1}{\gamma - 1} + \frac{1 - e^{\xi(\gamma-1)} + \xi(\gamma-1)e^{\xi(\gamma-1)}}{(\gamma-1)^2} \Big] d\xi \; =$$

$$= \int_0^{\alpha} n(\xi) e^{-\gamma\xi} \, \frac{1 - e^{\xi(\gamma-1)} + \xi(\gamma-1)e^{\xi(\gamma-1)}}{(\gamma-1)^2} d\xi > 0$$

  because $1 - e^{\xi(\gamma-1)} + \xi(\gamma-1)e^{\xi(\gamma-1)} > 0 \;\forall\, \xi > 0\;\; , \forall\, \gamma > 0$).

- $I_1 - I_3 > 0\,,\, I_3 > 0$ for the lemma.

In conclusion, the term $\beta_{12}\beta_{21}\Big(\cdots \Big)$ is ever positive for each possible choice of the parameters.

In the same way it is possible to study the sign of the term $\beta_{12}\beta_{22}\Big(\cdots \Big)$. Let us write all of this term:

$$
\begin{aligned}
&\beta_{12}\beta_{22}\Bigg(\int_\alpha^\omega n(\xi)e^{-\gamma\xi}\Big[\frac{1-e^{\alpha(\gamma-1)}+\alpha(\gamma-1)e^{\alpha(\gamma-1)}}{(\gamma-1)^2}+(1{+}\alpha)\,\frac{e^{\xi(\gamma-1)}-e^{\alpha(\gamma-1)}}{\gamma-1}\Big]\,d\xi\,\cdot\\
&\qquad\cdot\int_\alpha^\omega n(\xi)e^{-\gamma\xi}\Big[\frac{e^{\alpha(\gamma-1)}-1}{\gamma-1}-\alpha\frac{e^{\xi(\gamma-1)}-e^{\alpha(\gamma-1)}}{\gamma-1}+\\
&\qquad+\frac{\xi(\gamma-1)e^{\xi(\gamma-1)}-e^{\xi(\gamma-1)}-\big(\alpha(\gamma-1)e^{\alpha(\gamma-1)}-e^{\alpha(\gamma-1)}\big)}{(\gamma-1)^2}\Big]\,d\xi+\\
&+\int_\alpha^\omega n(\xi)e^{-\gamma\xi}\Big[(1{+}\alpha)\,\frac{e^{\xi(\gamma-1)}-e^{\alpha(\gamma-1)}}{\gamma-1}-\frac{\xi(\gamma-1)e^{\xi(\gamma-1)}-e^{\xi(\gamma-1)}-\big(\alpha(\gamma-1)e^{\alpha(\gamma-1)}-e^{\alpha(\gamma-1)}\big)}{(\gamma-1)^2}\Big]\,d\xi\cdot\\
&\cdot\int_\alpha^\omega n(\xi)e^{-\gamma\xi}\Big[-\frac{e^{\alpha(\gamma-1)}-1}{\gamma-1}+\frac{1-e^{\alpha(\gamma-1)}+\alpha(\gamma-1)e^{\alpha(\gamma-1)}}{(\gamma-1)^2}+\alpha\,\frac{e^{\xi(\gamma-1)}-e^{\alpha(\gamma-1)}}{\gamma-1}\Big]d\xi\Bigg)
\end{aligned}
\tag{3.18}
$$

The first and third of these integrals we have yet met in the lemma: they are positive $\forall 0<\alpha<\omega$, $\forall\,\gamma>0$. Again, we write as I_1 , I_2 , I_3 , I_4 the integrals in (3.18) with the order they appear, then:

$$
\beta_{12}\,\beta_{22}\big(I_1\,I_2\,+\,I_3\,I_4\,\big)=\beta_{12}\,\beta_{22}\Big(\big(I_1-I_3\big)\,I_2\,+\,I_3\,\big(I_2+I_4\big)\,\Big)>0
$$

because we have:

- I_1 , I_3 satisfy the lemma;
- $I_2>0$ because of the positivity of the function in this integral $\forall\,\xi>\alpha$;
- $I_2\,+\,I_4>0$, indeed we have:

$$
\begin{aligned}
I_2+I_4=&\int_\alpha^\omega n(\xi)e^{-\gamma\xi}\Big[\frac{e^{\alpha(\gamma-1)}-1}{\gamma-1}-\alpha\frac{e^{\xi(\gamma-1)}-e^{\alpha(\gamma-1)}}{\gamma-1}+\\
&+\frac{\xi(\gamma-1)e^{\xi(\gamma-1)}-e^{\xi(\gamma-1)}-\big(\alpha(\gamma-1)e^{\alpha(\gamma-1)}-e^{\alpha(\gamma-1)}\big)}{(\gamma-1)^2}\Big]\,d\xi+\\
&+\int_\alpha^\omega n(\xi)e^{-\gamma\xi}\Big[-\frac{e^{\alpha(\gamma-1)}-1}{\gamma-1}+\frac{1-e^{\alpha(\gamma-1)}+\alpha(\gamma-1)e^{\alpha(\gamma-1)}}{(\gamma-1)^2}+\alpha\,\frac{e^{\xi(\gamma-1)}-e^{\alpha(\gamma-1)}}{\gamma-1}\Big]d\xi=\\
&=\int_\alpha^\omega n(\xi)e^{-\gamma\xi}\,\frac{1-e^{\xi(\gamma-1)}+\xi(\gamma-1)e^{\xi(\gamma-1)}}{(\gamma-1)^2}\,d\xi>0
\end{aligned}
$$

Hence also the third piece is ever positive as all the parameters vary. In conclusion we have proved the following:

Proposition 8 *If $\beta(a, a')$ is in the functional form (3.13) and $n(\cdot)$ is monotone non-increasing then we have* $\det\big((\mathbf{I} - J_f(x^*))\big) > 0$ *for each possible choice of the parameters $\beta_{ij} > 0\,,\ \gamma > 0\,,\ 0 < \alpha < \omega\,,$ where x^* is a fixed point for the operator defined by (3.14).*

Chapter 4

Numerical simulations.

In the preceding chapter we have dealt with the question of the steady states of the system, which is relevant when studing the behaviour of the solution trajectories on the long run. In order to take a glance into the transient phase we now turn on system (1.8) and, following ([11, 16]) write a numerical algorithm based on a first-order implicit finite difference method along the characteristics for such system. We then utilize this algorithm to perform simulations on some examples of diffusion of an infectious disease in a population under the demographical assumptions till now adopted, that is with resident population in BRF and subject to immigration.

But before beginning in dealing with simulations, we must spend some time with the force of the infection.

4.1 Force of infection and the WAIFW matrix.

Let us remember that we choosed, following [2, 10, 19], as for the force of infection the functional form (1.2):

$$\lambda(a,t) = \int_0^\omega \beta(a,s) Y(s,t)\, ds \; . \tag{4.1}$$

in which age structure is taken into account by mean of the contact coefficient $\beta(a,s)$. Then now it raises the question of how to choose a convenient form for the $\beta(\cdot,\cdot)$ that fit our problem. What is nowaday usually done, as pointed out in [2], is to divide the population into n discrete age classes I_i, $i = 1 \dots n$, and to set

$$\beta(a,a') = \beta_{ij} \;\text{ iff }\; (a,a') \in \; I_i \times I_j \; ;$$

hence the transmission coefficient is represented by the $n \times n$ matrix $\{\beta_{ij}\}$, with β_{ij} the probability for an infective individual of age a' to have an effective contact with a susceptible of age a, per unit time. Such matrix is called WAIFW ("who acquires infection from whom") matrix.
Then we can write the force of infection $\lambda(a,t)$ as:

$$\lambda(a,t) = \int_0^\omega \beta(a,s)Y(s,t)\,ds = \sum_{i=1}^{n} \mathbf{1}_{I_i}(a)\lambda_i(t) \tag{4.2}$$

with $\mathbf{1}_{I_i}$ the indicator function of the i-th age class and $\lambda_i(t)$ the force of infection of the i-th age class at time t:

$$\lambda_i(t) = \sum_{j=1}^{n} \beta_{ij} \int_{I_j} Y(s,t)ds \ ,$$

which represents the proportion of the susceptibles in the i-th age class that become infective in the time unit (from a phisical point of view it has the dimension of a frequency).

The subsequent problem it raises is how to choose such matrix. It is known that is not an easy task to establish the value of the contact probabilities β_{ij}. The contact process is more obscure than reporting new cases classified by age. Then the data usually available is the vector $\{\lambda_i\}$, $i = 1 \dots n$, with λ_i the number of new cases of infection in the i-th age class per time unit, known by reporting age-structured data. The problem of determinig the $n(n+1)/2$ elements β_{ij} (remember that $\beta(a,a') = \beta(a',a)$, hence the $\{\beta_{ij}\}$ matrix is symmetric) from the known λ_i, $i = 1 \dots n$, was solved firstly by Schentzle in [19] by constructing the WAIFW matrix in such a way that it has only n distict entries.
An age-class subdivision, suitable for our purposes of analyzing the evolution of a childhood infectious disease, is $[0,3)$, $[3,6)$, $[6,11)$, $[11,19)$, $[19,\omega)$ as adopted in [13, 15, 9]; and as the WAIFW matrix we choose the default mixing matrix of [13, 15], properly conceived for measles, and "which assigns a dominant role to transmission in school and pre-school ages":

$$\begin{pmatrix} \beta_1 & \beta_1 & \beta_1 & \beta_1 & \beta_5 \\ \beta_1 & \beta_2 & \beta_4 & \beta_4 & \beta_5 \\ \beta_1 & \beta_4 & \beta_3 & \beta_4 & \beta_5 \\ \beta_1 & \beta_4 & \beta_4 & \beta_3 & \beta_5 \\ \beta_5 & \beta_5 & \beta_5 & \beta_5 & \beta_5 \end{pmatrix} \tag{4.3}$$

4.2 A numerical algorithm.

Let $\Delta t > 0$ be the age-time discretization parameter and let us X_j^n,Y_j^n,Z_j^n be

an approximation of $X(j\Delta t, n\Delta t)$, $Y(j\Delta t, n\Delta t)$, $Z(j\Delta t, n\Delta t)$ respectively, with j,n integer such that $n \geq 0$, $0 \leq j \leq \Omega$, $\Omega = \left[\frac{\omega}{\Delta t}\right]$ (in practice the usual choice for the parameter Δt is $1/2^k$, $k = 0, 1, 2 \ldots$).
To move from time step j to $j+1$ we solve explicitly the linear part of the equation from time t to time $t + \Delta t$, that is we rewrite (1.8) as:

$$\begin{cases} \frac{d}{dh}X(a+h,t+h) &= -\big(\lambda(a+h,t+h)+\mu(a+h)\big)X(a+h,t+h)+I_X(a+h) \\ \frac{d}{dh}Y(a+h,t+h) &= \lambda(a+h,t+h)X(a+h,t+h)-\big(\gamma+\mu(a+h)\big)Y(a+h,t+h)+I_Y(a+h) \\ \frac{d}{dh}Z(a+h,t+h) &= \gamma Y(a+h,t+h)-\mu(a+h)Z(a+h,t+h)+I_Z(a+h) \end{cases}$$

for $a \in (0, \omega)$, $t > 0$ and $h \in \mathbb{R}$ such that $a + h \in (0, \omega)$, $t + h > 0$. Such system has solution:

$$\begin{aligned} X(a+h,t+h) &= X(a,t)e^{-\int_0^h(\lambda(a+\tau,t+\tau)+\mu(a+\tau))d\tau}+ \\ &\quad +\int_0^h I_X(a+s)e^{-\int_s^h(\lambda(a+\tau,t+\tau)+\mu(a+\tau))d\tau}ds \\ Y(a+h,t+h) &= Y(a,t)e^{-\gamma h-\int_0^h\mu(a+\tau)d\tau}+ \\ &+\int_0^h\big(\lambda(a+s,t+s)X(a+s,t+s)+I_Y(a+s)\big)e^{-\gamma(h-s)-\int_s^h\mu(a+\tau)d\tau}ds \\ Z(a+h,t+h) &= Z(a,t)e^{-\int_0^h\mu(a+\tau)d\tau}+ \\ &+\int_0^h\big(\gamma Y(a+s,t+s)+I_Z(a+s)\big)e^{-\int_s^h\mu(a+\tau)d\tau}ds \end{aligned} \tag{4.4}$$

We now set $h = \Delta t$, $a = j\Delta t$, $t = n\Delta t$; and further:

$$\mu_j = \mu(j\Delta t)\,,\ \Lambda_j^n = \lambda(j\Delta t, n\Delta t)\,,$$

$$I_{X,j} = I_X(j\Delta t)\,,\ I_{Y,j} = I_Y(j\Delta t)\,,\ I_{Z,j} = I_Z(j\Delta t)\ for\ j = 0,1,\ldots,\Omega-1\,,\ n = 0,1,2,\ldots$$

and we write the following discretization scheme for the first two equation in (4.4):

$$
\begin{aligned}
X_{j+1}^{n+1} &= X_j^n\, e^{-(\Lambda_j^n+\Lambda_{j+1}^{n+1})\frac{\Delta t}{2}}\, e^{-(\mu_j+\mu_{j+1})\frac{\Delta t}{2}} + \\
&\quad + \frac{\Delta t}{2}\big(I_{X,j} e^{-\Lambda_j^n \Delta t} e^{-(\mu_j+\mu_{j+1})\frac{\Delta t}{2}} + I_{X,j+1}\big) \\
Y_{j+1}^{n+1} &= Y_j^n\, e^{-\gamma\Delta t} e^{-(\mu_j+\mu_{j+1})\frac{\Delta t}{2}} + \\
&+ \frac{\Delta t}{2}\Big(\big(\Lambda_j^n X_j^n + I_{Y,j}\big) e^{-\gamma\Delta t} e^{-(\mu_j+\mu_{j+1})\frac{\Delta t}{2}} + \Lambda_{j+1}^{n+1} X_{j+1}^{n+1} + I_{Y,j+1}\Big)
\end{aligned}
\tag{4.5}
$$

where to compute the integrals we use a second order quadratura formula (trapezes formula). But this scheme is not satisfactory in terms of time computing for at each time step a system of non linear equations should be solved; this in view of the fact that Λ_j^n is updated at each time step by mean of the formula:

$$
\Lambda_j^n = \sum_{k=0}^{\Omega} \tilde{\beta}_{jk}\, Y_k^n\, \Delta t \tag{4.6}
$$

with

$$
\tilde{\beta}_{jk} = \beta_{\bar{j}\bar{k}} \;\; if \;\; j\Delta t \in I_{\bar{j}}\,,\; k\Delta t \in I_{\bar{k}}\,,\; \bar{j},\bar{k} = 1,2,\ldots,N
$$

(here N is the number of age classes). Hence, with the (4.6) updating of the force of infection, the algorithm becomes recursive. To avoid such complication, in (4.5) we substitute Λ_{j+1}^{n+1} with Λ_j^n; in approximating $\int_0^h (\lambda(a+\tau, t+\tau)d\tau$ this corresponds to use of a first order quadratura formula (left point formula). Our scheme to approximate solutions of (1.8) is then:

$$
\begin{aligned}
X_{j+1}^{n+1} &= X_j^n\, e^{-\Lambda_j^n\Delta t} e^{-(\mu_j+\mu_{j+1})\frac{\Delta t}{2}} + \\
&\quad + \frac{\Delta t}{2}\big(I_{X,j} e^{-\Lambda_j^n \Delta t} e^{-(\mu_j+\mu_{j+1})\frac{\Delta t}{2}} + I_{X,j+1}\big) \\
Y_{j+1}^{n+1} &= Y_j^n\, e^{-\gamma\Delta t} e^{-(\mu_j+\mu_{j+1})\frac{\Delta t}{2}} + \\
&+ \frac{\Delta t}{2}\Big(\big(\Lambda_j^n X_j^n + I_{Y,j}\big) e^{-\gamma\Delta t} e^{-(\mu_j+\mu_{j+1})\frac{\Delta t}{2}} + \big(\Lambda_j^n X_{j+1}^{n+1} + I_{Y,j+1}\big)\Big) \\
Z_{j+1}^{n+1} &= Z_j^n\, e^{-(\mu_j+\mu_{j+1})\frac{\Delta t}{2}} + \\
&+ \frac{\Delta t}{2}\Big(\big(\gamma Y_j^n + I_{Z,j}\big) e^{-\gamma\Delta t} e^{-(\mu_j+\mu_{j+1})\frac{\Delta t}{2}} + \gamma Y_{j+1}^{n+1} + I_{Z,j+1}\Big)\,.
\end{aligned}
\tag{4.7}
$$

4.3 A look at the italian situation.

We now resort to simulations to enlight some features of the epidemiological model (1.8) under two main demographical assumptions:

(a) the SPI model context, that is we take as initial age profile that furnished from the equilibrium (1.6) of (1.5) that is we observe the evolution of the disease in the long term population result of the prosecution of the actual vital rates, in particular fertility below replacement, and stabilized by means of a constant migration inflow.

Long term age-profile typically employs a long period of time (more than one hundred years) to emerge; then we do the following too:

(b) we take the actual (2004) italian age profile as initial conditions to have a look in the transient phase.

Simulations are performed by numerically solving system (1.8) by mean of the algorithm (4.7).

Let us now do some assumptions on demographical and epidemiological rates and parameters in the model:

- Let us remeber that we assumed that immigrants suddenly acquire the same vital rates of the natives, that is fertility and mortality rates; and that our model is a one-sex model, only the female part of the population is considered. Hence as for the mortality rate we adopt that corresponding to the female italian population, observed in the year 2004; and as for the fertility rate we take the italian datum observed during the years 1996-2000. With these vital rates the net reproduction rate (the number of newborns an individual is expected to produce during his reproductive life) is:

$$R = \int_0^{\omega} \beta(a)\Pi(a)da \cong 0.59 \tag{4.8}$$

  (that is largely under the repalcement level $R = 1$).

- The contact pattern between individuals is determined by choosing the (4.3) WAIFW matrix, with the entries determined by means of the italian force of infection in the period preceding the beginning of vaccination and of the immigration; these fcts almost coincides: italian vaccination programme began in 1976, immigration in Itly began in the eighties)

- We assume that at the beginning of simulations measles is at the endemic state, for the resident population, furnished from the equilibrium of the SIR model without immigration and constant force of infection:

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) X &= -(\lambda + \mu(a))X \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) Y &= \lambda X - (\mu(a) + \gamma)Y \qquad \omega > a > 0\,,\, t > 0 \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) Z &= \gamma Y - \mu(a) Z \end{cases} \tag{4.9}$$

where λ it corresponds to the reciprocal of the average age at infection and is supposed different in the two populations (presumebly higher for the immigrants). By rescaling in (4.9) as did for (2.2), the equilibrium system of equations is:

$$\begin{cases} \frac{d}{da}x(a) &= -\lambda\, x(a) \\ \frac{d}{da}y(a) &= \lambda\, x(a) - \gamma\, y(a) \qquad 0 < a < \omega \\ \frac{d}{da}z(a) &= \gamma\, z(a) \\ \\ x(0) = 1 \;\;,\;\; y(0) = 0 \;\;,\;\; z(0) = 0 \end{cases} \tag{4.10}$$

whose solutions are given by:

$$\begin{aligned} x(a) &= e^{-\lambda a} \\ y(a) &= \frac{\lambda}{\gamma - \lambda}\left(e^{-\lambda a} - e^{-\gamma a}\right) \\ z(a) &= \frac{\gamma}{\gamma - \lambda}\left(1 - e^{-\lambda a}\right) - \frac{\lambda}{\gamma - \lambda}\left(1 - e^{-\gamma a}\right) \end{aligned} \tag{4.11}$$

Hence as initial condition we take:

$$X_0(a) = x(a)\, n(a) \;\;,\;\; ,Y_0(a) = y(a)\, n(a) \;\;,\;\; Z_0(a) = z(a)\, n(a) \tag{4.12}$$

$\big($notice that $1 = x(a) + y(a) + z(a)\ \forall\ a \in (0, \omega)\ \big)$ where $n(a)$ is the equilibrium age profile (1.6) of the population equation with immigration (1.5) if we are in the setting of the SPI model; or it is the italian (female) age-profile as given from ISTAT at 1/1/04 if we are looking at the transient phase.

- We assume that I immigrants enter the resident population in the unit time and take a relative age profile obtained by adapting the so called "double exponential curves" (as done in [17]) to observed migration data in Italy in the $'90^s$.
  Further, at the moment of their arrival, they are assumed in the same equilibrium situation given by (4.11) with respect to measles but with a higher (constant) λ, representing the fact they come from countries with higher force of infection and lower average age at infection. As for the fraction $x_i(\cdot)$, $y_i(\cdot)$, $z_i(\cdot)$ between the epidemiological classes hence we set:

$$I_X(a) = I\, x_i(a) \ \ , \ \ I_Y(a) = I\, y_i(a) \ \ , \ \ I_Z(a) = I\, z_i(a) \tag{4.13}$$

- As for γ, the reciprocal of the average duration of infection, we take $\gamma = 52.0\, year^{-1}$ (that is, duration of infection $\cong$ $1\, week$).

4.3.1 Long term age profiles.

Under various assumptions on the immigrants, as age profile and total number of yearly entries, we now look at the long term age profile assumed from the resident population in each case. We take two age profiles (given in figure 4.1) of the immigrants in the form $I f(a)$, where $f(a)$ is a relative age profile obtained by adapting the so-called double exponential curves of Rogers and Castro as in [17], to immigration data observed in Italy in the ninties; these relative age profiles have the same shape with the difference that peak at the age 21 under the "younger" hypothesis (call it PR1) and at the age 31 under the "older" hypothesis (call it PR2).

And as the annual number of entries we do three hypothesis. In the ninties the annual number of female entries was about I=25000, as reported in [14], we call such number $I1$; but in the last years such number has quite raised, so we do also the "middle" hypothesis $I = 50000$ (that we call $I2$) and the "big" hypothesis $I = 100000$ (that we call $I3$).
In [14] was noticed that the $I1$ yearly number of entries is very far from being a remedy to ageing of italian population caused by the prolonged BRF situation. In figure 4.2 we see the actual and the long term italian age profiles assumed under the opposite assumptions $I1$-PR2 and $I3$-PR1

where it is evident that even in the most optimistic assumption $I3$-PR1 italian population would undergo a substantial decrease; and with a quite old shape, in spite of the "rejuvenating effect" ([14]) caused by immigration. In figure 4.3 is shown an example of some phases of the transition to the long term age profile (notice that typically such age profiles employ quite long periods to be approached, more than 200 years).

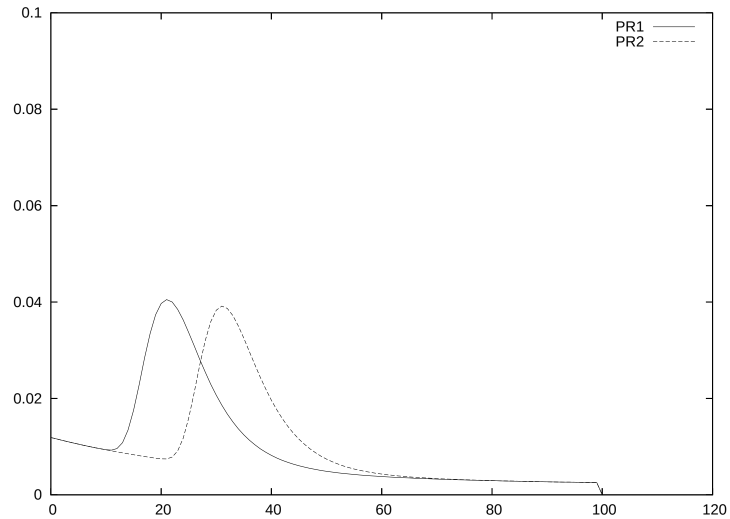

Figure 4.1: The two assumed (page 65) relative age profiles for the immigrants.

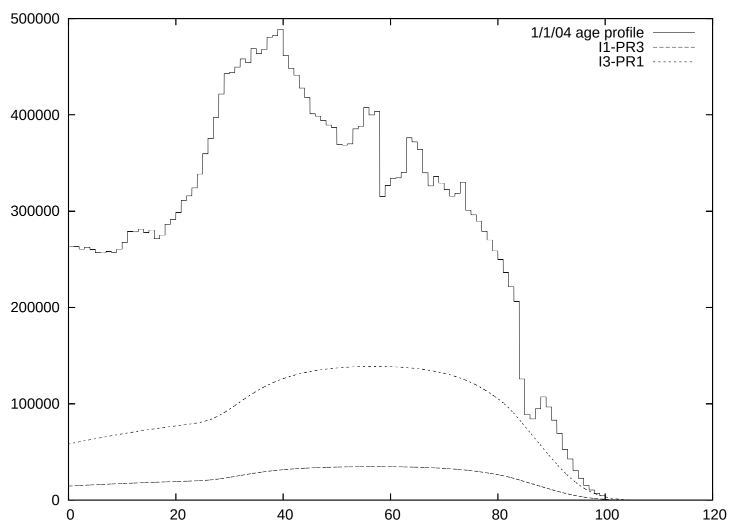

Figure 4.2: The age profiles of the actual female italian population ang the long term age profiles assumed under $I3$-pr1 and $I1$-pr3 ipothesys.

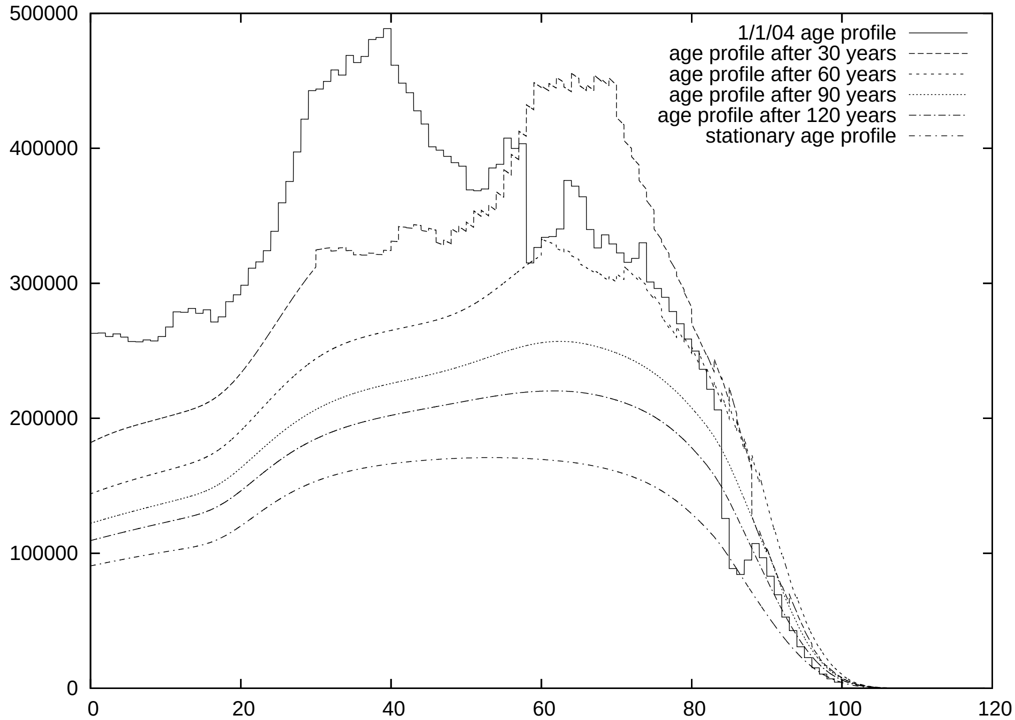

Figure 4.3: The current age profile and the age profiles that italian population will undergo in 30,60,90,120 years with the current vital rates and immigrant age-profile I3-PR1 (page 65); and the stationary age profile under I3-PR1.

4.3.2 Long term behaviour of the infection.

Aging of the population sure affects transmission of a disease like measles: raising of the average age of the population has the effect of lowering force of infection, due to the characteristic feature of measles transmission patterns of having higher coefficients at younger ages.

Remark. Aging is not the only way to lower force of infection: remember that as FOI was adopted the functional form (1.2)

$$\lambda(a,t) = \int_0^\omega \beta(a,s) Y(s,t)\, ds \ . \tag{4.14}$$

this FOI has the drawback that when population decreases, the FOI also will be lowered. It is ever reasonable such an effect? A functional form suitable to make the FOI less sensitive to demographical effects is

$$\lambda(a,t) = \int_0^\omega \beta(a,s) \frac{Y(s,t)}{n(s,t)}\, ds \ . \tag{4.15}$$

as studied in [15].

A characteristic feature of SIR models is the existence of thresholds, as indicated for example in [7, 6, 10, 3]. In SIR model with immigration, following [3], we speak of threshold-like results; following [6, 3], in section 3.4 we characterized the limiting behaviour (as the fraction immigrant approaches zero) of solutions of the operator equations for the study of the steady states, in terms of the spectral radius of a positive linear operator. We now show some examples to have a look at this behaviour: the total number of infectives on the long term under the various demographical assumptions made on the immigrants.

Figures 4.4-4.9 are put in order of increasing yearly number of entries and with the older profile before. All cases share a quite long initial phase characterized by oscillatory behaviour, which means that population is experiencing periodic epidemics. In figures 4.4,4.5,4.6 the trajectory looks becoming zero, but numerically speaking is not so: it stays at values below one which is meaningless if we are dealing with individuals. Presumebly we are under threshold ($r(T) < 1$), the few infected immigrant cannot induce an epidemic. The FOI has reached so little values that the burden of morbidity embodied in the new infected few entries cannot be employed; they recover and have very low chances to transmit the disease to others.
In figure 4.7 this trend changes. The eventual FOI is strong enough that the newly arrived infectives can transmit the disease to a number of individuals greater than the number of the infectors: a threshold value has been crossed.
Anyway, the forecast of the model is that the decrease of the population, caused by the prolonged BRF situation, causes a substantial decrease of the incidence

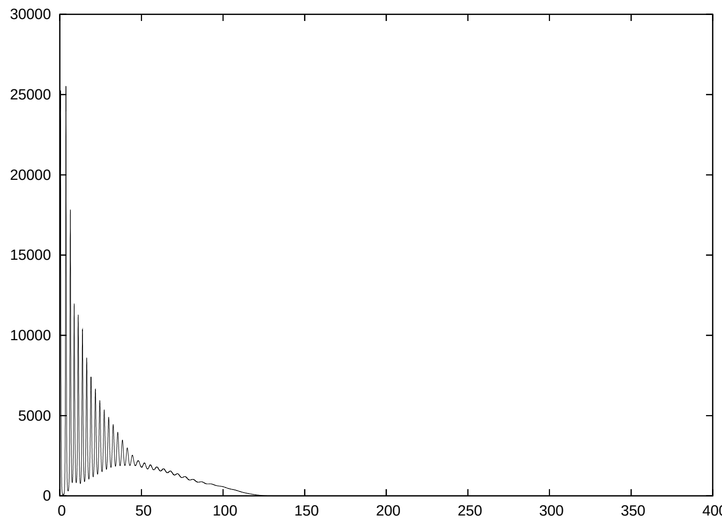

Figure 4.4: Long term behaviour of total number of infectives under demographical assumption I1-PR2 (page 65) on immigrants.

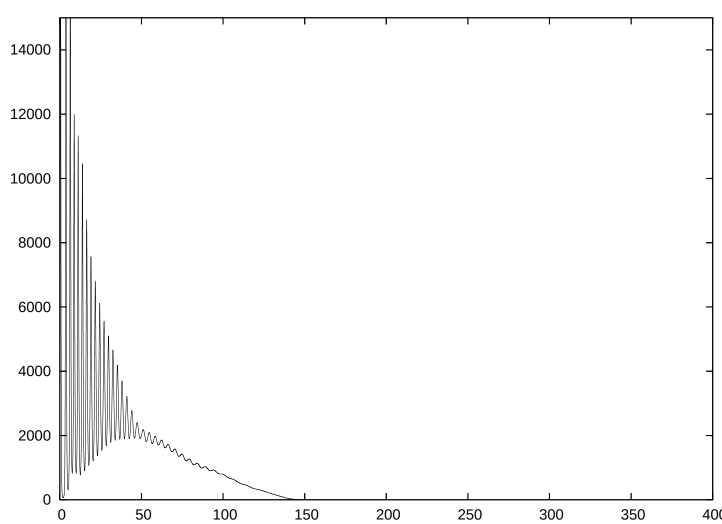

Figure 4.5: Long term behaviour of total number of infectives under demographical assumption I1-PR1 (page 65) on immigrants.

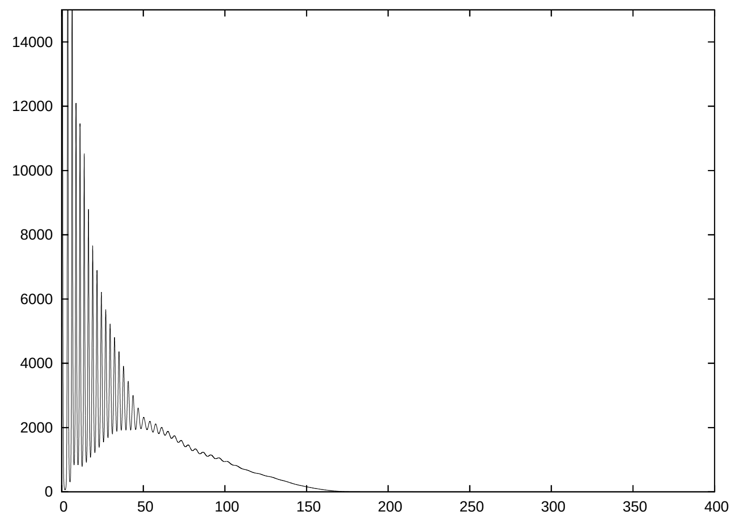

Figure 4.6: Long term behaviour of total number of infectives under demographical assumption I2-PR2 (page 65) on immigrants.

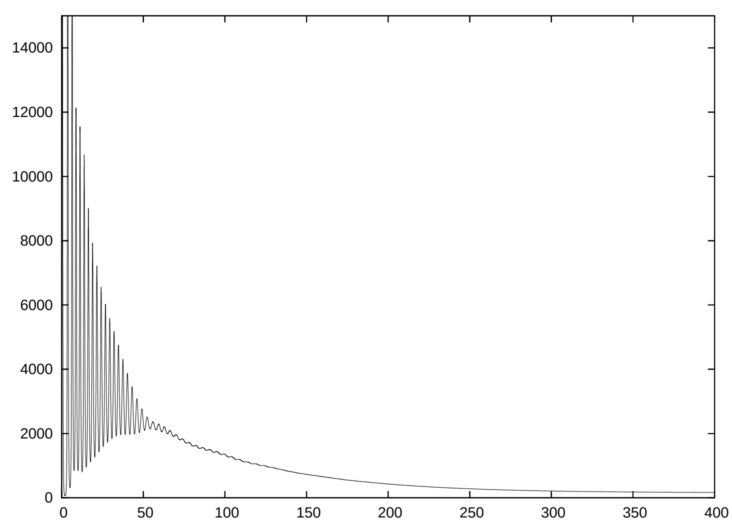

Figure 4.7: Long term behaviour of total number of infectives under demographical assumption I2-PR1 (page 65) on immigrants.

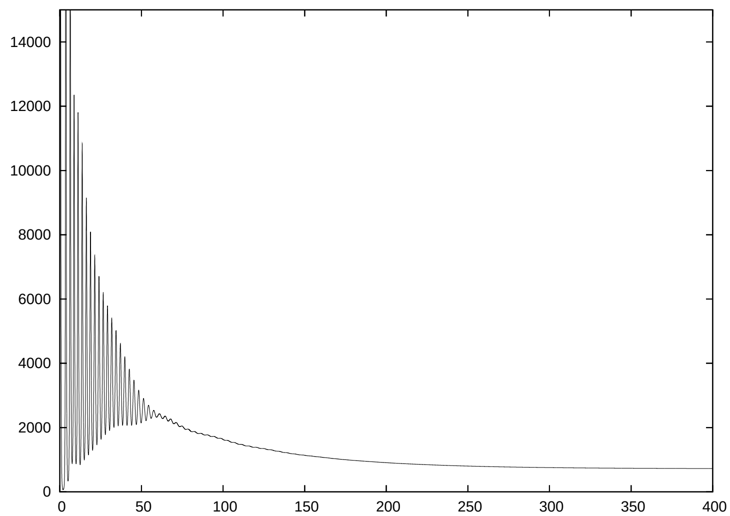

Figure 4.8: Long term behaviour of total number of infectives under demographical assumption I3-PR2 (page 65) on immigrants.

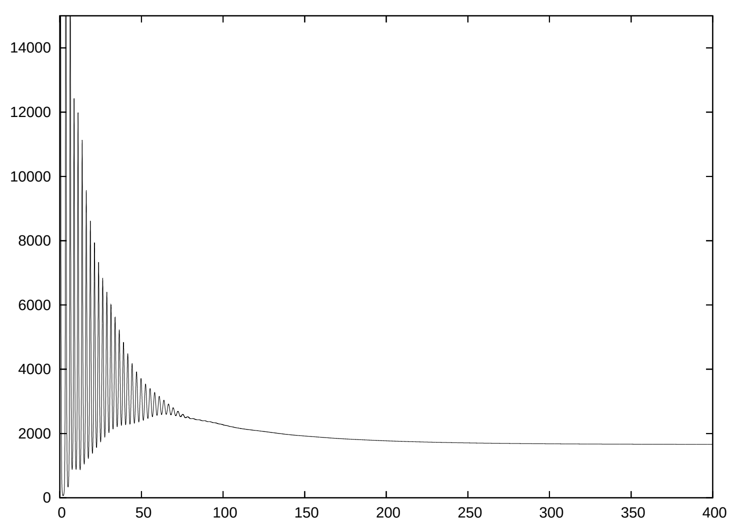

Figure 4.9: Long term behaviour of total number of infectives under demographical assumption I3-PR1 (page 65) on immigrants.

of the disaese.

4.3.3 Long term average age at infection.

Aging of the population and the consequent decline in the FOI affects also a relevant epidemiological parameter, that is the average age at infection. Average age at infection at time t $A(t)$ is given, for the SIR model represented by system (1.8), as in [2], by:

$$A(t) = \frac{\int_0^\omega a\,\lambda(a,t)X(a,t)da}{\int_0^\omega \lambda(a,t)X(a,t)da} \tag{4.16}$$

Remark. In the case considered we have $\gamma \gg \lambda(a,t)\ \forall\, a \in (0,\omega)\ \ \forall\, t > 0$; it is then possible to place us in the approximation ”short infection” ([2]) and rewrite (4.16) as:

$$A(t) = \frac{\int_0^\omega a\,Y(a,t)da}{\int_0^\omega Y(a,t)da} \tag{4.17}$$

as a consequence of the fact that we have:

$$Y(a,t) \backsimeq \frac{\lambda(a,t)X(a,t)}{\gamma}\ . \tag{4.18}$$

The expression (4.17) is that used to calculate $A(t)$ in the simulations. The ”short infection” approximation is reasonable in view of the fact we have $\gamma = 52$ and $\lambda(a,t) \leq 10^{-4}\ \ \forall a, t$.

Under each of the six different demographical situation we have considered for the immigrants, what is observed is a more or less marked increase of $A(t)$, as shown in figures 4.10,4.11. However it should be stressed the fact that in cases I1-PR1,I1-PR2,I2-PR2 it does not make much sense calculation of $A(t)$ when the total amount of infectives has reached a value below one (this happens more or less around the peak shown by such $A(t)$ trajectories).

Again, this fact is the consequence of the decrease of the population which, making smaller the term at denomintor in (4.16), magnifies $A(t)$.

We note that (even if it has not been an argument faced in this thesis), the solutions of (1.8) show an asymptotically stable behaviour, as suggested by figures 4.12,4.13 4.14

This on the long term: in the transient phase these solutions can assume very different values, as seen in figures 4.15,4.16 4.17

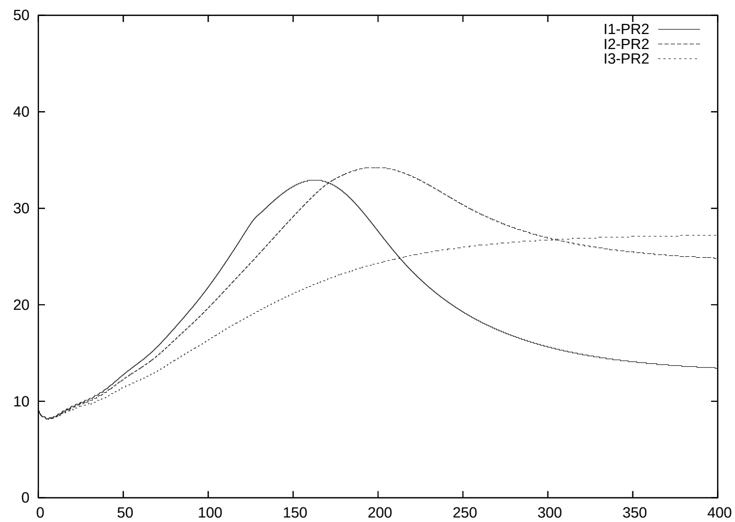

Figure 4.10: The average ages at infection under the demographical assumptions I1,I2,I3-PR2.

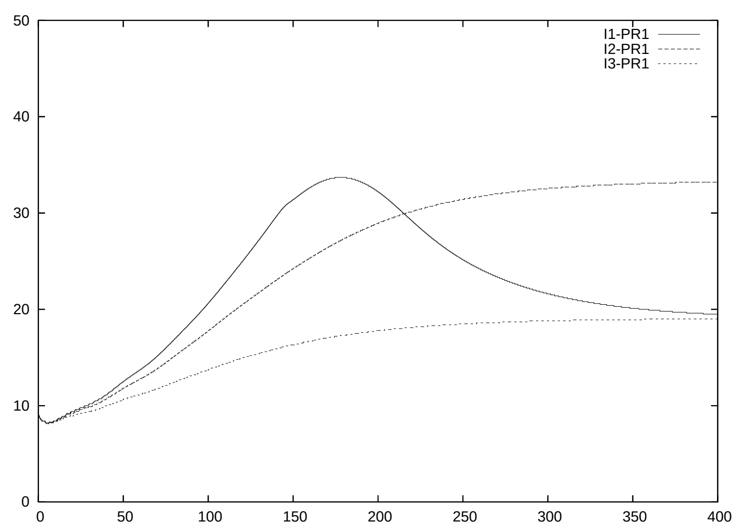

Figure 4.11: The average ages at infection under the demographical assumptions I1,I2,I3-PR1.

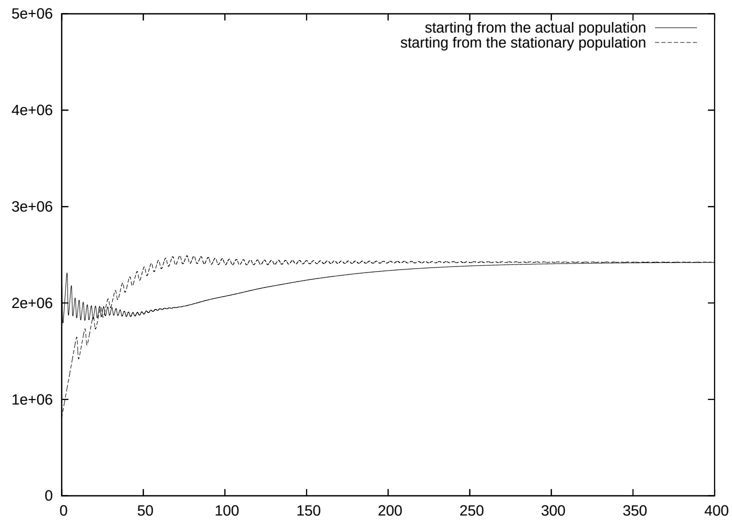

Figure 4.12: Long term behaviour of total number of susceptibles with initial condition the actual and the stationary population under the demographical assumption I3-PR1.

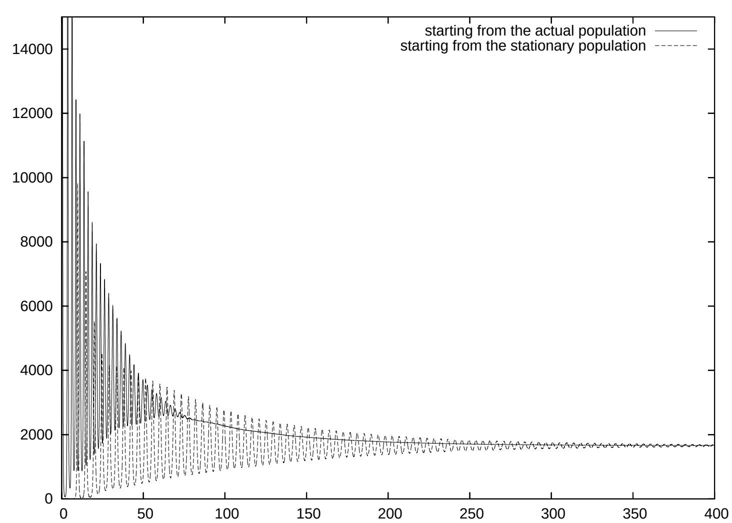

Figure 4.13: Long term behaviour of total number of infectives with initial condition the actual and the stationary population under the demographical assumption I3-PR1.

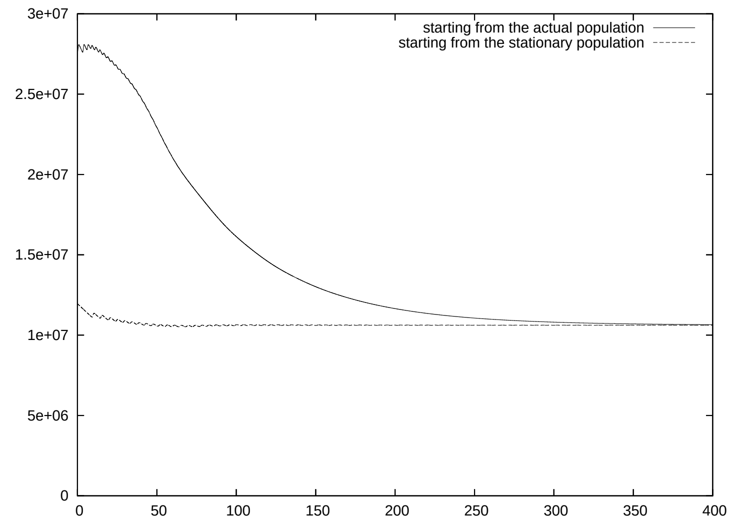

Figure 4.14: Long term behaviour of total number of removeds with initial condition the actual and the stationary population under the demographical assumption I3-PR1.

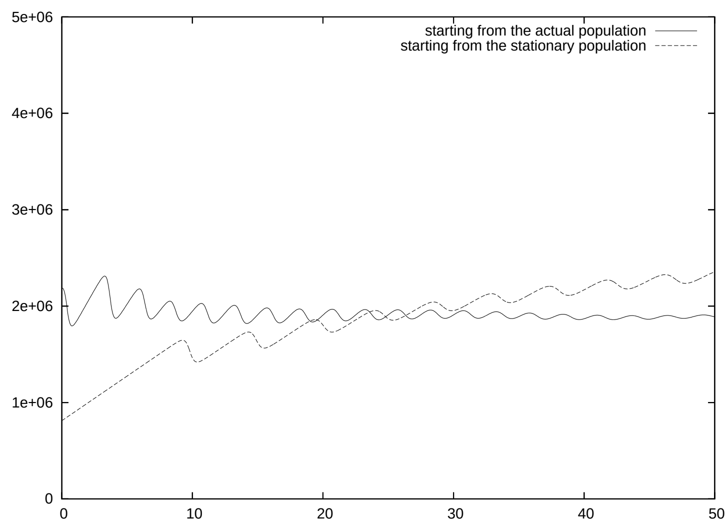

Figure 4.15: Long term behaviour of total number of susceptibles with initial condition tha actual and the stationary population under the demographical assumption I3-PR1.

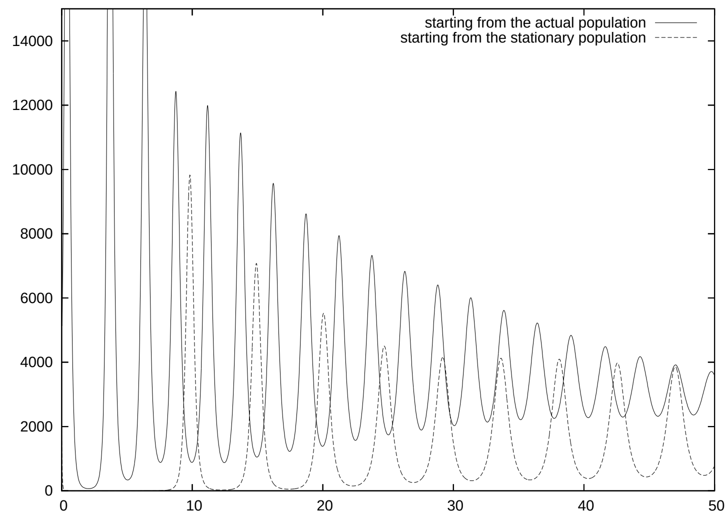

Figure 4.16: Long term behaviour of total number of infectives with initial condition the actual and the stationary population under the demographical assumption I3-PR1.

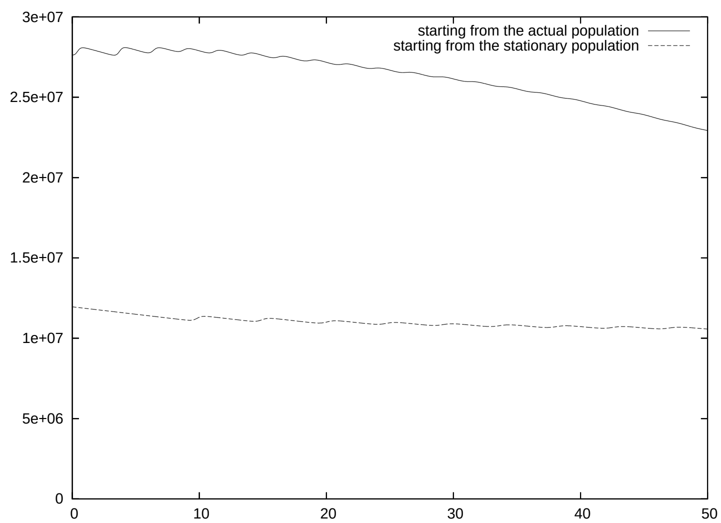

Figure 4.17: Long term behaviour of total number of removeds with initial condition the actual and the stationary population under the demographical assumption I3-PR1.

Bibliography

[1] H. Amann. On the number of solutions of nonlinear equations in ordered banach spaces. *Journal of functional analysis*, 11:346–384, 1972.

[2] Roy M. Anderson and Robert M. May. *Infectious diseases of humans: dynamics and control*. Oxford University Press, Oxford New York Tokio, 1991.

[3] F. Brauer and P.van den Driessche. Models for transmission of disease with immigration of infectives. *Mathematical Biosciences*, 171:143–154, 2001.

[4] H. Brezis. *Analisi funzionale: teoria e applicazioni*. Liguori Editore, Napoli, 1986.

[5] P. Cerone. On stable population theory with immigrations. *Demography*, 24/3:p. 431–438, 1987.

[6] D. Greenhalgh. Threshold and stability results for an epidemic model with an age structured meeting-rate. *IMA Journal of Mathematics applied in Medicine and Biology*, 5:81–100, 1988.

[7] H.W. Hethcote. Three basic epidemiological models. In Springer-Verlag, editor, *Applied mathematical ecology*, volume 18 of *Biomathematics Texts*, pages 119–144. S.A. Levin and T.G. Hallam and L.J. Gross, 1989.

[8] M. Iannelli. *Mathematical theory of age-structured population dynamics*. Giardini editori e stampatori in Pisa, Pisa, 1994.

[9] M. Iannelli and P. Manfredi. Demographic change and immigration in age-structured epidemic models. *preprint*, 2006.

[10] H. Inaba. Threshold and stability results for an age-structured epidemic model. *Journal of mathematical biology*, 28:p. 411–434, 1990.

[11] J.Douglas and F.A.Milner. Numerical methods for a model of population dynamics. *Calcolo*, 24:247–254, 1987.

[12] M.G. Krein and M.A. Rutman. Linear operators leaving invariant a cone in a banach space. *AMS translations*, 10:199–325, 1950.

[13] P. Manfredi et al. The pre-vaccination regional epidemiological landscape of measles in italy: contact patterns, effort needed for eradication, and comparison with other regions of europoe. *Population Health Metrics*, 3(1), 2005.

[14] P. Manfredi and A. Valentini. Populations with below replacement fertility and immigration: theoretical considerations and scenarios from the 'italian laboratory'. *Riv. Int. Sci. Soc.*, 108(1):61, 2000.

[15] P. Manfredi and John R. Williams. Realistic population dynamics in epidemiological models: the impact of population decline on the dynamics of childhood infectious dieases. measles as an example. *Mathematical Biosciences*, 192:153–175, 2004.

[16] M.Iannelli, F.A.Milner, and A.Pugliese. Analytical and numerical results for the age-structured s-i-s epidemic model with mixed inter-intracohort transmission. *SIAM J.MATH.ANAL.*, 23(3):662–688, 1992.

[17] A. Rogers and L.J. Castro. Model migration schedules. In *Research Report*, volume 30, International Institute for Applied System Analysis, Luxemburg, 1981.

[18] I. Sawashima. On spectral properties of some positive operators. *Nat. sci. report Ochanomizu Univ.*, 15:53–64, 1964.

[19] D. Schentzle. An age structured model of pre- and post-vaccination measles transmission. *IMA J.Math.Appl.Med.Biol.*, 1:169–191, 1984.

[20] G.F. Webb. *Theory of nonlinear population dynamics*. Marcel Dekker, New York and Basel, 1985.

[21] E. Zeidler. *Nonlinear functional analysis and its applications: fixed point theorems*, volume 1. Springer Verlag, New York Berlin Heidelberg Tokio, 1985.