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Quantum Field Theories in Curved Spacetime

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1. Generalities of Quantum Field Theory in Curved Spacetime. Quantum Field Theory in Curved Spacetime (QFTcs) is a generalization of Quantum Field Theory (QFT) in Minkowski spacetime under the assumptions that: (a) the metric g of the curved background spacetime M is a fixed classical field and (b) the field ϕ propagating in M is quantized. We shall describe first the concrete constructions and later we shall address more conceptual issues.
2. Basics on algebraic approach. The picture in curved spacetime changes dramatically in comparison with QFT in Minkowski spacetime. First the absence of Poincaré symmetry and, in general, of any continuous isometry, leads to the absence of a preferred, symmetry invariant, reference quantum state on a hand, on the other hand it implies the lack of symmetry-invariant notion of elementary particle. Secondly, appearance of unitarily *inequivalent* representations of the field operators algebra generally occurs (see the section 5 below). To tackle those features, in particular to treat the various unitary inequivalent representations of observables on the same footing, the most useful framework is the algebraic approach. In that framework, the observables are given by the self-adjoint elements of a suitable unital $*$-algebra, $\mathfrak{A}$. To some extent it is sometimes convenient to assume a stronger structure over $\mathfrak{A}$, that of a unital C^* algebra. In both cases an algebraic state is a linear functional $\omega : \mathfrak{A} \to \mathbb{C}$ which is positive (i.e. $\omega(a^*a) \geq 0$ for all $a \in \mathfrak{A}$) and normalized (i.e. $\omega(\mathbb{I}) = 1$, $\mathbb{I}$ being the unit of $\mathfrak{A}$). The celebrated GNS-reconstruction theorem proves that, for every such ω there is a triple (the GNS triple), $(\mathfrak{H}_\omega, \pi_\omega, \Psi_\omega)$, where $\mathfrak{H}_\omega$ is a complex Hilbert space, π_ω is a unit-preserving representation of $\mathfrak{A}$ in terms of linear operators over $\mathfrak{H}_\omega$ and $\Psi_\omega \in \mathfrak{H}_\omega$ is cyclic, i.e. $\pi_\omega(\mathfrak{A})\Psi_\omega$ is dense in $\mathfrak{H}_\omega$. Finally it holds $\langle \Psi_\omega | \pi_\omega(a) \Psi_\omega \rangle = \omega(a)$ for all $a \in \mathfrak{A}$, $\langle \cdot | \cdot \rangle$ denoting the scalar product in $\mathfrak{H}_\omega$. $(\mathfrak{H}_\omega, \pi_\omega, \Psi_\omega)$ is determined up to unitary transformations by $(\mathfrak{A}, \omega)$. (Operators $\pi_\omega(a)$ are defined on a common dense domain if $\mathfrak{A}$ is a $*$-algebra or, conversely, are bounded and everywhere defined when $\mathfrak{A}$ has the C^* structure. In the latter case π_ω is continuous and norm-decreasing referring to the uniform topology of the C^*-algebra of bounded linear operators on $\mathfrak{H}_\omega$.)
3. The simplest example: the real scalar field. Consider a real scalar field ϕ propagating in a smooth globally hyperbolic spacetime (M, g) (we use throughout the signature $-,+,+,+$) satisfying *Klein-Gordon equation* $P\phi = 0$ where $P := -\Box + V : C^\infty(M) \to C^\infty(M)$ denotes the Klein-Gordon operator. The symbol $\Box$ stands for the d'Alembert operator locally given by $\nabla_\mu \nabla^\mu$, ∇ representing the covariant derivative associated with the smooth metric g and

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$V \in C^\infty(M)$ being any fixed real function, in particular it can be fixed to be a constant $-m^2$ interpreted as minus the squared mass of the field. Now $\mathfrak{A}$ is generated by the unit $\mathbb{I}$ and formal field operators $\Phi(f)$ smeared with real functions $f \in C_0^\infty(M)$, satisfying:
(a) Linearity: i.e. $C_0^\infty(M) \ni f \mapsto \Phi(f)$ is linear,
(b) Field equation: i.e. $\Phi(Pf) = 0$ (Φ solves Klein-Gordon equation in distributional sense),
(c) Hermiticity: i.e. $\Phi(f) = \Phi(f)^*$,
(d) CCR: that is canonical commutations rules hold $[\Phi(f), \Phi(h)] = E(f \otimes h)\mathbb{I}$.
E is the *causal propagator*, i.e. the difference of the advanced and the retarded fundamental solutions of Klein-Gordon equation. The linear map $E : C_0^\infty(M) \to \mathcal{S}$ is onto, $\mathcal{S}$ being the real vector space of smooth solutions ψ of Klein-Gordon equation with compact support on Cauchy surfaces of M, and its kernel is made of the functions Pg with $g \in C_0^\infty(M)$. If $f \in C_0^\infty(M)$, $\psi = Ef$ is supported in the set $J(\text{supp} f)$ of the points of M which can be reached by future-directed or past-directed causal curves starting from the support of f. E is uniquely and globally defined on every globally hyperbolic spacetime, and enjoys suitable weak continuity properties which allow, as in (d) above, its interpretation as a distribution acting on functions of $C_0^\infty(M \times M)$ (see Dimock 1980 in [1]). There is a different approach to QFTcs for a free scalar field which is substantially, but not technically, equivalent to that sketched above. Consider the symplectic space $(\mathcal{S}, \sigma)$, σ being the non degenerate, isometry-invariant and Cauchy-surface independent symplectic form: $\sigma_M(\psi_1, \psi_2) := \int_\Sigma (\psi_2 \nabla_N \psi_1 - \psi_1 \nabla_N \psi_2)\; d\mu_g^{(S)}$, and Σ being any spacelike smooth Cauchy surface with unit normal future-pointing vector N and $d\mu_g^{(S)}$ the measure induced on Σ by the metric g. It is well known [1] that, for every real (nondegenerate) symplectic space, $(\mathcal{S}, \sigma)$ in our case, there is a unique – up to (isometric) $*$-isomorphisms – C^* algebra $\mathcal{W}(\mathcal{S}, \sigma)$ with nonvanishing generators $W(\psi)$, $\psi \in \mathcal{S}$, satisfying the following
Weyl relations: $W(-\psi) = W(\psi)^*$ and $W(\psi_1)W(\psi_2) = e^{i\sigma(\psi_1,\psi_2)/2} W(\psi_1 + \psi_2)$, $\forall \psi, \psi_1, \psi_2 \in \mathcal{S}$. $\mathcal{W}(\mathcal{S}, \sigma)$ is the *Weyl algebra* associated with $(\mathcal{S}, \sigma)$. Within this approach $\mathfrak{A} = \mathcal{W}(\mathcal{S}, \sigma)$ is the basic (C^*-) algebra of observables of the field ϕ. The Weyl generators $W(\psi)$ are formally understood as $W(\psi) := e^{-i\sigma(\Phi,\psi)}$, where $\sigma(\Phi, \psi)$ is the *field operator symplectically smeared* with elements of $\mathcal{S}(M)$. This interpretation makes mathematically sense in the GNS representations $(\mathfrak{H}_\omega, \pi_\omega, \Psi_\omega)$ of the states ω over $\mathcal{W}(\mathcal{S}, \sigma)$ which we call *regular*, i.e. those states such that $t \mapsto \pi_\omega(W(t\psi))$ is strongly continuous in $t = 0$ for every $\psi \in \mathcal{S}$. In this case $\sigma(\Phi_\omega, \psi)$ is the self-adjoint generator of $t \mapsto \pi_\omega(W(t\psi))$. If one defines $\Phi_\omega(f) := \sigma(\Phi_\omega, Ef)$, these fields fulfills (a),(b), (c) and (d) of the other approach as consequence of Weyl relations. *Quasifree states* are those that $\omega(\Phi(f)) = 0$ and the n-point functions $\omega(\Phi(f_1) \cdots \Phi(f_n))$ are completely determined via Wick expansion from the 2-point functions. A formally equivalent definition can be given dealing with Weyl approach [1]. The GNS triple of a quasifree state is a (symmetrized) Fock space representation, the cyclic vector being the vacuum vector [1]. This implies that quasifree state are regular [1]. If (M, g) admits a Killing vector ξ, the pull-back action on the functions of $C_0^\infty(M)$ (or respectively on the elements of $\mathcal{S}$) of the one-parameter group of isometries associated with ξ gives rise to a one-parameter group of $\mathfrak{A}$-automorphisms $\{\alpha_\tau\}_{\tau\in\mathbb{R}}$: every $\alpha_\tau : \mathfrak{A} \to \mathfrak{A}$ preserves the structure of $\mathfrak{A}$. If ξ is *timelike*, $\{\alpha_\tau\}_{\tau\in\mathbb{R}}$ may be interpreted as the set of (Killing) *time displacements*. In the general case *α-invariant states* are those that $\omega(\alpha_\tau a) = \omega(a)$ for all $a \in \mathfrak{A}$ and all $\tau \in \mathbb{R}$.

The most simple case of a (Killing) time displacement invariant state is a quasifree one. In that case the one-particle space of its GNS-Fock space is obtained by means of a generalization (due to Ashtekar and Magnon, and Kay see [1]) of Minkowskian QFT decomposition in positive and negative frequencies, now taken with respect to Killing-time displacements.
For the case of fields of different spin see [1] and references therein. It is worth mentioning that Verch has recently extended [2] the spin and statistics correspondence to general QFT in four dimensional curved spacetime using the notion of locally covariant quantum field theories we shall introduce later.
4. The emergence of a local algebraic structure. Also in the simple examples of a Klein-Gordon field the emergence of local structure is evident: the algebra $\mathfrak{A}$ can be organized in a class of sub-algebras $\{\mathfrak{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{I}}$, $\mathcal{I}$ being the set of open relatively-compact subsets of the globally hyperbolic spacetime (M,g). $\mathfrak{A}(\mathcal{O})$ is the $*$-algebra generated by field operators $\Phi(f)$, $f\in C_0^\infty(\mathcal{O})$ or, respectively, is the C^*-algebra generated by Weyl operators $W(Ef)$, $f\in C_0^\infty(\mathcal{O})$. Similarly to Haag-Kastler axioms in Minkowski spacetime, we have now the following properties.
Isotony : $\mathcal{O}\subset\mathcal{O}_1$ implies $\mathfrak{A}(\mathcal{O})\subset\mathfrak{A}(\mathcal{O}_1)$, $\forall\mathcal{O},\mathcal{O}_1\in\mathcal{I}$.
Causality: $[a,b]=0$ if $a\in\mathfrak{A}(\mathcal{O})$ and $b\in\mathfrak{A}(\mathcal{O}_1)$ if $\mathcal{O},\mathcal{O}_1\in\mathcal{I}$ are causally separed.
Time-Slice: Let $\mathcal{N}$ be any open neighborhood of any fixed Cauchy surface, $\forall\mathcal{O}\in\mathcal{I}$, $\exists\mathcal{O}_1\in\mathcal{I}$ with $\mathcal{O}_1\subset\mathcal{N}$ and $\mathfrak{A}(\mathcal{O})\subset\mathfrak{A}(\mathcal{O}_1)$. (There are also different formulations of this axiom.)
Isotony is consequence of support properties of E. The Time-Slice property in the form mentioned above was introduced by Dimock (see references in [1]) and it arises from the following result obtained by the properties of E [1] and properties of causal sets in globally hyperbolic spacetimes. Let $\mathcal{N}$ be as above, if $f\in C_0^\infty(M)$, there is $f_\mathcal{N}\in C_0^\infty(\mathcal{N})$ such that $Ef=Ef_\mathcal{N}$.
5. Well known results. Consider a globally-hyperbolic spacetime (M,g) which admits two timelike Killing vectors, one in the past region P of a Cauchy surface Σ_P and the other in the future region F of a Cauchy surface Σ_F, which lies in the future of Σ_P, and the timelike Killing vectors do not match each other in the intermediate region. Referring to a real scalar Klein-Gordon field ϕ in (M,g), two quasifree states ω_F and ω_R invariant under the respective Killing time displacements can be defined in the globally hyperbolic spacetimes $(F,g\restriction_F)$ and $(P,g\restriction_P)$ respectively. These states can actually be extended to the whole algebra $\mathfrak{A}$ on (M,g) due to the *Time-Slice property* mentioned above and it turns out that, generally speaking $\omega_F\neq\omega_P$. This phenomenon can easily be interpreted in terms of *spontaneous creation of future Fock-space particles from the past vacuum* ω_P due to the presence of gravitational classical field [1]. This is not the whole story. Referring to the GNS triples, it is possible to prove that [1] in the general case π_{ω_P} and π_{ω_F} are unitarily inequivalent representations: there is no unitary operator (a "S matrix") $U:\mathfrak{H}_{\omega_P}\to\mathfrak{H}_{\omega_F}$ such that $U\pi_{\omega_P}(\Phi(f))U^{-1}=\pi_{\omega_F}(\Phi(f))$ for all $f\in C_0^\infty(M)$, the same holds in terms of Weyl operators. In other words unitarily inequivalent representations of $\mathfrak{A}$ for the Klein-Gordon field may appear just due to the presence of the gravitational field.
In Minkowski spacetime, referring to a Minkowski frame (t,x,y,z) the (right) open *Rindler wedge* is defined by the condition $x>|t|$. $\mathbf{R}$ is globally hyperbolic and is invariant under the action of the one-parameter x-boost isometries whose Killing generator $B=a(x\partial_t-t\partial_x)$ is time-like in $\mathbf{R}$, $a>0$ being any fixed constant. The spacelike hypersurfaces in $\mathbf{R}$ normal to B may be interpreted as the common, synchronized, rest space of accelerating observers, with

acceleration a, whose histories are given by the integral lines of B. In coordinates adapted to the outlined structure, using the Killing parameter as time, the metric takes a form different from Minkowski. Obviously the metric is flat, but it simulates a curved metric due to the presence of inertial forces via equivalence principle. The algebra $\mathfrak{A}_{\mathbf{R}}$ associated with Klein-Gordon field propagating in $\mathbf{R}$ naturally identifies with a subalgebra of the algebra of Minkowski spacetime. Here the *Unruh effect* (see [1]) takes place. It was found out by Unruh – but also as explained by Sewell – by Bisognano and Wichmann who provided a rigorous proof of the phenomenon from an independent, apparently unrelated, mathematical point of view based on modular theory. The restriction to Minkowski vacuum ω_0 to $\mathbf{R}$, not only it is not the vacuum with respect to B-quantization procedure, but "accelerated observers in Minkowski vacuum feel themselves immersed in a thermal bath of particles": $\omega_0 \restriction_{\mathfrak{A}_{\mathbf{R}}}$ is a *thermal state* with respect to B-time displacement at the temperature $T_a = a/(2\pi)$. Thermal state means here that $\omega_0 \restriction_{\mathfrak{A}_{\mathbf{R}}}$ satisfies the KMS condition (see [1], KMS stands for Kubo-Martin-Schwinger) with respect the one-parameter group of automorphisms associated with B.
An important result achieved by Kay and Wald in 1991 (see [1]) is the following. Consider Kruskal spacetime (the result applies also to any globally hyperbolic spacetimes containing the structure of bifurcate Killing horizon generated by a global Killing field B as in Minkowski spacetime). Suppose that a state ω exists which is invariant under the one-parameter group of isometries associated with the Killing field B corresponding, in the two Schwarzschild wedges to Schwarzschild Killing time. If ω is Hadamard (see next section) and it is invariant under a discrete "wedge reflection" isometry (this is automatic in the analytic case), ω must be (a) unique an (b) thermal (i.e. KMS) with respect to B with temperature given by the famous *Hawking temperature*: $T_H := \kappa/(2\pi)$, κ being the surface gravity of the black hole included in Kruskal spacetime. As a consequence far from the black hole, where B becomes the standard Minkowskian temporal Killing vector ∂_t, the black hole is viewed as surrounded by a thermal bath of particles at Hawking temperature.
6. Hadamard states and their physical relevance in further developments. Which are the physical states in the plethora of algebraic states on $\mathfrak{A}$? The issue is historically related with the problem of computing the back reaction of quantum field in a given state to the background metric [1]. From a semiclassical point of view, the back reaction is perturbatively encompassed in Einstein equations when a contribution to stress-energy tensor $T_{\mu\nu}$ of the total matter in the spacetime is given by the expectation value of the quantum field Φ computed with respect to a quantum state ω of Φ. The stress-energy tensor cannot belong to the basic algebra $\mathfrak{A}$ of observables since its integral kernel should be a sum products of two (differentiated) copies of the field operator $\Phi(x)$ *evaluated on the same point* x. Formally speaking one would need a machinery to smear distributions $\omega(\Phi(x)\Phi(y))$ and similar differentiated objects – or vector-valued distributions $\Phi_\omega(x)\Phi_\omega(y)|\Psi_\omega\rangle$ – with distributions $\delta(x,y)f(x)$, $f \in C_0^\infty(M)$, when ω belongs to the class of physically meaningful states. Equivalently, it would be enough a procedure to restrict (vector-valued) distributions form $M \times M$ to the diagonal of $M \times M$. The rough idea is to subtract the divergent part (in Minkowski spacetime being embodied in the two-point function of Minkowski vacuum) of the considered kernels before performing the coincidence limit of arguments. This divergent part must be in common to all the physically meaningful

states. Generalizing, this way should lead naturally to a definition of Wick polynomials $\Phi^n(x)$ in curved spacetime. A class of states which, through several steps from '70 to nowadays, have been proved to be useful in this programme for linear QFT in curved globally hyperbolic spacetime are *Hadamard states* [1] whose rigorous definition has been fixed definitely in 1991 by Kay and Wald (see [1]). Barring technicalities, these states are individuated by requiring that, in a suitable open neighborhood of a Cauchy surface, the short distance behaviour of the kernel of their two-point function $\omega(\Phi(f)\Phi(g))$ must have the structure :

$$\omega(x,y) = \frac{U(x,y)}{\sigma(x,y)} + V(x,y)\ln\sigma(x,y) + \omega_{reg}(x,y).$$

Above $\sigma(x,y)$ is the squared geodesic distance of x from y and U and V are locally well-defined quantities depending on the local geometry only, ω_{reg} is not singular and determines the state ω. Minkowski vacuum is Hadamard. Fulling, Narcowich, Sweeny and Wald in 1978-1981 (see [1]) established the existence of a huge set of Hadamard states for every globally hyperbolic spacetime. In general these states are not unitarily equivalent: there is no a unique Hilbert space where they co-exist all together, however in 1994 Verch (see [1]) proved that, in a fixed globally hyperbolic spacetime, Hadamard states are *locally quasi equivalent*: if ω_1, ω_2 are Hadamard, for any open relatively compact but arbitrarily large region $\mathcal{O}$, $\omega_2\restriction_{\mathfrak{A}(\mathcal{O})}$ is represented by a vector or a density matrix in the GNS Hilbert space of $\omega_1\restriction_{\mathfrak{A}(\mathcal{O})}$. In 1992 Radzikowski [3] gave a microlocal characterization of Hadamard states in terms of *wavefront sets*. Radzikowski machinery produced a rapid evolution of the investigation of the mentioned issues making use of known Hörmander's microlocal results about the restriction to submanifolds of distributions. With a careful analysis of short-distances divergences of n-point functions of Hadamard states, Brunetti and Fredenhagen (see [4] and references therein), constructed (time-ordered) operator-valued products of Wick polynomials of free fields. As they showed, those Wick polynomial can be used as building blocks for a local (perturbative) definition of interacting fields. In this framework renormalization procedure amounts to extensions of expectation values of time-ordered products to all points of space-time. An improvement of the definition of Wick polynomials, in the direction of *general locality and covariance* (see section 7), was subsequently obtained by Hollands and Wald (see [5] and references therein) with a complete determination of the involved ambiguities of the divergence-subtraction procedure. Just to give a simple idea we only mention that, for $n = 2$, $\Phi^2(x)$ can be identified (by means of a suitable $*$-algebra isomorphism) to the restriction of the diagonal of $M \times M$ of the distribution $(\Phi_\omega(x)\Phi_\omega(y) - H(x,y)I)|\Psi_\omega\rangle$ for every fixed Hadamard state ω, H being the singular part of the two-point function of any Hadamard state. The mentioned ambiguities arises in this particular case from the fact that the singular part H of the two-point function is defined up to add smooth terms and these can be to some extent fixed by imposing suitable requirements (in particular about locality and covariance) of the Wick polynomials. Moretti [6] extended Hollands-Wald's definition to differentiated fields in order to rigorously define a stress-energy tensor operator of linear Klein-Gordon field which fulfills the constraint of conservation $\nabla_\mu T^{\mu\nu} = 0$ (this is not trivial because H fails to satisfy Klein-Gordon equation), takes into account the trace anomaly (see [1]), reduces to the classical form for classical fields and agrees with known Euclidean regularization procedures. The general issue of the conservation of the stress tensor in interacting quantum field theory in curved spacetimes within the generally local covariant perturbative approach was treated by Hollands and Wald in [7].

7. Locally covariant quantum field theories. The previous parts deliberately dismissed some conceptually deep problems. One of the main characteristic of the probabilistic interpretation of quantum theory is that we should be able to repeat experiments as many times as possible. But repetitions of measurements immediately call for a comparison tool, since they are obviously done in different spacetime locations. In flat Minkowskian spacetime continuous isometries can be used to compare experiments performed in different locations, but what about general (spacetime dependent and/or curved) backgrounds lacking any such isometries? What should replace isometries there? Even more dramatically, since experiments are done on bounded regions of spacetime, how can we be sure that these regions belong to a single specific spacetime? We attempt to give an answer to these questions (and others) in this part of the article.

It is a fact that (equivalence classes of) measurements corresponds to observables, in an idealistic sense. Hence, the description of a system, as done in the previous sections, as an algebra of observables can be considered as a fundamental tenet. However, this description is based on a definite choice of a spacetime manifold, something which we wish to avoid here, since there is no local measurement that can help in determining the global structure of the geometry. How can we quantize a system, i.e. in the sense described in the previous sections, without choosing a background? One possible, in a sense startling, suggestion to way out the dilemma would be to consider simultaneously all spacetimes, at least of a certain class. It is our wish to elevate this suggestion to a fundamental principle, on the base of which every theory should be discussed. Indeed, together with a careful choice of the structures and tools, it becomes a strengthening of the classical locality principle. Being here interested in quantizing field theories on curved spacetimes, we shall be dealing with a reasonable choice, i.e. we consider the class of globally hyperbolic spacetimes (of fixed dimension). The structure is flexible and changes can be envisaged, according to the needs; indeed, we wish to emphasize that one may as well treat any external classical background field adopting essentially the same conditions, an important example being that of QED under external spacetime-dependent electromagnetic fields.

The strengthening of the locality principle may be seen as deriving from the fact that globally hyperbolic spacetimes have subregions which are globally hyperbolic as well (relatively compact or not). For instance, in Minkowski spacetime we have double cones (the domains of dependence [1] of fixed-time three-dimensional open balls and their images under the action of the connected component of the Poincaré group) and wedges $\mathbf{R}$ (defined above), both globally hyperbolic; Schwarzschild spacetime is a globally hyperbolic patch of its Kruskal extension; other examples may be found, for instance, by looking into some construction of globally hyperbolic spacetimes in which there is only a black hole, i.e., they do not contain a white hole, and their maximal extensions to globally hyperbolic spacetimes containing bifurcate killing horizons (see reference Racz (1992) and Wald in [1]). The list can be continued, but the essence of the suggestion would be to consider as equivalent, in a sense to be precised, all theories (i.e. algebras of observables) defined on isometric (globally hyperbolic regions of different) globally hyperbolic spacetimes. The description of this equivalence is made precise by the use of the language of category theory. Indeed, globally hyperbolic spacetimes (d-dimensional, oriented and time oriented) can be taken as the objects of a category, say Loc, with morphisms all possible isometric and causal embeddings (i.e. provided the embedded spacetime is open, globally hyperbolic and keeps the same orientations); as algebras of observables we may consider the C^*-category Obs, where evidently C^*-algebras are the objects with injective *-homomorphisms the morphisms. Hence, we contend that a quantum field theory has to be considered as a covariant functor $\mathscr{A}$ from Loc to Obs, i.e., that to any M in Loc associates an algebra of observables $\mathscr{A}(M)$ and for which to any morphism $\psi : M \to M'$ in Loc there corresponds a morphism α_ψ in Obs yelding $\alpha_\psi(\mathscr{A}(M)) = \mathscr{A}(\psi(M))$.

8. Main Properties. The main properties we adopt for the functor $\mathscr{A}$ are the following;

System: To each $M \in \mathsf{Loc}$ there is associated an element $\mathscr{A}(M) \in \mathsf{Obs}$;

Subsystems: Let $\psi : M \to N$ a causality preserving isometric embedding of elements of Loc, then there exists an injective *-homomorphism $\alpha_\psi : \mathscr{A}(M) \to \mathscr{A}(N)$; ("causally preserving" means that if $p, q \in \psi(M)$ then each causal curve connecting $\psi(p)$ and $\psi(q)$ in N is the image through ψ of a corresponding causal curves in M connecting p and q.)

Covariance: If $\psi : M \to N$ and $\psi' : N \to O$ are two maps as defined above, then, $\alpha_{\psi'\psi} = \alpha_{\psi'}\alpha_\psi$;

Causality: If $\psi_1 : M_1 \to M$ and $\psi_2 : M_2 \to M$ are embeddings as above, such that $\psi_1(M_1)$ and $\psi_2(M_2)$ cannot be connected by a causal curve in M, then

$$\alpha_{\psi_1}(\mathscr{A}(M_1)) \vee \alpha_{\psi_2}(\mathscr{A}(M_2)) \equiv \alpha_{\psi_1}(\mathscr{A}(M_1)) \otimes \alpha_{\psi_2}(\mathscr{A}(M_2))$$

where $\vee$ indicates the generated subalgebra of $\mathscr{A}(M)$.

Time-Slice: Let $\psi : M \to N$ be an embedding as above such that $\psi(M)$ contains a whole Cauchy surface of N, then $\alpha_\psi(\mathscr{A}(M)) = \mathscr{A}(N)$.

These properties describe a causal covariant functor $\mathscr{A}$ from Loc to Obs obeying the Time-Slice property.

The main reasons in favour of the contention are the following: First of all by this procedure we describe automatically the presence of subsystems, the equivalent of a local structure, and the fact that physics should not depend by the embedding of a spacetime into another. Indeed, as already emphasized, one views the new structure as the implementation, and merging, of the concepts of locality and general covariance. For instance, the example done before for Schwarzschild is classical, in that the physics beyond the horizon should not depend in the way Schwarzschild is embedded into Kruskal. Secondly, the previously mentioned Haag-Kastler description can be recovered as a subcase when one specialize to a fixed background. As a third, a posteriori, reason, several crucial new results have been uncovered by the use of this structure, for instance a refinement [7] of perturbation theory on curved spacetime [4], a rigorous description of the renormalization group ideas (see reference in [7]), a description of the spin-statistics connection [2], a first discussion of the superselection structure [9] for locally covariant theories, new ideas on the quantum energy inequalities [10].

A similar functorial relation can be used to give a formulation of a state space associated to the quantum field theory in the sense just explained. Referring to the definition of states on a unital C^*-algebra given in section 2, a new category can be defined, that is, the category of state spaces Sts, whose objects $\mathcal{S}$ are states ω on some unital C^*-algebra $\mathcal{A}$, with the further properties of being closed under taking finite linear convex combinations and operations $\omega(\cdot) \mapsto \omega_A(\cdot) = \omega(A^* \cdot A)/\omega(A^*A)$, $A \in \mathscr{A}$. Morphisms between elements $\mathcal{S}'$ and $\mathcal{S}$ are maps $\gamma^* : \mathcal{S}' \to \mathcal{S}$ that arise as duals of the morphisms $\gamma : \mathcal{A} \to \mathcal{A}'$ between unital C^*-algebras, via the relation $\gamma^*\omega'(A) = \omega'(\gamma(A))$, with $\omega' \in \mathcal{S}'$ and $A \in \mathcal{A}$. The composition of morphism is obvious. Now, it is clear that Sts is dual, in the described sense, to the category Obs. Then a state space for the locally covariant quantum field theory $\mathscr{A}$ is a <u>contravariant functor</u> $\mathfrak{S}$ from Loc to Sts, i.e., a correspondence $\mathsf{Loc} \ni M \to \mathfrak{S}(M) \in \mathsf{Sts}$ such that for any two given embeddings $\psi : M \to M'$ and $\psi' : M' \to M''$, we have $\alpha^*_{\psi' \circ \psi} = \alpha^*_{\psi'} \circ \alpha^*_{\psi}$, where α^*_ψ is the dual map of the morpism α_ψ in Obs.

Also in this case there are nice results, but we address the interested reader to the original literature for more information. The only important bit of information is that only *local folia* of states are invariant under the isometric causality-preserving embeddings. We remind that the *folium* $\mathbf{F}(\pi_\omega)$ of a state ω on a unital C^*-algebra $\mathcal{A}$ contains the states on $\mathcal{A}$ which can be obtained as trace class operators in the GNS representation $(\mathfrak{H}_\omega, \pi_\omega, \Psi_\omega)$ of ω: $\omega' \in \mathbf{F}(\pi_\omega)$ iff $\omega'(A) = tr(\rho\pi_\omega(A))$ of every $A \in \mathcal{A}$. For a globally hyperbolic spacetime M consider two states ω, ω' on $\mathscr{A}(M)$. We say that a state ω' belongs to the *local folium* of ω if, for every relatively-compact globally-hyperbolic subregion $O \subset M$ such that the inclusion map $\imath_{O,M} : O \to M$ is causality-preserving, it holds $\omega' \circ \alpha_{\imath_{O,M}} \in \mathbf{F}(\pi_\omega \circ \alpha_{\imath_{O,M}})$.

A straightforward example of the structure so far invoked is given by the extension of the one already taken in the previous sections. Indeed, by quantizing the free scalar field on any globally hyperbolic spacetime M in Loc we get a corresponding Weyl algebra (as defined in section 3) $\mathscr{A}(M) = \mathcal{W}(E(C_0^\infty(M)), \sigma)$ (where, as before, E is the unique advanced-minus retarded fundamental solution of the Klein-Gordon equation) and to any isometric causality-preserving embedding $\psi : M \to M'$ there correspond, by known theorems, a C^*-algebraic injective homorphism $\alpha_\psi : \mathscr{A}(M) \to \mathscr{A}(M')$ that satisfies the covariance properties (Brunetti, Fredenhagen and Verch [8]). The functor so defined satisfies all properties listed above, including of course, causality and the time-slice property (an important feature on which the result stands is the covariance of the fundamental solution E). The same reasoning leads to a complete characterization of a state space for this theory by looking at folia of quasifree Hadamard states and recalling that wave front sets are covariant under diffeomorphisms.

To summarize the discussion, by adopting this new point of view of quantizing a theory, we answered at least that part of the questions at the beginning that was looking for a reasonable definition of a quantum field theory that does not adopt a priori any background. All possible backgrounds should be included, at least inside a reasonable class. The question about the comparability of experiments in different spacetime locations can be answered by resorting to another tool in category theory, that of <u>natural transformations</u>. These are further covariant structures that work between functors defined over Loc.

First of all, we may say that two different locally covariant theories are equivalent if the defining functors are naturally equivalent. Namely, if $\mathscr{A}_1$ and $\mathscr{A}_2$ are the two functors from Loc to Obs, then they are called *naturally equivalent* if there exists a family β_M of *-isomorphisms between $\mathscr{A}_1(M)$ and $\mathscr{A}_2(M)$ such that $\beta_M \alpha_{1,\psi} = \alpha_{2,\psi}\beta_M$ for any admissible embedding ψ of M into another spacetime. One can readily prove (see [8]) that, if the two functors are associated to scalar field theories with different masses ($V \equiv m^2$), then the two theories are never equivalent in the above sense.

However, the most important class of natural transformations is that related to the concept of quantum fields. Indeed, if $\mathfrak{D}$ is the functor that associate to any element of Loc the topological vector space of test functions over M, i.e. $\mathfrak{D}(M) \equiv C_0^\infty(M)$, then a *locally covariant quantum field* Φ is a family of linear continuous maps $\Phi_M : \mathfrak{D}(M) \to \mathscr{A}(M)$ such that, if $\psi : M \to N$ is any admissible embedding, then

$$\alpha_\psi \Phi_M = \Phi_N \psi_*$$

where ψ_* is the push-forward on the test function space

$$(\psi_* f)(x) = \begin{cases} f(\psi^{-1}(x)) \ , & x \in \psi(M) \ , \\ 0 \ , & \text{else.} \end{cases}$$

It is precisely the above covariance condition that makes this new definition of quantum fields useful for providing a solution to the problem of comparison of experiments in different spacetime regions, even between different spacetimes (via the embeddings). This new notion was unforeseen before. Of course, even in this case, one can show an example by resorting to the scalar field case. However, the power of the new definition has been largely extended to the case of any Wick polynomial of the scalar field, and even to their time ordered products. This new results, due to Hollands and Wald [5], are the basic building blocks of their refinement of the perturbation techniques developped by Brunetti and Fredenhagen [4]. A more recent improvement, in case interactions contain derivatives terms, builds along the same lines and is due again to the efforts of Hollands and Wald [7] (besides some previous attempts of Moretti [6]).

By dropping the conditions of continuity and linearity one may enlarge the definition of locally covariant quantum fields. Typical examples are the notion of a local scattering matrix in the sense of Stückelberg-Bogolubov-Epstein-Glaser (see [4][5]), and Fewster's energy inequalities [10].

As far as the scattering matrix (S-matrix) theory is concerned, S-matrices are unitaries $S_M(\lambda)$ with $M \in \mathsf{Loc}$ and $\lambda \in \mathfrak{D}(M)$ which satisfy the conditions

$$S_M(0) = 1$$

$$S_M(\lambda + \mu + \nu) = S_M(\lambda + \mu) S_M(\mu)^{-1} S_M(\mu + \nu)$$

for $\lambda, \mu, \nu \in \mathfrak{D}(M)$ such that the supports of λ and ν can be separated by a Cauchy surface of M with the support of λ in the future of the surface.

The importance of these S-matrices relies on the fact that they can be used to define a new quantum field theory. The new theory is locally covariant if the original theory was and if the local S-matrices satisfy the condition of a locally covariant field above.

Lack of space prevents us from giving details of other examples, but they can be easily traced back in the recent literature.

Further reading

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