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Quantum ground states holographically induced by asymptotic flatness: Invariance under spacetime symmetries, energy positivity and Hadamard property.

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Abstract. This paper continues the analysis of the quantum states introduced in previous works and determined by the universal asymptotic structure of four-dimensional asymptotically flat vacuum spacetimes at null infinity M. It is now focused on the quantum state λ_M, of a conformally coupled scalar field propagating in M. λ_M is "holographically" induced in the bulk by the universal BMS-invariant state λ at infinity $\Im^+$ of M. It is done by means of the correspondence between observables in the bulk and those on the boundary at null infinity discussed in previous papers. This induction is possible when some requirements are fulfilled. This happens in particular whenever the spacetime M and the associated unphysical one, $\tilde{M}$, are globally hyperbolic and M admits future infinity i^+. As is known λ_M coincides with Minkowski vacuum if M is Minkowski spacetime. It is now proved that, in the general case, λ_M enjoys the following further remarkable properties.
(i) λ_M is invariant under the (unit component of the Lie) group of isometries of the bulk spacetime M.
(ii) λ_M fulfills a natural energy-positivity condition with respect to every notion of Killing time (if any) in the bulk spacetime M: If M admits a complete time-like Killing vector, the associated one-parameter group of isometries is represented by a strongly-continuous unitary group in the GNS representation of λ_M. The unitary group has positive self-adjoint generator without zero modes in the one-particle space.
(iii) λ_M is (globally) Hadamard and thus it can be used as starting point for perturbative renormalization procedure.

1 Introduction

In this paper we continue the analysis of the states determined by the asymptotic structure of four-dimensional asymptotically flat spacetimes at null infinity started in [DMP06] and fully developed in [Mo06]. Part of those result will be summarized in Sec.2. In [DMP06] and [Mo06] it has been established that the null boundary at future infinity $\Im^+$ of an asymptotically flat spacetime admits a natural formulation of bosonic linear QFT living therein. A preferred quasifree pure state λ has been picked out in the plethora of algebraic states defined on the algebra of Weyl observables $\mathcal{W}(\Im^+)$ of the QFT on $\Im^+$. That state enjoys remarkable properties, in particular it is invariant under the action of the natural group of symmetries of $\Im^+$ – the so-called (infinite-dimensional) BMS group – describing the asymptotic symmetries of the physical spacetime M. λ is the vacuum state for BMS-massless particles if one analyzes the unitary representation of the BMS group within the Wigner-Mackey approach [DMP06]. λ is uniquely determined by a positive BMS-energy requirement in addition to the above-mentioned BMS invariance [Mo06] (actually the latter requirement can be weakened considerably). Finally, in the folium of λ there are no further pure BMS-invariant (not necessarily quasifree or positive energy) states. λ is universal: it

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does not depend on the particular asymptotically flat spacetime under consideration but it is defined in terms of the asymptotic extent which is the same for all asymptotically flat spacetimes.
However, in every fixed asymptotically flat spacetime M and under suitable hypotheses on M and $\tilde{M}$, λ it induces a preferred quasifree state λ_M on the algebra $\mathcal{W}(M)$ of the Weyl observables of a (bosonic massless conformally coupled linear) field propagating in the bulk M. This is because, due to a sort of holographic correspondence discussed in [DMP06, Mo06], the algebra of bulk observables $\mathcal{W}(M)$ is one-to-one mapped into a subalgebra of boundary observables $\mathcal{W}(\Im^+)$ by means of a isometric $*$-algebra homomorphism $\imath : \mathcal{W}(M) \to \mathcal{W}(\Im^+)$. In this way λ_M is defined as $\lambda_M(a) := \lambda(\imath(a))$ for all $a \in \mathcal{W}(M)$. The existence of $\imath$ is assured if some further conditions defined in [DMP06] are fulfilled for the spacetime M and the associated unphysical spacetime $\tilde{M}$ (see (b) of Proposition 2.1). In particular both M and $\tilde{M}$ are required to be globally hyperbolic. Those requirements are valid when M is Minkowski spacetime and, in that case, λ_M turns out to coincide with Minkowski vacuum [DMP06]. However, it has been established in [Mo06], that the conditions are verified in a wide class of spacetimes (supposed to be globally hyperbolic with the unphysical spacetime $\tilde{M}$) individuated by Friedrich [Fri86-88]: the asymptotically flat vacuum spacetimes admitting future infinity i^+.
This paper is devoted to study the general features of the state λ_M. In particular, is Sec.3 we focus on isometry-invariance properties of λ_M and on the energy positivity condition with respect to timelike Killing vectors in any bulk spacetime M (Theorem 3.1): we show that λ_M is invariant under the unit component of the Lie group of isometries of the bulk spacetime M. This fact holds true also replacing λ_M with any other state λ'_M uniquely defined by assuming that $\lambda'_M(a) := \lambda'(\imath(a))$ for all $a \in \mathcal{W}(M)$, where λ' is any (not necessarily quasi free or pure) BMS invariant state defined on $\mathcal{W}(\Im^+)$. Furthermore λ_M fulfills a natural energy-positivity condition with respect to every notion of Killing time in the bulk spacetime M: If M posses a complete time-like Killing vector ξ, the associated one-parameter group of isometries is represented by a strongly continuous unitary group in the GNS representation of λ_M, that group admits a positive self-adjoint generator H and H has no zero modes in the one particle space. In this sense the quasifree state λ_M is a *regular ground state* for H [KW91]. Actually these properties are proved to hold also if ξ is causal and future directed, but not necessarily timelike.
Sec.4 is devoted to discuss the validity of the Hadamard condition for the state λ. First we show that the two-point function of λ_M is a proper distribution of $\mathscr{D}'(M \times M)$ (Theorem 4.2) when the asymptotically flat spacetime M admits future infinity i^+. It is interesting noticing that the explicit expression of the two point function of λ_M we present strongly resembles that of Hadamard states in manifolds with bifurcate Killing horizons studied by Kay and Wald [KW91] when restricted to the algebra of observables localized on a Killing horizon.
The last result we establish in this work is that λ_M is Hadamard (Theorem 4.3). In this case the kernel of the two-point function of λ_M satisfies the *global Hadamard condition* [KW91]. The proof – performed within the microlocal framework taking advantage of some well known result established by Radzikowski [Ra96a, Ra96b] – is based on a "from local to global" argument and the analysis of the wavefront sets of the involved distributions.
The rest of this section is devoted to remind the reader the basic geometric structures in asymptotically flat spacetimes.

1.1. *Notations, mathematical conventions.* Throughout $\mathbb{R}^+ := [0, +\infty)$, $\mathbb{N} := \{0, 1, 2, \ldots\}$. For smooth manifolds M, N, $C^\infty(M; N)$ (omitting N whenever $N = \mathbb{R}$) is the space of smooth functions $f : M \to N$. $C_0^\infty(M; N) \subset C^\infty(M; N)$ is the subspace of compactly-supported functions. If $\chi : M \to N$ is a diffeomorphism, χ^* is the natural extension to tensor bundles (counter-, co-variant and mixed) from M to N (Appendix C in [Wa84]). A spacetime is a four-dimensional semi-Riemannian (smooth if no specification is supplied) connected manifold (M, g), whose metric has signature $- + ++$, and it is assumed

to be oriented and time oriented. We adopt definitions of causal structures of Chap. 8 in [Wa84]. If $S \subset M \cap \tilde{M}$, (M, g) and $(\tilde{M}, \tilde{g})$ being spacetimes, $J^{\pm}(S; M)$ ($I^{\pm}(S; M)$) and $J^{\pm}(S; \tilde{M})$ ($I^{\pm}(S; \tilde{M})$) indicate the causal (chronological) sets associated to S and respectively referred to the spacetime M or $\tilde{M}$. Concerning distribution and wavefront-set theory we essentially adopt standard definitions and notation used in [Hö89, Hör71] and in the last chapter of [FJ98].

1.2. *Asymptotic flatness at future null infinity and* $\Im^+$. Following [AH78, As80, Wa84], a smooth spacetime (M, g) is called **asymptotically flat vacuum spacetime at future null infinity** if there is a second smooth spacetime $(\tilde{M}, \tilde{g})$ such that M can be viewed as an open embedded submanifold of $\tilde{M}$ with boundary $\Im^+ \subset \tilde{M}$. $\Im^+$ is an embedded submanifold of $\tilde{M}$ satisfying $\Im^+ \cap J^-(M; \tilde{M}) = \emptyset$. $(\tilde{M}, \tilde{g})$ is required to be strongly causal in a neighborhood of $\Im^+$ and it has to hold $\tilde{g}\restriction_M = \Omega^2 \restriction_M g\restriction_M$ where $\Omega \in C^\infty(\tilde{M})$ is strictly positive on M. On $\Im^+$ one has $\Omega = 0$ and $d\Omega \neq 0$. Moreover, defining $n^a := \tilde{g}^{ab}\partial_b\Omega$, there must be a smooth function, ω, defined in $\tilde{M}$ with $\omega > 0$ on $M \cup \Im^+$, such that $\tilde{\nabla}_a(\omega^4 n^a) = 0$ on $\Im$ and the integral lines of $\omega^{-1}n$ are complete on $\Im^+$. The topology of $\Im^+$ has to be that of $\mathbb{S}^2 \times \mathbb{R}$. Finally vacuum Einstein equations are assumed to be fulfilled for (M, g) in a neighborhood of $\Im^+$ or, more weakly, "approaching" $\Im^+$ as discussed on p.278 of [Wa84].
Summarizing $\Im^+$ is a 3-dimensional submanifold of $\tilde{M}$ which is the union of integral lines of the nonvanishing null field $n^\mu := \tilde{g}^{\mu\nu}\nabla_\nu\Omega$, these lines are complete for a certain regular rescaling of n, and $\Im^+$ is equipped with a degenerate metric $\tilde{h}$ induced by $\tilde{g}$. $\Im^+$ is called *future infinity* of M.
Remark 1.1. For brevity, from now on **asymptotically flat spacetime** means *vacuum spacetime asymptotically flat at future null infinity.*

Minkowski spacetime and Schwarzschild spacetime are well-known examples of asymptotically flat spacetimes. It is simply proved – for instance reducing to the Minkowski space case – that, with our conventions, the null vector n is always *future directed* with respect to the time-orientation of $(\tilde{M}, \tilde{g})$ induced from that of (M, g).
As far as the only geometric structure on $\Im^+$ is concerned, changes of the unphysical spacetime $(\tilde{M}, \tilde{g})$ associated with a fixed asymptotically flat spacetime (M, g), are completely encompassed by **gauge transformations** $\Omega \to \omega\Omega$ valid in a neighborhood of $\Im^+$, with ω smooth and strictly positive. Under these gauge transformations the triple $(\Im^+, \tilde{h}, n)$ transforms as

$$\Im^+ \to \Im^+ \,, \quad \tilde{h} \to \omega^2\tilde{h} \,, \quad n \to \omega^{-1}n \,. \tag{1}$$

If C is the class of the triples $(\Im^+, \tilde{h}, n)$ transforming as in (1) for a fixed asymptotically flat spacetime, there is no general physical principle to single out a preferred element in C. On the other hand, C is *universal* for all asymptotically flat spacetimes [Wa84]: If C_1 and C_2 are the classes of triples associated respectively to (M_1, g_2) and (M_2, g_2), there is a diffeomorphism $\gamma : \Im_1^+ \to \Im_2^+$ such that for suitable $(\Im_1^+, \tilde{h}_1, n_1) \in C_1$ and $(\Im_2^+, \tilde{h}_2, n_2) \in C_2$: $\quad \gamma(\Im_1^+) = \Im_2^+ \,, \quad \gamma^*\tilde{h}_1 = \tilde{h}_2 \,, \quad \gamma^* n_1 = n_2$.
Choosing ω such that $\tilde{\nabla}_a(\omega^4 n^a) = 0$ – this choice exists in view of the very definition of asymptotically flat spacetime – and using the fact that vacuum Einstein's equations are fulfilled in a neighborhood of $\Im^+$, the tangent vector n turns out to be that of *complete null geodesics* with respect to $\tilde{g}$ (see Sec. 11.1 in [Wa84]). ω is completely fixed by requiring that, in addition, the non-degenerate metric on the transverse section of $\Im^+$ is the standard metric of $\mathbb{S}^2$ in $\mathbb{R}^3$ constantly along geodesics. We indicate by ω_B and $(\Im^+, \tilde{h}_B, n_B)$ that value of ω and the associated triple respectively. For $\omega = \omega_B$, a **Bondi frame** on $\Im^+$ is a global coordinate system $(u, \zeta, \overline{\zeta})$ on $\Im^+$, where $u \in \mathbb{R}$ is an affine parameter of the complete null $\tilde{g}$-geodesics whose union is $\Im^+$ ($n = \partial/\partial u$ in these coordinates) and $\zeta, \overline{\zeta} \in \mathbb{S}^2 \equiv \mathbb{C} \cup \{\infty\}$ are complex coordinates on the cross section of $\Im^+$: $\zeta = e^{i\varphi}\cot(\theta/2)$ with θ, φ usual spherical coordinates of $\mathbb{S}^2$. With these choices, the metric on the transverse section of $\Im^+$ reads $2(1+\zeta\overline{\zeta})^{-2}(d\zeta \otimes d\overline{\zeta} + d\overline{\zeta} \otimes d\zeta) = d\theta \otimes d\theta + \sin^2\theta\, d\varphi \otimes d\varphi$.
By definition $\chi : \Im^+ \to \Im^+$ belongs to the **BMS group**, G_{BMS} [Pe63, Pe74, Ge77, AS81], if χ is a

diffeomorphism and $\chi^*\tilde{h}$ and $\chi^* n$ differ from $\tilde{h}$ and n by a rescaling (1) at most. Henceforth, whenever it is not explicitly stated otherwise, *we consider as admissible realizations of the unphysical metric on* $\Im^+$ *only those metrics* $\tilde{h}$ *which are accessible from a metric with associate triple* $(\Im^+, \tilde{h}_B, n_B)$*, by means of a transformations in* G_{BMS} .

In coordinates of a fixed Bondi frame $(u, \zeta, \overline{\zeta})$, the group G_{BMS} is realized as semi-direct group product $SO(3,1)\uparrow\!\!\ltimes C^\infty(\mathbb{S}^2)$, where $(\Lambda, f) \in SO(3,1)\uparrow \times C^\infty(\mathbb{S}^2)$ acts as

$$u \;\rightarrow\; u' := K_\Lambda(\zeta,\overline{\zeta})(u + f(\zeta,\overline{\zeta}))\,, \tag{2}$$

$$\zeta \;\rightarrow\; \zeta' := \Lambda\zeta := \frac{a_\Lambda \zeta + b_\Lambda}{c_\Lambda \zeta + d_\Lambda}\,, \quad \overline{\zeta} \rightarrow \overline{\zeta}' := \Lambda\overline{\zeta} := \frac{\overline{a_\Lambda}\overline{\zeta} + \overline{b_\Lambda}}{\overline{c_\Lambda}\overline{\zeta} + \overline{d_\Lambda}}\,. \tag{3}$$

K_Λ is the smooth positive function on $\mathbb{S}^2$

$$K_\Lambda(\zeta,\overline{\zeta}) := \frac{(1+\zeta\overline{\zeta})}{(a_\Lambda\zeta + b_\Lambda)(\overline{a_\Lambda}\overline{\zeta} + \overline{b_\Lambda}) + (c_\Lambda\zeta + d_\Lambda)(\overline{c_\Lambda}\overline{\zeta} + \overline{d_\Lambda})} \quad \text{and} \quad \begin{bmatrix} a_\Lambda & b_\Lambda \\ c_\Lambda & d_\Lambda \end{bmatrix} = \Pi^{-1}(\Lambda)\,. \tag{4}$$

Above Π is the well-known surjective covering homomorphism $SL(2,\mathbb{C}) \rightarrow SO(3,1)\uparrow$ (see [DMP06] for further details). Two Bondi frames are connected each other through the transformations (2),(3) with $\Lambda \in SU(2)$. Conversely, any coordinate frame $(u', \zeta', \overline{\zeta}')$ on $\Im^+$ connected to a Bondi frame by means of an arbitrary BMS transformation (2),(3) is *physically equivalent* to the latter from the point of view of General Relativity, but it is not necessarily a Bondi frame in turn. A global reference frame $(u', \zeta', \overline{\zeta}')$ on $\Im^+$ related with a Bondi frame $(u, \zeta, \overline{\zeta})$ by means of a BMS transformation (2)-(3) will be called **admissible frame**. By construction, the action of G_{BMS} takes the form (2)-(3) in admissible fames too. The notion of Bondi frame is useful but conventional. *Any physical object must be invariant under the whole BMS group and not only under the subgroup of* G_{BMS} *connecting Bondi frames.*

The local one-parameter group of diffeomorphisms generated by a (smooth) vector field ξ defined in an asymptotically flat spacetime (M, g) is called **asymptotic Killing symmetry** if (i) ξ extends smoothly to a field $\tilde{\xi}$ tangent to $\Im^+$ and (ii) $\Omega^2 \pounds_\xi g$ has a smooth extension to $\Im^+$ which vanishes there. This is the best approximation of a Killing symmetry for a generic asymptotically flat spacetime which does *not* admits proper Killing symmetries (see e.g. [Wa84]). The following well-known result illustrates how G_{BMS} describes asymptotic Killing symmetries valid for every asymptotically flat spacetime. [Ge77, Wa84].

Proposition 1.1. *Consider any asymptotically flat spacetime* (M, g)*. The one-parameter group of diffeomorphisms generated by a vector field* ξ *tangent ot* $\Im^+$ *is a subgroup of* G_{BMS} *if and only if* $\tilde{\xi}$ *is the smooth extension of some vector field of* (M, g) *defining an asymptotic Killing symmetry.*

2 Summary of some previously achieved results.

2.1. *Quantum fields on* $\Im^+$. Let us summarize how a natural linear QFT can be defined on $\Im^+$ employing the algebraic approach and the GNS reconstruction theorem. Motivations for the following theoretical construction and more details can be found in [DMP06, Mo06].
Referring to a fixed Bondi frame on $\Im^+$, consider the real symplectic space $(\mathsf{S}(\Im^+), \sigma)$, where

$$\mathsf{S}(\Im^+) := \left\{ \psi \in C^\infty(\Im^+) \;\middle|\; \psi\,, \partial_u\psi \in L^2(\mathbb{R}\times\mathbb{S}^2, du \wedge \epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta})) \right\}\,, \quad \epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}) := \frac{2d\zeta\wedge d\overline{\zeta}}{i(1+\zeta\overline{\zeta})^2}\,, \tag{5}$$

$\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta})$ being the standard volume form of the unit 2-sphere, and the nondegenerate symplectic form σ is given by, if $\psi_1,\psi_2 \in \mathsf{S}(\Im^+)$

$$\sigma(\psi_1,\psi_2) := \int_{\mathbb{R}\times\mathbb{S}^2} \left(\psi_2 \frac{\partial \psi_1}{\partial u} - \psi_1 \frac{\partial \psi_2}{\partial u}\right) du \wedge \epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}) \,. \tag{6}$$

There is a natural representation of G_{BMS} acting on $(\mathsf{S}(\Im^+),\sigma)$ discussed in [DMP06, Mo06]. Start from the representation A of G_{BMS} made of transformations on functions $\psi \in C^\infty(\Im^+)$

$$(A_g\psi)(u,\zeta,\overline{\zeta}) := K_\Lambda \left(g^{-1}(u,\zeta,\overline{\zeta})\right)^{-1} \psi\left(g^{-1}(u,\zeta,\overline{\zeta})\right) \,, \quad \text{where } g=(\Lambda,f). \tag{7}$$

It turns out that $A_g(\mathsf{S}(\Im^+)) \subset \mathsf{S}(\Im^+)$. Moreover, due to the weight K_Λ^{-1}, the G_{BMS} representation A *preserves* the symplectic form σ. As a consequence the space $(\mathsf{S}(\Im^+),\sigma)$ does not depend on the used Bondi frame. In this context it is convenient to assume that the elements of $\mathsf{S}(\Im^+)$ are densities which transform under the action of A when one changes admissible frame. In the following the restriction $A_g\restriction_{\mathsf{S}(\Im^+)}$ will be indicated by A_g for sake of simplicity. Naturalness and relevance of the representation A follows from the content of proposition 3.4 below as discussed in [DMP06, Mo06].

As is well known [BR021, BR022], it possible to associate canonically any symplectic space, for instance $(\mathsf{S}(\Im^+),\sigma)$, with a **Weyl C^*-algebra**, $\mathcal{W}(\mathsf{S}(\Im^+),\sigma)$. This is the, unique up to (isometric) $*$-isomorphisms, C^*-algebra with generators $W(\psi)\neq 0$, $\psi\in\mathsf{S}(\Im^+)$, satisfying **Weyl commutation relations** (we use here conventions adopted in [Wa94])

$$W(-\psi) = W(\psi)^* \,, \qquad W(\psi)W(\psi') = e^{i\sigma(\psi,\psi')/2}W(\psi+\psi') \,. \tag{8}$$

Here $\mathcal{W}(\mathsf{S}(\Im^+)) := \mathcal{W}(\mathsf{S}(\Im^+),\sigma)$ has the natural interpretation of the algebra of observables for a linear bosonic QFT defined on $\Im^+$ as discussed in [DMP06, Mo06] (see also the appendix A of [Mo06]).
As discussed in [DMP06, Mo06], the representation A induces [BR022] a $*$-automorphism G_{BMS}-representation $\alpha : \mathcal{W}(\mathsf{S}(\Im^+)) \to \mathcal{W}(\mathsf{S}(\Im^+))$, uniquely individuated (by linearity and continuity) by the requirement $\alpha_g(W(\psi)) := W(A_{g^{-1}}\psi)$ for all $\psi\in\mathsf{S}(\Im^+)$ and $g\in G_{BMS}$.

Since we expect that physics is BMS-invariant we face the issue about the existence of α-invariant algebraic states on $\mathcal{W}(\mathsf{S}(\Im^+))$. To this end it has been established in [DMP06] that there is at least one algebraic quasifree[2] pure state λ defined on $\mathcal{W}(\mathsf{S}(\Im^+))$ which is invariant under G_{BMS}. It is that uniquely induced by linearity and continuity from:

$$\lambda(W(\psi)) = e^{-\mu_\lambda(\psi,\psi)/2} \,, \quad \mu_\lambda(\psi_1,\psi_2) := -i\sigma(\overline{\psi_{1+}},\psi_{2+}) \,, \quad \psi\in\mathsf{S}(\Im^+) \tag{9}$$

the bar over ψ_+ denotes the complex conjugation, ψ_+ being the **positive u-frequency part** of ψ computed with respect to the Fourier-Plancherel transform defined in section 4.2:

$$\psi_+(u,\zeta,\overline{\zeta}) := \int_{\mathbb{R}} \frac{e^{-iku}}{\sqrt{2\pi}}\Theta(k)\widehat{\psi}(k,\zeta,\overline{\zeta})dk \,, \quad (k,\zeta,\overline{\zeta})\in\mathbb{R}\times\mathbb{S}^2 \,.$$

Everything is referred to an arbitrarily fixed Bondi frame $(u,\zeta,\overline{\zeta})$ and $\Theta(k)=0$ for $k<0$ and $\Theta(k)=1$ for $k\geq 0$. Consider the GNS representation of λ, $(\mathfrak{H},\Pi,\Upsilon)$. Since λ is quasifree, $\mathfrak{H}$ is a bosonic Fock space $\mathcal{F}_+(\mathcal{H})$ with cyclic vector Υ given by the Fock vacuum and 1-particle Hilbert $\mathcal{H}$ space generated by

[2] We adopt the definition of quasifree state given in [KW91] and also adopted in [DMP06, Mo06], see the appendix A of [Mo06].

the positive-frequency parts of u-Fourier-Plancherel transforms $\widehat{\psi}_+ := \Theta\widehat{\psi}$. In other words one has that $\mathcal{H} \equiv L^2(\mathbb{R}^+ \times \mathbb{S}^2; 2kdk \wedge \epsilon_{\mathbb{S}^2})$. Indeed it arises from (9):

$$\langle \psi_+, \psi'_+ \rangle = \int_{\mathbb{R}\times\mathbb{S}^2} 2k\Theta(k)\overline{\widehat{\psi}(k,\zeta,\overline{\zeta})}\widehat{\psi}'(k,\zeta,\overline{\zeta})dk \wedge \epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}) \,. \tag{10}$$

In [DMP06, Mo06] we used a different, *but unitarily equivalent*, definition of positive frequency part in Fourier variables (therein we used positive frequency parts defined, in Fourier variables, as $\widetilde{\psi}_+(E,\zeta,\overline{\zeta}) := \sqrt{2E}\widehat{\psi}_+(E,\zeta,\overline{\zeta})$ so that $\mathcal{H} \equiv L^2(\mathbb{R}^+ \times \mathbb{S}^2; dE \wedge \epsilon_{\mathbb{S}^2})$.)

λ is a *regular* state, that is self-adjoint **symplectically-smeared field operators** $\sigma(\Psi,\psi)$ are defined via Stone's theorem: $\Pi(W(t\psi)) = e^{it\sigma(\Psi,\psi)}$ with $t \in \mathbb{R}$ and $\psi \in \mathsf{S}(\Im^+)$.

As a remarkable result, it has been established in [DMP06] that, equipping G_{BMS} with a suitable Fréchet topology, the unique unitary representation U of G_{BMS} leaving Υ invariant and implementing α is strongly continuous. Its restriction to $\mathcal{H}$ (which is invariant under U) is an irreducible and strongly continuous Wigner-Mackey representation associated with a scalar representation of the little group given by the double covering of $2D$ Euclidean group. The little group is the same as in the case of massless Poincaré particles. As a matter of facts, in the space of characters of G_{BMS}, where a generalization of Mackey machinery works [MC72-75, AD05, Da04, Da05, Da06] (notice that G_{BMS} is not locally compact), there is a a notion of *mass*, m_{BMS}, which is invariant under the action of G_{BMS}. It turns out that the found G_{BMS} representation is defined over an orbit in the space of characters with $m_{BMS} = 0$. So we are dealing with BMS-invariant massless particles.

λ enjoys some further properties, in particular a uniqueness property, which will be re-visited shortly from a point of view different from that adopted in [Mo06].

2.2. *Interplay with massless particles propagating in the bulk spacetime.* We want now to summarize some achieved results in [DMP06, Mo06]) on the interplay of QFT defined on $\Im^+$ and that defined in the bulk M, for a massless conformally coupled scalar field. Consider an asymptotically flat spacetime (M,g) with associated unphysical spacetime $(\tilde{M}, \tilde{g} = \Omega^2 g)$. In addition to asymptotic flatness assume also that both $M, \tilde{M}$ be *globally hyperbolic*. Consider standard bosonic QFT in (M,g) based on the symplectic space $(\mathcal{S}(M), \sigma_M)$, where $\mathcal{S}(M)$ is the space of real smooth, compactly supported on Cauchy surfaces, solutions ϕ of massless, conformally-coupled, Klein-Gordon equation in M:

$$P\phi = 0 \,, \quad \text{where } P \text{ is the Klein-Gordon operator } P = \Box_g - \tfrac{1}{6}R \,, \tag{11}$$

with Cauchy-surface independent symplectic form:

$$\sigma_M(\phi_1,\phi_2) := \int_S \left(\phi_2 \nabla_N \phi_1 - \phi_1 \nabla_N \phi_2\right) \, d\mu_g^{(S)} \tag{12}$$

S being any Cauchy surface of M with normal unit future-directed vector N and 3-volume measure $d\mu_g^{(S)}$ induced by g. Henceforth the **Weyl algebra associated with the symplectic space** $(\mathcal{S}(M), \sigma_M)$, whose **Weyl generators** are indicated by $W_M(\phi)$, $\phi \in \mathcal{S}(M)$, will be denoted by $\mathcal{W}(M)$. That C^*-algebra represents the basic set of quantum observables associated with the bosonic field ϕ propagating in the bulk spacetime (M,g). The generators $W_M(\phi)$ are formally interpreted as the exponentials $e^{-i\sigma_M(\Phi,\phi)}$ where $\sigma_M(\Phi,\phi) = -\sigma_M(\phi,\Phi)$ is the **field operator symplectically smeared** with a solution $\phi \in \mathcal{S}(M)$ of field equations (concerning the sign of σ we employ conventions used in [Wa94] which differ from those adopted in [KW91]). The interpretation has a rigorous meaning referring to a GNS representation of $\mathcal{W}(M)$. If the considered state ω is regular, $-i\sigma_M(\Phi,\phi)$ can be defined as the self-adjoint generator

of the subgroup $\mathbb{R} \ni t \mapsto \Pi_\omega(W(t\psi))$. The more usual field operator $\Phi(f)$ **smeared with functions** $f \in C_0^\infty(M)$ is related with $\sigma_M(\Phi, \phi)$ by means of $\Phi(f) := \sigma_M(\Phi, E(f))$, where the **causal propagator** $E : C_0^\infty(M) \to C^\infty(M)$ is the difference of the advanced and retarded fundamental solutions of Klein-Gordon equation which exist in every globally hyperbolic spacetime [Le53, Di80, BGP96]. Φ solves Klein-Gordon equation in distributional sense: $\Phi(Pf) = 0$ because $E \circ P = 0$ by definition.
The relation between QFT in M and that defined on $\Im^+$ can be now illustrated as follows (simplified form of proposition proposition 1.1 in [Mo06]) joined to proposition 2.5 in [DMP06].

Proposition 2.1. *Assume that both the asymptotically flat spacetime (M, g) and the unphysical spacetime $(\tilde{M}, \tilde{g})$ are globally hyperbolic. The following holds.*
(a) *Every $\phi \in \mathcal{S}(M)$ vanishes approaching $\Im^+$ but $(\omega\Omega)^{-1}\phi$ extenders to a smooth field, ω being any (arbitrarily fixed) positive function defined in a neighborhood of $\Im^+$ allowed by gauge transformation of the geometry on $\Im^+$ (see section 1.1). For the special case $\omega = \omega_B$ we define the $\mathbb{R}$-linear map*

$$\Gamma_M : \mathcal{S}(M) \ni \phi \mapsto \left((\omega_B\Omega)^{-1}\phi\right) \restriction_{\Im^+}.$$

(b) *If Γ_M fulfills both the following requirements:*
(i) $\Gamma_M(\mathcal{S}(M)) \subset \mathsf{S}(\Im^+)$ *and* **(ii)** *symplectic forms are preserved by Γ_M, that is, for all $\phi_1, \phi_2 \in \mathcal{S}(M)$, it holds $\sigma_M(\phi_1, \phi_2) = \sigma(\Gamma_M\phi_1, \Gamma_M\phi_2)$, then $\mathcal{W}(M)$ can be identified with a sub C^*-algebra of $\mathcal{W}(\Im^+)$ by means of a C^*-algebra isomorphism $\imath$ uniquely determined by the requirement*

$$\imath(W_M(\phi)) = W(\Gamma_M\phi)\,, \quad \textit{for all } \phi \in \mathcal{S}(M)\,. \tag{13}$$

In other words, if (i) and (ii) are valid, the field observables of the bulk M can be identified with observables of the boundary $\Im^+$. This is a sort of holographic correspondence.
If (M, g) is Minkowski spacetime (so that $(\tilde{M}, \tilde{g})$ is Einstein closed universe), hypotheses (i) and (ii) are fulfilled so that $\imath$ exists [DMP06]. However there is a large class of asymptotically flat spacetimes which fulfill hypotheses (i) and (ii) as proved in Theorem 4.1 in [Mo06]. They are the asymptotically flat spacetimes, which are globally hyperbolic together with the associated unphysical spacetime and such that *admit future time infinity* i^+ in the sense of Friedirich [Fri86-88]. Roughly speaking we may define an **asymptotically flat vacuum spacetime with future time infinity** i^+ as an asymptotically flat vacuum spacetime at future null infinity (M, g) such that there is a point $i^+ \in \tilde{M} \cap I^+(M)$ ($i^+ \notin \Im^+$) such that the geometric extent of $\Im^+ \cup \{i^+\}$ about i^+ "is the same as that in a region about the tip i^+ of a light cone in a (curved) spacetime". The precise definition is stated in the appendix B (see also the discussion in [Mo06]).

3 The state λ_M: invariance under isometries and energy positivity.

A straightforward but very important consequence of proposition 2.1 is that, whenever (i) and (ii) are fulfilled, *the G_{BMS}-invariant quasifree pure state λ defined on $\Im^+$ can be pulled back to a state λ_M (quasifree by construction) acting on bulk observables* defined by:

$$\lambda_M(a) := \lambda(\imath(a)) \quad \text{for all } a \in \mathcal{W}(M)\,. \tag{14}$$

If (M, g) is Minkowski spacetime, it turns out that λ_M *coincides with Minkowski vacuum.* The main goal of this paper is to study the general properties of λ_M whenever it can be defined, i.e. for those asymptotically flat spacetimes which fulfill the requirements (i) and (ii) of proposition 2.1, in particular,

asymptotically flat spacetimes admitting i^+.

3.1. *The spaces of supertranslations, 4-translations and interplay with bulk symmetries.* In this section we introduce some notions and results, missed in [DMP06, Mo06], which play a central role in studying the properties of λ_M. We focus on the **internal action** of G_{BMS} $G_{BMS} \ni \alpha \mapsto g \circ \alpha \circ g^{-1}$, for any fixed $g \in G_{BMS}$. The decomposition of $h \in G_{BMS}$ as a pair $(\Lambda, f) \in SO(3,1)\uparrow \times C^\infty(\mathbb{S}^2)$ depends on the used admissible frame. However the factor $\Sigma := C^\infty(\mathbb{S}^2)$ is BMS-**invariant**, i.e. invariant under the above-mentioned internal action for every fixed $g \in G_{BMS}$[3] and thus it is well-defined independently from the used admissible frame, since admissible frames are connected to each other by BMS transformations: If $h \in G_{BMS}$ belongs to $C^\infty(\mathbb{S}^2)$ (i.e. has the form (I, α) with $\alpha \in \Sigma$) when referring to an admissible frame, the same result holds referring to any other admissible frame. Σ is called the **group of supertranslations**. However there is another, more important normal subgroup of both G_{BMS} and Σ. That is the **group of 4-translations**:

$$T^4 := \left\{ \alpha = \sum_{j=0,1} \sum_{|m|\leq j} c_{jm} Y_{jm} \;\middle|\; c_{jm} \in \mathbb{C}\,, \alpha(p) \in \mathbb{R}\,, \forall p \in \mathbb{S}^2 \right\}, \tag{15}$$

Y_{jm} being the standard spherical harmonics normalized with respect to the measure of the unit sphere $\mathbb{S}^2$. T^4 turns out to be –once-again – BMS-**invariant** and thus, like Σ, it is well-defined independently from the used admissible frame. Notice that T^4 enjoys the structure of real vector space in addition to that of additive group. By direct inspection one sees that the internal action of G_{BMS} on T^4 defines in fact a representation of G_{BMS} made of *linear* transformations with respect to the real-vector-space structure of T^4. It is possible to pass from the complex basis of T^4, $\{Y_{jm}\}_{j=0,1,|m|\leq J}$ to a real basis $\{Y_\mu\}_{m=0,1,2,3}$ (see [DMP06] and references cited therein for more details[4]), with $Y_0 := \sqrt{2/\pi}$, $Y_1 = -\sqrt{2/\pi}\sin\theta\cos\varphi$, $Y_2 = -\sqrt{2/\pi}\sin\theta\sin\varphi$, $Y_3 = -\sqrt{2/\pi}\cos\theta$. Referring to that basis, if $\alpha := \sum_\mu \alpha^\mu Y_\mu$ and $\tilde{\alpha} := \sum_\mu \tilde{\alpha}^\mu Y_\mu$ the Lorentzian scalar product $< \alpha, \tilde{\alpha} >_{BMS} := -\alpha^0\tilde{\alpha}^0 + \alpha^1\tilde{\alpha}^1 + \alpha^2\tilde{\alpha}^2 + \alpha^3\tilde{\alpha}^3$, turns out to be BMS-**invariant** with respect to the above-mentioned internal (and linear) action of G_{BMS}. As a consequence T^4 results to be equipped with a *light cone structure*: there is a BMS-**invariant** decomposition of $T^4\setminus\{0\}$ into **spacelike**, **timelike** and **null** 4-translations. Every fixed admissible frame $(u, \zeta, \overline{\zeta})$ individuates a time-orientation of T^4. Indeed, consider the BMS diffeomorphism associated with a positive rigid translations of $\Im^+$, $\alpha_\tau : \mathbb{R}\times\mathbb{S}^2 \ni (u, \zeta, \overline{\zeta}) \mapsto (u+\tau, \zeta, \overline{\zeta})$, ($\tau > 0$ fixed). Looking at (2)-(3) one finds that, trivially, α_τ identifies with $\tau\sqrt{\pi/2}Y_0 \in T^4$. Since $\tau > 0$, α_τ picks out the same half of the light-cone not depending on τ. This choice for time-orientation is not affected by changes in the used admissible frame. This is because $n = \partial/\partial u$ is always future-directed with respect to the time-orientation of $(\tilde{M}, \tilde{g})$ induced by that of (M, g) when working in a Bondi frame. The action of BMS group, to pass to a generic admissible frame, does not changes the extent as a consequence of (2)-(3) as one can check by direct inspection. Therefore the light cone in T^4 has a natural preferred time-orientation. With our definition of time-orientation of T^4, if $\alpha \in T^4$ is causal and future-directed, its action on $\Im^+$ displaces the points towards the very future defined in $(\tilde{M}, \tilde{g})$ by the time orientation of (M, g).

The G_{BMS}-subgroup $SO(3,1)\uparrow\!\!\ltimes T^4$ is isomorphic to the proper orthochronous Poincaré group. However, differently from T^4, that group is not normal and different admissible frames select different copies of G_{BMS}-subgroup isomorphic to the proper orthochronous Poincaré group.

We are now ready to state a key result concerning the interplay of BMS group and symmetries. The following proposition is obtained by collecting together several known results but spread in the literature.

[3] In other words Σ and the subsequent subgroup T^4 are *normal* subgroups of G_{BMS}.

[4] In Eq. (3.19) in [DMP06] the statement "if $1 < k \leq l$" has to be corrected to "if $1 \leq k \leq l$", whereas in the right-hand side of subsequent Eq. (3.20), $Y_{l\,-k}$ and $Y_{l\,k}$ have to be corrected to $Y_{l\,-(k-l)}$ and $Y_{l\,(k-l)}$ respectively.

In the Appendix B there is a proof of the statement (c). The results in (a)-(b) can be made much more strong as established in [AX78]. However we do not need here stronger statements than (a)-(b).

Proposition 3.1. *Let (M, g) be an asymptotically flat spacetime. The following facts hold.*
(a) *[Ge77] If ξ is a Killing vector field of (M, g), then ξ smoothly extends to a vector field on $\tilde{M}$. The restriction to $\Im^+$, $\tilde{\xi}$, of such an extension is tangent to $\Im^+$, is uniquely determined by ξ, and generates a one-parameter subgroup of G_{BMS}.*
(b) *[AX78] The linear map $\xi \mapsto \tilde{\xi}$ defined in (a) fulfills the following properties:*
(i) it is injective ($\tilde{\xi}$ is the zero vector field on $\Im^+$ only if ξ is the zero vector field in M);
(iii) if, for a fixed ξ, the one-parameter G_{BMS}-subgroup generated by $\tilde{\xi}$ lies in Σ then, more strictly, it must be a subgroup of T^4.
(c) *Consider a one-parameter subgroup of G_{BMS}, $\{g_t\}_{t\in\mathbb{R}} \subset \Sigma$. Suppose that $\{g_t\}_{t\in\mathbb{R}}$ arises from the integral curves of a smooth vector $\tilde{\xi}$ tangent to $\Im^+$. In any fixed Bondy frame:*

$$g_t : \mathbb{R} \times \mathbb{S}^2 \ni (u, \zeta, \overline{\zeta}) \mapsto \big(u + tf(\zeta, \overline{\zeta}), \zeta, \overline{\zeta}\big) \ ,$$

where the function $f \in C^\infty(\mathbb{S}^2) \equiv \Sigma$ individuates completely the subgroup.

Remark 3.1. The fields $\tilde{\xi}$ associated with one-parameter subgroup of G_{BMS} are always complete since the parameter of the generated one-parameter subgroup ranges in the whole real line by definition. This would be false in case of incompleteness of the field n.

The following proposition can be established by direct inspection from (c) in Proposition 3.1 and (2)-(3).

Proposition 3.2. *Consider a nontrivial one-parameter subgroup of G_{BMS}, $\{g_t\}_{t\in\mathbb{R}} \subset T^4$ generated by a smooth complete vector $\tilde{\xi}$ tangent to $\Im^+$. The following facts hold true referring to the the time-oriented light-cone structure of T^4 defined above.*
(a) *$\{g_t\}_{t\in\mathbb{R}}$ is made of future-directed timelike 4-translations if and only if there is an admissible frame $(u, \zeta, \overline{\zeta})$ such that the action of $\{g_t\}_{t\in\mathbb{R}}$ reduces there to: $g_t : (u, \zeta, \overline{\zeta}) \mapsto (u + t, \zeta, \overline{\zeta})$, $\quad \forall t \in \mathbb{R}$.*
(b) *$\{g_t\}_{t\in\mathbb{R}}$ is made of future-directed causal 4-translations if and only if there is a Bondi frame $(u, \zeta, \overline{\zeta})$ and constants $c > 0$, $a \in \mathbb{R}$ with $|a| \leq 1$, such that the action of $\{g_t\}_{t\in\mathbb{R}}$ reduces there to*

$$g_t : (u, \zeta, \overline{\zeta}) \mapsto \left(u + tc\left(1 - a\frac{\zeta\overline{\zeta} - 1}{\zeta\overline{\zeta} + 1}\right), \zeta, \overline{\zeta}\right) \ , \quad \forall t \in \mathbb{R} \ . \tag{16}$$

These translations are null if and only if $|a| = 1$. They are timelike for $|a| < 1$.
(c) *A 4-translation of $T^4 \setminus \{0\}$ viewed as a function $f \in C^\infty(\mathbb{S}^2)$ in any arbitrarily fixed admissible frame:*
(i) is spacelike if and only if f attains both signs,
(ii) is timelike and future-directed if and only if f is strictly positive,
(iii) is null and future-directed if and only if f is positive and vanishes on a single point of $\mathbb{S}^2$.

Propositions 3.2 and 3.1 have the following technical consequence relevant for our goal whose proof is in the appendix B.

Proposition 3.3. *Let ξ be a Killing vector of an asymptotically flat spacetime (M, g). Then:*
(a) *ξ defines an asymptotic Killing symmetry as expected;*
(b) *If ξ is everywhere causal future-oriented, the associated one-parameter subgroup of G_{BMS} is made of causal future-directed elements of T^4.*

3.2. *Isometry invariance and energy positivity of λ_M.* We go to prove that the state λ_M is invariant under any isometry generated by every complete Killing vector ξ of the bulk spacetime M. Moreover we prove that the spectrum of the self-adjoint generator associated with ξ is positive whenever ξ is timelike and thus the generator may be interpreted as an Hamiltonian with positive energy as it is expected from physics. Positivity of the spectrum of the Hamiltonian is a *stability requirement*: it guarantees that, under small (external) perturbations, the system does not collapse to lower and lower energy states.
The proof of invariance of λ_M is based on the following remarkable result whose proof is in the Appendix B.

Proposition 3.4. *Assume that both the asymptotically flat spacetime (M, g) and the unphysical spacetime $(\tilde{M}, \tilde{g})$ are globally hyperbolic and consider the linear map $\Gamma_M : \mathcal{S}(M) \to C^\infty(\Im^+)$ in proposition 2.1 and the BMS representation A defined in (7).*
If the vector field ξ on (M, g) is complete, smoothly extends to $\tilde{M}$ and defines the asymptotic Killing symmetry $\{g_t^{(\xi)}\}_{t\in\mathbb{R}}$, then the action of that asymptotic symmetry on the field ϕ in M is equivalent to the action of a BMS-symmetry on the associated field $\psi := \Gamma_M\phi$ on $\Im^+$ via the representation A:

$$\Gamma_M(\phi \circ g_{-t}^{(\xi)}) = A_{g_t^{(\tilde{\xi})}}(\psi) \quad \textit{for all } t \in \mathbb{R} \textit{ if } \psi = \Gamma_M\phi \textit{ with } \phi \in \mathcal{S}(M)\,, \tag{17}$$

where $\{g_t^{(\tilde{\xi})}\}_{t\in\mathbb{R}}$ is the one-parameter subgroup of G_{BMS} generated by the smooth extension $\tilde{\xi}$ to $\Im^+$ of ξ.

Notice that, in general, $\phi \circ g_{-t}^{(\xi)}$ does not belong to $\mathcal{S}(M)$ if ϕ does. However it happens when $g_t^{(\xi)}$ is an isometry, since Klein-Gordon equation and thus $\mathcal{S}(M)$ are invariant under isometries of (M, g).
We now prove one of the main results of this work. As is known the *identity component* $\mathcal{G}_1$ of a Lie group $\mathcal{G}$ is the subgroup made of the connected component of $\mathcal{G}$ containing the unit element of $\mathcal{G}$.

Theorem 3.1. *Assume that both the asymptotically flat spacetime (M, g) and the unphysical spacetime $(\tilde{M}, \tilde{g})$ are globally hyperbolic and conditions (i) and (ii) in (b) of proposition 2.1 are fulfilled. Consider the quasifree state λ_M canonically induced on $\mathcal{W}(M)$ from the BMS-invariant quasifree pure state λ defined on $\Im^+$. The following facts are valid.*
(a) *λ_M coincides with (free) Minkowski vacuum if (M, g) is Minkowski spacetime.*
(b) *λ_M is invariant under the identity component $\mathcal{G}_1$ of the Lie group $\mathcal{G}$ of isometries of M:*

$$\lambda_M(\beta_g a) = \lambda_M(a)\,, \quad \textit{for all } a \in \mathcal{W}(M) \textit{ and every } g \in \mathcal{G}_1\,, \tag{18}$$

where β is the (isometric) $*$-algebra isomorphism representation of $\mathcal{G}$ uniquely induced (imposing linearity and continuity) by the requirement on Weyl generators

$$\beta_g(W(\phi)) := W(\phi \circ g^{-1})\,, \quad \textit{for every } \phi \in \mathcal{S}(M) \textit{ and } g \in \mathcal{G}\,.$$

Thus, in particular the Lie-subgroup $\mathcal{G}_1$ admits unitary implementation in the GNS representation of λ_M.
(b)' *The statement (b) holds true as it stands replacing λ_M with any other state λ'_M with $\lambda'_M(a) := \lambda'(\imath(a))$, for all $a \in \mathcal{W}(M)$, where λ' is any BMS invariant state (not necessarily quasifree or pure or satisfying some positivity-energy condition) defined on $\mathcal{W}(\Im^+)$.*
(c) *Assume that (M, g) admits a complete causal future-directed Killing vector ξ. The unitary one-parameter group which leaves the cyclic vector fixed and implements the one-parameter group of isometries generated by ξ in the GNS (Fock) space of λ_M satisfies the following properties:*
(i) it is strongly continuous,
(ii) the associated self-adjoint generator, $H^{(\xi)}$, has nonnegative spectrum,

(iii) the restriction of $H^{(\xi)}$ to the one-particle space has no zero modes.

Remark 3.2. Concerning in particular the statement (c), when ξ is timelike and future-directed, $H^{(\xi)}$ provides a natural (positive) notion of *energy*, associated with ξ displacements.
Since λ_M is quasifree, its GNS representation is a Fock representation. When ξ is timelike, the collection of properties (i), (ii) are summarized [KW91] by saying that λ_M is a **ground state**. Then property (iii) states that λ_M is a **regular** ground state if adopting terminology in [KW91].

Proof of Theorem 3.1. (a) It was proved in theorem 4.5 [DMP06]. (b) It is well-known [O'N83] that there is a unique way to assign a Lie-group structure to the group of isometries $\mathcal{G}$ of a (semi-)Riemannian manifold (M, g) in order that the action of the one-parameter subgroup is jointly smooth when acting on the manifold. Moreover the Lie algebra of $\mathcal{G}$ is that of complete Killing vectors of (M, g). Finally using the exponential map one sees that every element of the identity component $\mathcal{G}_1$ can be obtained as a finite product of elements which belong to one-parameter subgroups. As a consequence, to establish the validity of (b) it is sufficient to prove that λ_M is invariant under the one-parameter subgroups generated by complete Killing vectors of (M, g). Let us prove it. Let ξ be a complete Killing vector of (M, g) and $\tilde{\xi}$ the associated generator of G_{BMS} on $\Im^+$ in view of Proposition 3.1. Employing the same notation as in Proposition 3.4 and using the definition (14), one achieves:

$$\lambda_M \left(W_M(\phi \circ g_{-t}^{(\xi)}) \right) = \lambda \left(W(A_{g_t^{(\tilde{\xi})}}(\psi)) \right) .$$

The right hand side is, by definition,

$$\lambda \left(\alpha_{g_t^{(\tilde{\xi})}}(W(\psi)) \right) = \lambda \left(W(\psi) \right) ,$$

where, in the last step we have used the invariance of λ under the representation α of BMS-group defined in section 2.1. Since $\psi = \Gamma_M \phi$ and using (14) again we have finally obtained that

$$\lambda_M \left(W_M(\phi \circ g_{-t}^{(\xi)}) \right) = \lambda_M \left(W_M(\phi) \right) .$$

By linearity and continuity this result extends the the whole algebra $\mathcal{W}(M)$:

$$\lambda_M(\beta_{g_t^{(\xi)}}(a)) = \lambda_M(a) , \quad \text{for every } a \in \mathcal{W}(M) .$$

Since the state is invariant, in its GNS representation, there is a unique unitary implementation of the representation β which leaves fixed the cyclic vector (e.g. see [Ar99]). The proof of (b)' is the same as that given for (b), replacing λ_M with λ'_M.
(c) As λ_M being quasifree, its GNS representation is a Fock representation (e.g. see the appendix A of [Mo06] and references cited therein, especially [KW91]). As a consequence it is sufficient to prove the positivity property for the restriction of the unitary group which represents the group of isometries in the one-particle space $\mathcal{H}_M$. The GNS triple of λ_M is obtained as follows. Consider the GNS triple of λ, $(\mathfrak{H}, \Pi, \Upsilon)$ where $\mathfrak{H} = \mathcal{F}_+(\mathcal{H})$ is the bosonic Fock space with one-particle space $\mathcal{H}$. As said above, that space, is isomorphic to the space of (Fourier transforms of the) u-positive frequency parts $L^2(\mathbb{R}_+ \times \mathbb{S}^2, 2kdk \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}))$ when a Bondi frame $(u, \zeta, \overline{\zeta})$ is fixed, k being the Fourier variable associated with u. Consider the Hilbert subspace $\mathcal{H}_M$ of $\mathcal{H}$ obtained by taking the closure of the complex span of the u-positive-frequency parts of the wavefunctions $\Gamma_M \phi$, for every $\phi \in \mathcal{S}(M)$. Let $\mathfrak{H}_M = \mathcal{F}_+(\mathcal{H}_M)$ be the Fock space generated by $\mathcal{H}_M$ which, in turn, is a Hilbert subspace of $\mathfrak{H}$. Notice that we are assuming that the vacuum vectors Υ_M and Υ coincide. By construction $\mathfrak{H}_M$ is invariant under Π and $\Pi_M := \Pi\restriction_{\mathfrak{H}_M}$

is a $*$-representation of $\imath(\mathcal{W}(M))$. Moreover $\Pi_M(\imath(\mathcal{W}(M)))\Upsilon_M = \Pi_M(\imath(\mathcal{W}(M)))\Upsilon$ is dense in $\mathfrak{H}_M$ by construction. By the uniqueness (up to unitary maps) property of the GNS triple, we conclude that $(\mathfrak{H}_M, \Pi_M, \Upsilon_M)$ is the GNS triple of λ_M[5].

Consider the unique unitary G_{BMS} representation U which acts on $\mathfrak{H}$ implementing α and leaving $\Upsilon(=\Upsilon_M)$ fixed. It is the unitary BMS representation defined by linearity and continuity by the requirement:

$$U_g\Pi(W(\psi))\Upsilon := \Pi(\alpha_g(W(\psi)))\Upsilon\ , \quad \text{for all } \psi \in \mathsf{S}(\Im^+) \text{ and } g \in G_{BMS}\ . \tag{19}$$

Since the space $\mathcal{S}(M)$ is invariant under the group of isometries $g_t^{(\xi)}$ and (17) holds true, it arises that $\alpha_{g_t^{(\tilde{\xi})}}(W(\psi)) \in \imath(\mathcal{W}(M))$ if $W(\psi) \in \imath(\mathcal{W}(M))$ and thus (19) entails

$$U_{g_t^{(\tilde{\xi})}}\Pi_M(W_M(\phi))\Upsilon_M := \Pi(\beta_{g_t^{(\xi)}}(W_M(\phi)))\Upsilon_M\ , \quad \text{for all } \phi \in \mathcal{S}(M) \text{ and } t \in \mathbb{R}\ . \tag{20}$$

As a consequence of (20) we can conclude that the unique unitary representation $U^{(\xi)}$ of $\{g_t^{(\xi)}\}_{t\in\mathbb{R}}$ on $\mathfrak{H}_M$ which leaves Υ_M invariant, is nothing but the restriction of U to $\{g_t^{(\tilde{\xi})}\}_{t\in\mathbb{R}} \subset G_{BMS}$ and $\mathfrak{H}_M \subset \mathfrak{H}$. This result allows us to compute explicitly the self-adjoint generator of the unitary representation of $\{g_t^{(\xi)}\}_{t\in\mathbb{R}}$. The representation U is obtained by tensorialization of a unitary representation of G_{BMS} working in the one particle space [DMP06, Mo06] (notice that in those papers, as one-particle space, we used the unitarily isomorphic space $L^2(\mathbb{R}^+ \times \mathbb{S}^2; dk \wedge \epsilon_{\mathbb{S}^2})$ instead of $L^2(\mathbb{R}^+ \times \mathbb{S}^2; 2kdk \wedge \epsilon_{\mathbb{S}^2})$ therefore the expression above looks different, but it is equivalent to that given in [DMP06, Mo06]):

$$\left(U^{(1)}_{(\Lambda,f)}\varphi\right)(k,\zeta,\overline{\zeta}) = e^{ikK_\Lambda(\Lambda^{-1}(\zeta,\overline{\zeta}))f(\Lambda^{-1}(\zeta,\overline{\zeta}))}\varphi\left(kK_\Lambda\left(\Lambda^{-1}(\zeta,\overline{\zeta})\right), \Lambda^{-1}(\zeta,\overline{\zeta})\right)\ , \tag{21}$$

for every $\varphi \in L^2(\mathbb{R}^+ \times \mathbb{S}^2; 2kdk \wedge \epsilon_{\mathbb{S}^2})$ and $G_{BMS} \ni g \equiv (\Lambda, f)$. The restriction of $U^{(1)}$ to $\{g_t^{(\tilde{\xi})}\}_{t\in\mathbb{R}}$ and $\mathcal{H}_M$ defines a unitary representation of $\{g_t^{(\xi)}\}_{t\in\mathbb{R}}$ whose tensorialization on $\mathcal{F}_+(\mathcal{H}_M)$ is the very representation $U^{(\xi)}$. Notice that $U^{(1)}$ restricted to $\{g_t^{(\tilde{\xi})}\}_{t\in\mathbb{R}}$ leaves invariant $\mathcal{H}_M$ by construction because $\mathcal{H}_M$ is the closure of the span of vectors $d/dt|_{t=0}\Pi(W(t\psi))\Upsilon$ for $\psi \in \Gamma_M(\mathcal{S}(M))$ (the derivative being computed using the Hilbert topology). To conclude the proof it is sufficient to prove that the self-adjoint generator of $\left\{U^{(1)}{}_{g_t^{(\tilde{\xi})}} \restriction_{\mathcal{H}_M}\right\}_{t\in\mathbb{R}}$ exists and has positive spectrum.

In our hypotheses $\{g_t^{(\tilde{\xi})}\}$ is a one-parameter group of causal future-directed 4-translations. As a consequence, selecting the Bondi frame as in (b) in Proposition 3.2, we have that there are a fixed real a with with $|a| \leq 1$ and a fixed real $c > 0$ such that, for every $t \in \mathbb{R}$

$$g_t^{(\tilde{\xi})} : (u,\zeta,\overline{\zeta}) \mapsto \left(u + tc\left(1 - a\frac{\zeta\overline{\zeta}-1}{\zeta\overline{\zeta}+1}\right), \zeta, \overline{\zeta}\right)\ .$$

Therefore, if $\varphi \in \mathcal{H} \equiv L^2(\mathbb{R}^+ \times \mathbb{S}^2; 2kdk \wedge \epsilon_{\mathbb{S}^2})$

$$\left(U^{(1)}_{g_t^{(\tilde{\xi})}}\varphi\right)(k,\zeta,\overline{\zeta}) = e^{itkc\left(1-a\frac{\zeta\overline{\zeta}-1}{\zeta\overline{\zeta}+1}\right)}\varphi\left(k,\zeta,\overline{\zeta}\right)\ . \tag{22}$$

Strong continuity is obvious (also after restriction to $\mathcal{H}_M$). Finally, using Lebesgue's dominate convergence to evaluate the strong-operator topology derivative at $t = 0$ of $U^{(1)}_{g_t^{(\tilde{\xi})}}$, one obtains that this derivative is $ih^{(\xi)}$ where the self-adjoint operator $h^{(\xi)}$ is

$$(h^{(\xi)}\varphi)(k,\zeta,\overline{\zeta}) := kc\left(1 - a\frac{\zeta\overline{\zeta}-1}{\zeta\overline{\zeta}+1}\right)\varphi(k,\zeta,\overline{\zeta})\ , \tag{23}$$

[5]Notice that Π_M may be reducible also if Π is irreducible: in other words λ_M may be a mixture also if λ is pure.

defined in the dense domain $\mathcal{D}(h^{(\xi)})$ made of the vectors $\varphi \in L^2(\mathbb{R}^+ \times \mathbb{S}^2; 2kdk \wedge \epsilon_{\mathbb{S}^2})$ such that the right-hand side of (23) belongs to $L^2(\mathbb{R}^+ \times \mathbb{S}^2; 2kdk \wedge \epsilon_{\mathbb{S}^2})$ again. In view of Stone theorem $h^{(\xi)}$ is the self-adjoint generator of $U^{(1)}_{g_t^{(\xi)}}$. Notice that passing to work in polar coordinates:

$$kc\left(1 - a\frac{\zeta\overline{\zeta} - 1}{\zeta\overline{\zeta} + 1}\right) = kc\,(1 - a\cos\theta) \geq 0 \tag{24}$$

because $k \in [0 + \infty)$, $c > 0$ and $a \in \mathbb{R}$ with $|a| \leq 1$. Therefore interpreting the integral below as a Lebesgue integral in $F := [0, +\infty) \times [0, \pi] \times [-\pi, \pi]$:

$$\left\langle \varphi, h^{(\xi)}\varphi \right\rangle = 2c \int_F |\varphi(k, \theta, \phi)|^2\,(1 - a\cos\theta)\,k^2 \sin^2\theta\, dkd\theta d\phi \geq 0\,, \quad \text{for all } \varphi \in \mathcal{D}(h^{(\xi)}). \tag{25}$$

This fact entails that the spectrum of $h^{(\xi)}$ is included in $[0, +\infty)$ via spectral theorem. The result remains unchanged when restricting $U^{(1)}_{g_t^{(\xi)}}$ (and thus $h^{(\xi)}$) to the invariant Hilbert-subspace $\mathcal{H}_M$.
Suppose there is a zero mode of $h^{(\xi)}$, that is $\varphi \in \mathcal{H}_M \setminus \{0\}$ with $h^{(\xi)}\varphi = 0$. By (25),

$$\int_F |\varphi(k, \theta, \phi)|^2\,(1 - a\cos\theta)\,k^2 \sin^2\theta\, dkd\theta d\phi = 0\,,$$

The integrand is nonnegative on F by construction in particular because (24) is valid, therefore we conclude that the integrand vanishes almost everywhere in the Lebesgue measure of $\mathbb{R}^3$. Since the function $(k, \theta, \phi) \mapsto (1 - a\cos\theta)\,k^2 \sin^2\theta$ is almost-everywhere strictly positive on F, φ must vanish almost everywhere therein, so that $\varphi = 0$ when one thinks of φ as an element of $L^2(\mathbb{R}^+ \times \mathbb{S}^2; 2kdk \wedge \epsilon_{\mathbb{S}^2})$. In other words $h^{(\xi)}$ has no zero modes. □

3.3. *Reformulation of the uniqueness theorem for λ.* It is clear that there are asymptotically flat spacetimes which do not admit isometries at all. In that case the invariance property stated in (a) and the positivity energy condition (c) of Theorem 3.1 are meaningless. However those statements remain valid if referring to the asymptotic theory based on QFT on $\Im^+$ and the universal state λ. Indeed λ is invariant under the whole G_{BMS} group – which represents *asymptotic symmetries* of every asymptotically flat spacetime – and λ satisfies a positivity energy condition with respect to every smooth one-parameter subgroup og G_{BMS} made of future-directed timelike or null 4-translations – which correspond to Killing-time evolutions whenever the spacetime admits a timelike Killing field, as established above.
As proved in theorem 3.1 in [Mo06], the energy positivity condition with respect to timelike 4-translations determines uniquely λ. We may restate it into a more invariant form as follows. The possibility of such a re-formulation was already noticed in a comment in [Mo06], here we do it explicitly[6].

Theorem 3.2. *Consider a nontrivial one-parameter subgroup of G_{BMS}, $G := \{g_t\}_{t\in\mathbb{R}}$ made of future-directed timelike 4-translations, associated with a smooth complete vector tangent to $\Im^+$ and let $\alpha^{(G)}$ be the one-parameter group of $*$-isomorphisms induced by G on $\mathcal{W}(\Im^+)$.*
(a) *The BMS-invariant state λ is the unique pure quasifree state on $\mathcal{W}(\Im^+)$ satisfying both:*
(i) it is invariant under $\alpha^{(G)}$,
(ii) the unitary group which implements $\alpha^{(G)}$ leaving fixed the cyclic GNS vector is strongly continuous with nonnegative generator.

[6]The author is grateful to A. Ashtekar for suggesting this improved formulation of the theorem.

(b) *Let ω be a pure (not necessarily quasifree) state on $\mathcal{W}(\Im^+)$ which is BMS-invariant or, more weakly, $\alpha^{(G)}$-invariant. ω is the unique state on $\mathcal{W}(\Im^+)$ satisfying both:*
(i) it is invariant under $\alpha^{(G)}$,
(ii) it belongs to the folium of ω.

Proof. The proof is that given for theorem 3.1 in [Mo06] working in the admissible frame, individuated in (a) of Proposition 3.2, where G reduces there to

$$g_t : (u, \zeta, \overline{\zeta}) \mapsto (u + t, \zeta, \overline{\zeta}) \,, \quad \forall t \in \mathbb{R} \,. \quad \square$$

4 The Hadamard property.

4.1. *Hadamard states.* It is well known that Hadamard states [KW91, Wa94] have particular physical interest in relation with the definition of physical quantities which, as the stress-energy tensor operator (e.g. see [Mo03, HW04]), cannot be represented in terms of elements of the Weyl algebra or the associated $*$-algebra of products of smeared field operators. In the last decade the deep and strong relevance of Hadamard states in local generally covariant QFT in curved spacetime has been emphasized from different points of view (e.g. see [HW01, BFV03]). The rigorous definition of Hadamard state ω – referring to a quantum scalar real bosonic field ϕ propagating in a globally hyperbolic spacetime (M, g) satisfying Klein-Gordon equation with Klein-Gordon operator $P := \Box + V(x)$ (V being any fixed smooth real function) – has been given in [KW91] in terms of a requirement on the behaviour of the singular part of the integral kernel of two-point function defined as the bi-linear functional $\omega : C_0^\infty(M) \times C_0^\infty(M) \to \mathbb{C}$ with:

$$\omega(f, g) := -\frac{\partial^2}{\partial s \partial t} \left\{ \omega \left(W_M(sEf + tEg) \right) e^{ist\sigma_M(Ef,Eg)/2} \right\}\Big|_{s=t=0}, \quad f, g \in C_0^\infty(M) \times C_0^\infty(M). \tag{26}$$

Above $E : C_0^\infty(M) \to \mathcal{S}(M) \subset C^\infty(M)$ is the previously mentioned *causal propagator*, the two-point function $\omega(f, g)$ exists if and only if the right-hand side makes sense for every pair f, g. This happens in particular whenever the GNS representation of ω is a Fock representation (for instance, that is the case if ω is quasifree [KW91] see also appendix A in [Mo06]). In that case – the proof is straightforward and it provides an heuristic motivation for the definition (26) – one finds

$$\omega(f, g) = \langle \Upsilon_\omega, \Phi(f)\Phi(g)\Upsilon_\omega \rangle \,, \tag{27}$$

where Υ_ω is the cyclic GNS vector which, in this case, coincides with the Fock vacuum vector and $\Phi(f)$ denotes the self-adjoint field operator smeared with the smooth function f defined in the GNS Hilbert space $\mathfrak{H}_\omega$. Notice that, since $E \circ P = P \circ E = 0$ if the two-point function exists one gets [Wa94]:

$$\omega(Pf, g) = \omega(f, Pg) = 0 \quad \text{(KG)}$$

and, directly from (26),

$$\omega(f, g) - \omega(g, f) = -i \int_M f(x) \left(E(g) \right)(x) \, d\mu_g(x) \quad \text{(Com)} \,.$$

If the a two-point function $\omega(f, g)$ is well-defined, the integral kernel $\omega(x, y)$ is defined (if it exists at all) as the function, generally singular and affected by some $\epsilon \to 0^+$ prescription, such that

$$\int_{M \times M} \omega(x, y) f(x) g(y) \, d\mu_g(x) d\mu_g(y) = \omega(f, g) \,, \quad \text{for all } f, g \in C_0^\infty(M) \,. \tag{28}$$

Referring to a quantum scalar real bosonic field ϕ propagating in a globally hyperbolic spacetime (M, g) satisfying Klein-Gordon equation, a quasifree state ω which admits two-point function (indicated with the same symbol) is said to be **Hadamard** if the integral kernel of its two-point function $\omega(x, y)$ exists and satisfies the **global Hadamard prescription**. That prescription requires that the integral kernel $\omega(x, y)$ takes a certain – quite complicated – form in a neighborhood of a Cauchy surface of the spacetime discussed in details in Sec. 3.3 of [KW91]. The global Hadamard condition implies the **local Hadamard condition** which states that, for every point $p \in M$ there is a (geodesically convex normal) neighborhood G_p, such that

$$\omega(x,y) = \text{w-}\lim_{\epsilon\to 0^+} \left\{ \frac{U(x,y)}{\sigma(x,y) + 2i\epsilon(T(x)-T(y)) + \epsilon^2} + V(x,y)\ln(\sigma(x,y) + 2i\epsilon(T(x)-T(y)) + \epsilon^2) \right\} + \omega_{reg}(x,y)\,, \quad \text{if } (x,y) \in G_p \times G_p, \tag{LH}$$

where $\sigma(x, y)$ is the squared geodesic distance of x from y, T is any, arbitrarily fixed, time function increasing to the future and U and V are locally well-defined quantities depending on the local geometry only. Finally ω_{reg} is smooth and is, in fact the part of the two-point function determining the state. w-$\lim_{\epsilon\to 0^+}$ indicates that the limit as $\epsilon \to 0^+$ as to be understood in weak sense, i.e. *after* the integration of $\omega(x, y)$ with smooth compactly supported functions f and g.
In a pair of very remarkable papers [Ra96a, Ra96b] Radzikowski established several important results about Hadamard states, in particular he found out a *microlocal characterization* of Hadamard states (part of theorem 5.1 in [Ra96a]):

Proposition 4.1. *In a globally hyperbolic spacetime (M, g), consider a quasifree state with two-point function ω (so that (KG) and (Com) are valid) defining a distribution of $\mathscr{D}'(M \times M)$. The state is Hadamard if and only if the wavefront set $WF(\omega)$ of the distribution is*

$$WF(\omega) = \{((x, k_x), (y, -k_y)) \in T^*M \setminus \mathbf{0} \times T^*M \setminus \mathbf{0} \mid (x, k_x) \sim (y, k_y)\,, k_x \rhd 0\} \tag{29}$$

where $(x, k_x) \sim (y, k_y)$ means that there is a null geodesic joining x and y with co-tangent vectors at x and y given by k_x and k_y respectively, whereas $k \rhd 0$ means that k is causal and future directed. $\mathbf{0}$ is the zero section of the cotangent bundle.

A second result by Radzikowski, which in fact proved a conjecture by Kay, establishes that (immediate consequence of Corollary 11.1 in [Ra96b]):

Proposition 4.2. *In a globally hyperbolic spacetime (M, g), if the two-point function of a quasifree state is a distribution of $\mathcal{D}(M \times M)$ whose (Schwartz) kernel satisfies (LH) in some neighborhood G_P of every point $p \in M$, then the state is Hadamard.*

In the following we shall prove that, in the presence of i^+, the following results hold true. (i) λ_M is a distribution of $\mathcal{D}'(M \times M)$ and, making use of Radzikowski results, (ii) λ_M is Hadamard. To tackle the item (i) we have to introduce some notions concerning a straightforward extension of Fourier-Plancherel transform theory for functions and distributions defined on $\Im^+ \equiv \mathbb{R} \times \mathbb{S}^2$.

4.2. *Fourier-Plancherel transform on $\mathbb{R} \times \mathbb{S}^2$.* Define $\mathscr{S}(\Im^+)$ as the complex linear space of the smooth functions $\psi : \Im^+ \to \mathbb{C}$ such that, in a fixed Bondi frame, ψ with all derivatives vanish as $|u| \to +\infty$, uniformly in $\zeta, \overline{\zeta}$, faster than $|u|^{-k}$, $\forall k \in \mathbb{N}$. The space $\mathscr{S}(\Im^+)$ generalizes straightforwardly Schwartz' distribution space on $\mathbb{R}^n$, $\mathscr{S}'(\mathbb{R}^n)$. It can be equipped with the Hausdorff topology induced from the

countable class seminorms – *they depend on the Bondi frame but the topology does not* – $p, q, m, n \in \mathbb{N}$,

$$||\psi||_{p,q,m,n} := \sup_{(u,\zeta,\overline{\zeta})\in\Im^+} \left| |u|^p \partial_u^q \partial_\zeta^m \partial_{\overline{\zeta}}^n \psi(u,\zeta,\overline{\zeta}) \right| .$$

$\mathscr{S}(\Im^+)$ is dense in both $L^1(\mathbb{R}\times\mathbb{S}^2, du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}))$ and $L^2(\mathbb{R}\times\mathbb{S}^2, du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}))$ (with the topology of these spaces which are weaker than that of $\mathscr{S}(\Im^+)$), because it includes the dense space $C_0^\infty(\mathbb{R}\times\mathbb{S}^2;\mathbb{C})$ of smooth compactly-supported complex-valued functions. We also define the space of *distributions* $\mathscr{S}'(\Im^+)$ containing all the linear functionals from $\mathbb{R}\times\mathbb{S}^2$ to $\mathbb{C}$ which are weakly continuous with respect to the topology of $\mathscr{S}(\Im^+)$. Obviously $\mathscr{S}(\Im^+)\subset\mathscr{S}'(\Im^+)$ and $L^p(\mathbb{R}\times\mathbb{S}^2, du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}))\subset\mathscr{S}'(\Im^+)$ for $p=1,2$. We introduce the Fourier transforms $\mathscr{F}_\pm[f]$ of $f\in\mathscr{S}(\Im^+)$

$$\mathscr{F}_\pm[f](k,\zeta,\overline{\zeta}) := \int_{\mathbb{R}} \frac{e^{\pm iku}}{\sqrt{2\pi}} f(u,\zeta,\overline{\zeta})du\,, \quad (k,\zeta,\overline{\zeta})\in\mathbb{R}\times\mathbb{S}^2\,.$$

$\mathscr{F}_\pm$ enjoy the properties listed below which are straightforward extensions of the analogs for standard Fourier transform in $\mathbb{R}^n$. The proof of the following theorem is in the Appendix B. It is possible, but useless for our present goal, to state an analog of Riemann-Lebesgue lemma in $L^1(\mathbb{R}\times\mathbb{S}^2, du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}))$[7].

Theorem 4.1. *The maps $\mathscr{F}_\pm$ satisfy the following properties.*
(a) *for all $p,m,n\in\mathbb{N}$ and every $\psi\in\mathscr{S}(\Im^+)$ it holds*

$$\mathscr{F}_\pm\left[\partial_u^p\partial_\zeta^m\partial_{\overline{\zeta}}^n\psi\right](k,\zeta,\overline{\zeta}) = (\pm i)^p k^p \partial_\zeta^m\partial_{\overline{\zeta}}^n\psi\mathscr{F}_\pm[\psi](k,\zeta,\overline{\zeta})\,.$$

(b) *$\mathscr{F}_\pm$ are continuous bijections onto $\mathscr{S}(\Im^+)$ and $\mathscr{F}_- = (\mathscr{F}_+)^{-1}$.*
(c) *If $\psi,\phi\in\mathscr{S}(\Im^+)$ one has*

$$\int_{\mathbb{R}} \overline{\mathscr{F}_\pm[\psi](k,\zeta,\overline{\zeta})}\mathscr{F}_\pm[\phi](k,\zeta,\overline{\zeta})dk = \int_{\mathbb{R}} \overline{\psi(u,\zeta,\overline{\zeta})}\phi(u,\zeta,\overline{\zeta})du\,,\ \textit{for all } (\zeta,\overline{\zeta})\in\mathbb{S}^2\,, \tag{30}$$

$$\int_{\mathbb{R}\times\mathbb{S}^2} \overline{\mathscr{F}_\pm[\psi](k,\zeta,\overline{\zeta})}\mathscr{F}_\pm[\phi](k,\zeta,\overline{\zeta})dk\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}) = \int_{\mathbb{R}\times\mathbb{S}^2} \overline{\psi(u,\zeta,\overline{\zeta})}\phi(u,\zeta,\overline{\zeta})du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta})\,. \tag{31}$$

(d) *If $T\in\mathscr{S}'(\Im^+)$ the definition $\mathscr{F}_\pm T[f] := T\left(\mathscr{F}_\pm[f]\right)$,for all $f\in\mathscr{S}'(\Im^+)$ is well-posed, gives rise to the unique weakly continuous linear extension of $\mathscr{F}_\pm$ to $\mathscr{S}'(\Im^+)$ and one has, with the usual definition of derivative of a distribution,*

$$\mathscr{F}_\pm\left[\partial_u^p\partial_\zeta^m\partial_{\overline{\zeta}}^n T\right] = (\pm i)^p k^p\partial_\zeta^m\partial_{\overline{\zeta}}^n\mathscr{F}_\pm[T]\,,\quad \textit{for all } p,m,n\in\mathbb{N}\,.$$

(e) Plancherel theorem. *$\mathscr{F}_\pm$ extend uniquely to unitary transformations from the Hilbert space $L^2(\mathbb{R}\times\mathbb{S}^2, du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}))$ to $L^2(\mathbb{R}\times\mathbb{S}^2, du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}))$ and the extension of $\mathscr{F}_-$ is the inverse of that of $\mathscr{F}_+$. These extensions coincide respectively with the restrictions to $L^2(\mathbb{R}\times\mathbb{S}^2, du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}))$ of the action of $\mathscr{F}_\pm$ on distributions as in (4).*
(f) *If $\tilde{\mathscr{F}}_\pm : L^2(\mathbb{R},du)\to L^2(\mathbb{R},du)$ denotes the standard Fourier transform on the line, for every $\psi\in L^2(\mathbb{R}\times\mathbb{S}^2, du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}))$ it holds:*

$$\mathscr{F}_\pm[\psi](k,\zeta,\overline{\zeta}) = \tilde{\mathscr{F}}_\pm(\psi(\cdot,\zeta,\overline{\zeta}))(k)\,,\quad \textit{almost everywhere on } \mathbb{R}\times\mathbb{S}^2. \tag{32}$$

[7]The corresponding statement of (6) in Theorem C1 in Appendix C of [Mo06] is erroneous, but this fact affects by no means the results achieved in [Mo06] since that statement did not enter the paper anywhere.

As a consequence, if $\psi, \phi \in L^2(\mathbb{R}\times\mathbb{S}^2, du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}))$, one may say that almost everywhere in $(\zeta,\overline{\zeta})\in\mathbb{S}^2$:

$$\int_{\mathbb{R}} \overline{\mathscr{F}_\pm[\psi](k,\zeta,\overline{\zeta})}\mathscr{F}_\pm[\phi](k,\zeta,\overline{\zeta})dk \quad = \quad \int_{\mathbb{R}} \overline{\psi(u,\zeta,\overline{\zeta})}\phi(u,\zeta,\overline{\zeta})du\,. \tag{33}$$

(g) *If $m\in\mathbb{N}$ and $T\in\mathscr{S}'(\Im^+)$, $\mathscr{F}_+[T]$ is a measurable function satisfying*

$$\int_{\mathbb{R}\times\mathbb{S}^2}(1+|k|^2)^m|\mathscr{F}_+[T]|^2 dk\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta})<+\infty$$

if and only if the u-derivatives of T in the sense of distributions, are measurable functions with

$$\partial_u^n T\in L^2(\mathbb{R}\times\mathbb{S}^2, du\wedge\epsilon_{\mathbb{S}^2})\,,\ \ \textit{for } n=0,1,\ldots,m.$$

From now on $\mathscr{F}:\mathscr{S}'(\Im^+)\to\mathscr{S}'(\Im^+)$ denotes the extension to distributions of $\mathscr{F}_+$ as stated in (d) in theorem 4.1 whose inverse, $\mathscr{F}^{-1}$, is the analogous extension of $\mathscr{F}_-$. We call $\mathscr{F}$ **Fourier-Plancherel transformation**, also if, properly speaking this name should be reserved to its restriction to $L^2(\mathbb{R}\times\mathbb{S}^2, du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}))$ defined in (e) in theorem 4.1. We also use the formal distributional notation for $\mathscr{F}$ (and the analog for $\mathscr{F}^{-1}$)

$$\mathscr{F}[\psi](k,\zeta,\overline{\zeta}):=\int_{\mathbb{R}}\frac{e^{iku}}{\sqrt{2\pi}}\psi(u,\zeta,\overline{\zeta})du\,,$$

regardless if f is a function or a distribution. Throughout the paper the notation $\widehat{\psi}(k,\zeta,\overline{\zeta})$ is also used for the Fourier(-Plancherel, extension to distributions) transform $\mathscr{F}[\psi](k,\zeta,\overline{\zeta})$.

4.3. *The integral kernel of λ_M is a distribution when (M,g) admits i^+.* Since the considered spacetimes are equipped, by definitions, with metrics and thus preferred volume measures, here we assume that distributions of $\mathscr{D}'(M\times M)$ work on smooth compactly-supported scalar fields of $\mathscr{D}(M):=C_0^\infty(M)$ as in [Fr75] instead of smooth compactly-supported scalar densities as in [Hör71]. As is well known this choice is pure matter of convention.
First of all we prove that λ_M individuates a distribution in $\mathscr{D}'(M\times M)$, i.e. it is continuous in the relevant weak topology [Fr75], whenever the spacetime (M,g) is a vacuum asymptotically flat at null infinity spacetime and admits future temporal infinity i^+. We give also a useful explicit expression for the distribution.

Theorem 4.2. *Assume that the spacetime (M,g) is an asymptotically flat vacuum spacetime with future time infinity i^+ (Definition A.1) and that both (M,g) and the unphysical spacetime $(\tilde{M},\tilde{g})$ are globally hyperbolic. Let $E:C_0^\infty(M)\to\mathcal{S}(M)$ be the causal propagator associated with the real massless, conformally coupled Klein-Gordon operator P defined by Eq.(11) on (M,g). Then the following facts are valid concerning the state λ_M defined in Eq.(14).*
(a) *Referring to a Bondi frame $(u,\zeta,\overline{\zeta})$ one has*

$$\lambda_M(f,g)=\lim_{\epsilon\to 0^+}-\frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{S}^2}\frac{\psi_f(u,\zeta,\overline{\zeta})\psi_g(u',\zeta,\overline{\zeta})}{(u-u'-i\epsilon)^2}\,du\wedge du'\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta})\,, \tag{34}$$

where $\psi_h:=\Gamma_M(Eh)$ for all $h\in C_0^\infty(M)$ with $\Gamma_M:\mathcal{S}(M)\to\mathsf{S}(\Im^+)$ defined in Proposition 2.1.
(b) *The two-point function of the state λ_M individuates a distribution of $\mathscr{D}'(M\times M)$.*

Remark 4.1. It is intriguing noticing that the expression (34) is the same as that for two-point functions of quasifree *Hadamard* states obtained in [KW91] (Eq. (4.13)) in globally hyperbolic spacetimes with bifurcate Killing horizon. In that case the null 3-manifold $\Im^+$ is replaced by a bifurcate Killing horizon, the 2-dimensional cross section $\mathbb{S}^2$ with spacelike metric corresponds to the bifurcation surface Σ with spacelike metric and the null geodesics forming $\Im^+$, parametrized by the affine parameter u, correspond to the null geodesics forming the Killing horizon parametrized by the affine parameter U.

Proof of theorem 4.2. We start with a useful lemma whose proof is in the Appendix B.

Lemma 4.1. *In the hypotheses of theorem above, if $h \in C_0^\infty(M)$, the following holds.*
(a) *ψ_h can be written in terms of the causal propagator $\tilde{E}$ for the massless conformally coupled Klein-Gordon operator $\tilde{P}$ in $(\tilde{M}, \tilde{g} = \Omega^2 g)$ and the smooth function $\omega_B > 0$ defined on $\Im^+$ introduced in Section 1.2:*

$$\psi_h(u,\zeta,\overline{\zeta}) = \omega_B(u,\zeta,\overline{\zeta})^{-1}\tilde{E}(\Omega^{-3}h)\restriction^+_{\Im}(u,\zeta,\overline{\zeta})\,, \quad \text{for } u \in \mathbb{R} \text{ and } (\zeta,\overline{\zeta}) \in \mathbb{S}^2. \tag{35}$$

(b) *For any compact $K \subset M$ there is $u_0 \in \mathbb{R}$ such that $\psi_h(u,\zeta,\overline{\zeta}) = 0$ for all $(\zeta,\overline{\zeta}) \in \mathbb{S}^2$ and $u < u_0$ if $\operatorname{supp} h \subset K$.*

Let us pass to the main proof. From now on we use the content section 4.2.
(a) We start from the fact that, as found in the proof of Theorem 3.1, the Fock GNS triple of λ_M, $(\mathfrak{H}_M, \Pi_M, \Upsilon_M)$ is such that $\Upsilon_M = \Upsilon$ and $\Pi_M(W_M(\psi)) = \Pi_M(W(\Gamma_M(\psi)))$ In our hypotheses, since λ_M is quasifree, one has referring to its GNS representation $(\mathfrak{H}_M, \Pi_M, \Upsilon_M)$:

$$\lambda_M(f,g) = \langle \Upsilon_M, \Phi(f)\Phi(g)\Upsilon_M\rangle = \langle \Upsilon, \sigma(\Psi,\Gamma_M(Ef))\sigma(\Psi,\Gamma_M(Eg))\Upsilon\rangle = \langle \psi_{f+}, \psi_{g+}\rangle\,.$$

where ψ_{h+} is the u-positive frequency part of $\Gamma_M(Eh)$. Using (10), if $\widehat{\psi_f}$ is the Fourier-Plancherel transform of ψ_h one has finally:

$$\lambda_M(f,g) = \int_{\mathbb{R}_+\times\mathbb{S}^2} 2k\overline{\widehat{\psi_f}(k,\zeta,\overline{\zeta})}\widehat{\psi_g}(k,\zeta,\overline{\zeta})dk \wedge \epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta})\,.$$

If $\Theta(k) = 0$ for $k \leq 0$ and $\Theta(k) = 1$ for $k > 0$, the identity above can be rewritten as

$$\lambda_M(f,g) = \int_{\mathbb{R}\times\mathbb{S}^2} \overline{\widehat{\psi_f}(k,\zeta,\overline{\zeta})}2k\Theta(k)\widehat{\psi_g}(k,\zeta,\overline{\zeta})dk \wedge \epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta})\,. \tag{36}$$

We remind the reader that, by definition of $\mathsf{S}(\Im^+)$, ψ_f and ψ_g are real, smooth and $\psi_f, \psi_g, \partial_u\psi_f, \partial_u\psi_g$ belong to $L^2(\mathbb{R}\times\mathbb{S}^2, dk \wedge \epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}))$. Using the fact that Fourier-Plancherel transform on the real line is unitary one gets:

$$\int_{\mathbb{R}} \overline{\widehat{\psi_f}(k,\zeta,\overline{\zeta})}2k\Theta(k)\widehat{\psi_g}(k,\zeta,\overline{\zeta})dk \wedge \epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}) = \int_{\mathbb{R}} \psi_f(u,\zeta,\overline{\zeta})\mathscr{F}^{-1}[2\Theta k\widehat{\psi_g}](u,\zeta,\overline{\zeta})du \wedge \epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta})\,. \tag{37}$$

Notice that the identity above makes sense because both $\psi_f, \partial_u\psi_g \in L^2(\mathbb{R}\times\mathbb{S}^2, dk\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}))$, by definition of the space $\mathsf{S}(\Im^+)$, so that the Fourier-Plancherel transform of $\partial_u\psi_g$, which is $k\widehat{\psi_g}$ up to a constant factor, and the restriction to the latter to $k \in \mathbb{R}^+$ do it. Now, since $\Theta(k)e^{-k\epsilon}\widehat{\psi_g}(k,\zeta,\overline{\zeta})$ converges, as $\epsilon \to 0^+$, to $\Theta(k)\widehat{\psi_g}(k,\zeta,\overline{\zeta})$ in the sense of $L^2(\mathbb{R}\times\mathbb{S}^2, dk \wedge \epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}))$, and using the fact that the (inverse) Fourier-Plancherel transform is continuous, one has

$$\mathscr{F}^{-1}[\Theta e^{-k\epsilon}k\widehat{\psi_g}] \to \mathscr{F}^{-1}[\Theta k\widehat{\psi_g}]\,, \quad \text{as } \epsilon \to 0^+ \text{ in the topology of } L^2(\mathbb{R}\times\mathbb{S}^2, dk \wedge \epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}))\,. \tag{38}$$

The left-hand side can be computed by means of convolution theorem (the convolution restricted to the variable u) since both functions $k \mapsto \widehat{\psi_g}(k,\zeta,\overline{\zeta})$ and $k \mapsto \Theta(k)e^{-\epsilon k}$ belong to $L^2(\mathbb{R}\times\mathbb{S}^2, dk)$ by construction almost everywhere in $(\zeta,\overline{\zeta})$ fixed (for the former function it follows from Fubini-Tonelli theorem using the fact that $\widehat{\psi_g} \in L^2(\mathbb{R}\times\mathbb{S}^2, dk\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}))$ since $\psi_g \in L^2(\mathbb{R}\times\mathbb{S}^2, du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta})))$. In this way, by direct inspection one finds

$$\mathscr{F}^{-1}[\Theta e^{-k\epsilon}k\widehat{\psi_g}](u,\zeta,\overline{\zeta}) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\partial_{u'}\psi_g(u',\zeta,\overline{\zeta})}{u-u'-i\epsilon}du' \,.$$

Inserting in (38) we have achieved that, as $\epsilon \to 0^+$ in the topology of $L^2(\mathbb{R}\times\mathbb{S}^2, dk\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}))$,

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{\partial_{u'}\psi_g(u',\zeta,\overline{\zeta})}{u-u'-i\epsilon}du' \to \mathscr{F}^{-1}[\Theta k\widehat{\psi_g}] \,.$$

Inserting in the right-hand side of (37) we have:

$$\lambda_M(f,g) = \frac{1}{\pi}\int_{\mathbb{R}\times\mathbb{S}^2} du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}) \lim_{\epsilon\to 0^+}\int_{\mathbb{R}} \frac{\psi_f(u,\zeta,\overline{\zeta})\partial_{u'}\psi_g(u',\zeta,\overline{\zeta})}{u-u'-i\epsilon}du' \,, \tag{39}$$

then, using the continuity of the scalar product of the Hilbert space $L^2(\mathbb{R}\times\mathbb{S}^2, dk\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}))$ one obtains:

$$\lambda_M(f,g) = \lim_{\epsilon\to 0^+}\frac{1}{\pi}\int_{\mathbb{R}\times\mathbb{S}^2} du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}) \int_{\mathbb{R}} \frac{\psi_f(u,\zeta,\overline{\zeta})\partial_{u'}\psi_g(u',\zeta,\overline{\zeta})}{u-u'-i\epsilon}du' \,. \tag{40}$$

Since both $\psi_g, \partial_u\psi_g$ belong to $C^\infty(\mathbb{R})\cap L^2(\mathbb{R}, du)$ by hypotheses, almost everywhere in $(\zeta,\overline{\zeta})$ fixed, one has $\psi_g(u,\zeta,\overline{\zeta}) \to 0$ for $u \to \pm\infty$[8]. Integrating by parts the last integral one obtains in that way

$$\lambda_M(f,g) = \lim_{\epsilon\to 0^+} -\frac{1}{\pi}\int_{\mathbb{R}\times\mathbb{S}^2} du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}) \int_{\mathbb{R}} \frac{\psi_f(u,\zeta,\overline{\zeta})\psi_g(u',\zeta,\overline{\zeta})}{(u-u'-i\epsilon)^2}du' \tag{41}$$

To conclude the proof it is sufficient to show that, for $\epsilon > 0$ the function

$$(u,u'\zeta,\overline{\zeta}) \mapsto \frac{|\psi_f(u,\zeta,\overline{\zeta})||\psi_g(u',\zeta,\overline{\zeta})|}{(u-u')^2+\epsilon^2} =: H(u,u',\zeta,\overline{\zeta})$$

is integrable in the joint measure of $\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2$. Since the function is positive, it is equivalent to prove that the function is integrable under iterated integrations, first in du' and then respect to $du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta})$. We decompose the iterated integration in four terms:

$$\int_{[0,u_1)\times\mathbb{S}^2} du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta})\int_{[0,u_1)} du' H(u,u',\zeta,\overline{\zeta}) + \int_{[u_1,+\infty)\times\mathbb{S}^2} du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta})\int_{[0,u_1)} du' H(u,u',\zeta,\overline{\zeta})$$

$$+\int_{[0,u_1)\times\mathbb{S}^2} du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta})\int_{[u_1,+\infty)} du' H(u,u',\zeta,\overline{\zeta}) + \int_{[u_1,+\infty)\times\mathbb{S}^2} du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta})\int_{[u_1,+\infty)} du' H(u,u',\zeta,\overline{\zeta}) \,. \tag{42}$$

[8] Work at fixed $(\zeta,\overline{\zeta})$. Using elementary calculus, by continuity of $\partial_u\psi_g$ and Cauchy-Schwarz inequality, one has $|\psi_g(u')-\psi_g(u)| \le ||\psi_g||_{L^2(\mathbb{R},du)}|u-u'|$ so that $u \mapsto \psi_g(u)$ is uniformly continuous. If were $\psi_g \not\to 0$ as $u \to +\infty$ (the other case is analogous) one would find a sequence of intervals I_k centered on $k = 1,2,\ldots$ with $\int_{I_k} du > \epsilon$ and $|\psi_g|\restriction_{I_k} > M$ for some $M > 0$ and $\epsilon > 0$. As a consequence it would be $\int_{\mathbb{R}} |\psi_g(u)|^2 du = +\infty$.

Above we have fixed the origin of u and u' away in the past of the support of ψ_f and ψ_g on $\Im^+$. This is possible due to the last statement in Lemma 4.1. The point u_1 is taken as follows. It has been established in the proof of Lemma 4.4 in [Mo06] the following result that we restate in improved form.

Lemma 4.2. *Assume that the spacetime (M,g) is an asymptotically flat vacuum spacetime with future time infinity i^+ (Definition A.1). Referring to a Bondi frame, for every $\beta \in [1,2)$ there are $u_1 > 0$, a compact ball B centered in i^+ defined with respect a suitable coordinate patch x^1, x^2, x^3, x^4 in $\tilde{M}$ centered on i^+, and constants $a, b > 0$ such that if $u \geq u_1$, $(\zeta, \overline{\zeta}) \in \mathbb{S}^2$:*

$$\left|\omega_B^{-1}\Psi \restriction_{\Im}^{+} (u,\zeta,\overline{\zeta})\right| \leq \frac{aM_\Psi}{|u-b|}, \quad \left|\partial_u(\omega_B^{-1}\Psi \restriction_{\Im}^{+})(u,\zeta,\overline{\zeta})\right| \leq \frac{aM_\Psi}{|u-b|^\beta}\,, \tag{43}$$

where

$$M_\Psi := \max\left(\sup_B |\Psi|, \sup_B |\partial_{x^1}\Psi|, \cdots, \sup_B |\partial_{x^4}\Psi|\right)\,. \tag{44}$$

for every $\Psi \in C^\infty(\tilde{M})$.

Proof. The proof is exactly that given for Lemma 4.4 in [Mo06]. There the smooth function $\Psi \in C^\infty(\tilde{M})$ was specialized to the case $\Psi = \Gamma_M(\phi)$ for some $\phi \in \mathcal{S}(M)$, however such a restriction can be removed without affecting the proof as it is evident from the proof of Lemma 4.4 in [Mo06]. The improvement concerning the exponent β is obtained by noticing that in the last estimation before Eq. (44) in [Mo06], $e^{-\lambda(4+\epsilon)}$ can be replaced by the improved bound $e^{-\beta\lambda(4+\epsilon)}$ for every $\beta \in [1,2)$ provided the free parameter $\epsilon > 0$ fulfills $\epsilon < 4(2-\beta)/(\beta+4)$. □

If $h \in C_0^\infty(M)$, the lemma above entails that (with $\beta = 1$), for some constants $a, b > 0$: taking :

$$|\psi_h(u,\zeta,\overline{\zeta})|, |\partial_u\psi_h(u,\zeta,\overline{\zeta})| \leq \frac{aM_h}{u-b}\,, \tag{45}$$

where

$$M_h := \max\left(\sup_B |\tilde{E}(h)|, \sup_B |\partial_{x^1}\tilde{E}(h)|, \cdots, \sup_B |\partial_{x^4}\tilde{E}(h)|\right)\,. \tag{46}$$

Enlarging u_1 if necessary, we can always assume that $u_1 > u_0, b$. In the decomposition (42) we use that value for u_1. Therein the first integral converges trivially. Concerning the last integral, due to Eq. (45), we have the estimation in its domain of integration

$$H(u,u',\zeta,\overline{\zeta}) \leq \frac{a^2}{(u-u')^2+\epsilon^2}\frac{M_f M_g}{(u-b)(u'-b)}\,.$$

Using that and the fact that the volume of $\mathbb{S}^2$ is finite, by direct computation one finds

$$\int_{[u_1,+\infty)\times\mathbb{S}^2} du \wedge \epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}) \int_{[u_1,+\infty)} du' H(u,u',\zeta,\overline{\zeta})$$

$$\leq \int_{u_1}^{+\infty} du \frac{4\pi a^2 M_f M_g}{(u-b)(1+u-b)} \left\{ \frac{\frac{\pi}{2} + \tan^{-1}\left(\frac{u-u_1}{\epsilon}\right)}{\epsilon} - \frac{1}{2(u-b)} \ln \frac{(u_1-b)^2}{(u_1-u)^2+\epsilon^2} \right\} < +\infty\,.$$

By Fubini-Tonelli theorem (H is positive) the second iterated integral in (42) converges if the third does. Concerning the third one we have the estimation in its domain of integration (notice that ψ_f is smooth in $[0,u_1]\times\mathbb{S}^2$ and thus bounded and $u'\geq u_1>b$.)

$$H(u,u',\zeta,\overline{\zeta})\leq \frac{C}{[(u-u')^2+\epsilon^2](u'-b)}\leq \frac{C'}{(u-u')^2+\epsilon^2}$$

for some constants $C,C'\geq 0$. Therefore, computing the integral in u' and using the finite volume of $\mathbb{S}^2$ we found:

$$\int_{[0,u_1)\times\mathbb{S}^2} du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta})\int_{[u_1,+\infty)} du' H(u,u',\zeta,\overline{\zeta})\leq C'\int_0^{u_1} du\, \frac{\frac{\pi}{2}+\tan^{-1}\left(\frac{u-u_1}{\epsilon}\right)}{\epsilon}<+\infty\,.$$

We conclude that the function H is integrable in the joint measure of $\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2$ so that (41) entails (34).
(b) Due to Schwartz kernel theorem, the statement (b) is equivalent to prove that (i) for every $g\in C_0^\infty(M)$, $C_0^\infty(M)\ni f\mapsto \lambda_M(f,g)$ is continuous in the topology of $C_0^\infty(M)$ and (ii) the linear map $C_0^\infty(M)\ni g\mapsto \lambda_M(\cdot,g)\in\mathscr{D}'(M)$ is weakly continuous. (ii) means that, for every fixed $f\in C_0^\infty(M)$, if $\{g_n\}_{n\in\mathbb{N}}\subset C_0^\infty(M)$ converges to 0, as $n\to+\infty$, in the topology of $C_0^\infty(M)$, then $\lambda_M(f,g_n)\to 0$ as $n\to+\infty$. To prove that the couple of requirements is fulfilled notice that, by Cauchy-Schwarz inequality and (36) one finds

$$|\lambda_M(f,g)|\leq \left|\left|\widehat{\psi_f}\right|\right|_{L^2(\mathbb{R}\times\mathbb{S}^2,dk\wedge\epsilon_{\mathbb{S}^2})}\left|\left|k\Theta\widehat{\psi_g}\right|\right|_{L^2(\mathbb{R}\times\mathbb{S}^2,dk\wedge\epsilon_{\mathbb{S}^2})}\leq C_g\,||\psi_f||_{L^2(\mathbb{R}\times\mathbb{S}^2,du\wedge\epsilon_{\mathbb{S}^2})} \tag{47}$$

and

$$|\lambda_M(f,g)|\leq \left|\left|k\Theta\widehat{\psi_f}\right|\right|_{L^2(\mathbb{R}\times\mathbb{S}^2,dk\wedge\epsilon_{\mathbb{S}^2})}\left|\left|\widehat{\psi_g}\right|\right|_{L^2(\mathbb{R}\times\mathbb{S}^2,dk\wedge\epsilon_{\mathbb{S}^2})}\leq C_f\,||\psi_g||_{L^2(\mathbb{R}\times\mathbb{S}^2,du\wedge\epsilon_{\mathbb{S}^2})} \tag{48}$$

where, in the last passages $C_f:=||k\Theta\widehat{\psi_f}||_{L^2(\mathbb{R}\times\mathbb{S}^2)}$, $C_g:=||k\Theta\widehat{\psi_g}||_{L^2(\mathbb{R}\times\mathbb{S}^2)}$ and we have used the fact that Fourier-Plancherel transform is isometric. Thus, the statement (b) is true if $||\psi_{g_n}||_{L^2(\mathbb{R}\times\mathbb{S}^2)}\to 0$ for $g_n\to 0$ in the topology of $C_0^\infty(M)$. Let us prove this fact exploiting (35) and (45) for $h=g_n$. It is known that the causal propagator defined in a globally hyperbolic spacetime $\tilde{E}:C_0^\infty(\tilde{M})\to C^\infty(\tilde{M})$ is continuous in the standard compactly-supported test-function topology in the domain and the natural Fréchet topology in $C^\infty(\tilde{M})$ (see [Le53, Di80, BGP96]). Fix $f\in C_0^\infty(M)$, a compact set $K\subset M$ and a sequence $\{g_n\}_{n\in\mathbb{N}}\subset C_0^\infty(M)$ supported in K. From (b) in Lemma 4.1 there is $u_0\in\mathbb{R}$ such that the support of every ψ_{g_n} is included in the set $u\geq u_0$. Moreover form Lemma 4.2, we know that, if $u_1>0$ is sufficiently large, there is a compact ball B centered in i^+ defined with respect a suitable coordinate patch centered on i^+, and constants $a,b>0$ such that if $u\geq u_1$, $(\zeta,\overline{\zeta})\in\mathbb{S}^2$ (45) hold for $u=g_n$ (for every n), where

$$M_{g_n}:=\max\left(\sup_B|\tilde{E}(g_n)|,\sup_B|\partial_{x^1}\tilde{E}(g_n)|,\cdots,\sup_B|\partial_{x^4}\tilde{E}(g_n)|\right)\,.$$

Enlarging u_1 if necessary, we can always assume that $u_1>u_0,b$.
Continuity of $\tilde{E}$ implies that $M_{g_n}\to 0$ as $n\to+\infty$. If $B'\subset\tilde{M}$ is another compact set such that $B'\supset\{(u,\zeta,\overline{\zeta})\in\mathfrak{I}^+\mid u_1>u>u_0\}$, since ω_B^{-1} is bounded therein, continuity of $\tilde{E}$ entails by (35) that ψ_{g_n} vanishes uniformly as $n\to+\infty$ in B'. Now

$$||\psi_{g_n}||^2_{L^2(\mathbb{R}\times\mathbb{S}^2,du\wedge\epsilon_{\mathbb{S}^2})}=\int_{[u_0,+\infty)\times\mathbb{S}^2}\left|\psi_{g_n}(u,\zeta,\overline{\zeta})\right|^2 du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta})$$

Decompose the last integral into two terms, the former corresponding to the integration from u_0 to u_1 and the latter from u_1 to $+\infty$. Both parts vanish as $n \to +\infty$. The former vanishes because ψ_{g_n} vanishes uniformly on $\{(u,\zeta,\overline{\zeta}) \in \Im^+ \mid u_1 > u > u_0\}$ as $n \to +\infty$, the latter vanishes as consequence of (45) with $h = g_n$, since $M_{g_n} \to 0$ as $n \to +\infty$ and

$$\int_{[u_1,+\infty)\times\mathbb{S}^2} \left|\psi_{g_n}(u,\zeta,\overline{\zeta})\right|^2 du \wedge \epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}) \leq a^2 M_{g_n}^2\, 4\pi \int_{u_1}^{+\infty} \frac{1}{(u-b)^2} du\,.$$

This concludes the proof. □

4.4. *λ_M is Hadamard when (M,g) admits i^+.*. We are in place to state and prove the main result of this section and perhaps of this work.

Theorem 4.3. *Assume that the spacetime (M,g) is an asymptotically flat vacuum spacetime with future time infinity i^+ (Definition A.1) and that both (M,g) and the unphysical spacetime $(\tilde{M},\tilde{g})$ are globally hyperbolic. Consider the quasifree state λ_M – on the Wayl algebra $\mathcal{W}(M)$ of the massless conformally coupled real scalar field propagating in M – canonically induced by the BMS-invariant state λ on $\Im^+$.*
λ_M is Hadamard.

Proof. The proof is based on properties of the restriction of λ_M to sets $I^-(p;M) \cap I^+(q;M)$.
Since the class of all the sets $I^-(r;M) \cap I^+(s;M)$ define a topological base of the topology of a strongly causal spacetime (M,g), and since the geodesically convex normal neighborhoods define an analogous base, if p' is sufficiently close to q', $I^-(p';M) \cap I^+(q';M)$ must be contained in a geodesically convex normal neighborhood U. Taking $p,q \in I^-(p';M) \cap I^+(q';M)$ with $p \in I^+(q,M)$ we have $J^-(p;M) \cap J^+(q;M) \subset I^-(p';M) \cap I^+(q';M) \subset U$.
In the following, a set $I^-(p;M) \cap I^+(q;M) \subset M$ such that both $I^-(p;M) \cap I^+(q;M)$ and $J^-(p;M) \cap J^+(q;M)$ are contained in a geodesically convex normal neighborhood will be called a **standard domain**. Standard domains form a base of the topology of every strongly causal spacetime. A strongly causal spacetime (M,g) is globally hyperbolic if and only if every set $J^-(p;M) \cap J^+(q;M)$ is compact [Wa84]. In this case $J^-(p;M) \cap J^+(q;M) = \overline{I^-(p;M) \cap I^+(q;M)}$ holds as well [Wa84]. Therefore, if (M,g) is globally hyperbolic, the standard domains form a topological base made of globally hyperbolic (when considered spacetimes in their own right if equipped with the restriction of the metric g) open set with compact closure.

In the hypotheses of Theorem 4.3, consider a standard domain $N \subset M$ and the restriction of the two point function of λ_M to $C_0^\infty(N) \times C_0^\infty(N)$. Since we know that λ_M is a distribution by Theorem 4.2, this is equivalent to restrict the distribution $\lambda_M \in \mathcal{D}'(M \times M)$ to $C_0^\infty(M \times M)$ producing a distribution of $\mathcal{D}'(N \times N)$. We have a first central result whose proof, given in the Appendix B, relies on the know wavefront set of the causal propagator $\tilde{E}$, on several pieces of information on the wavefront set of λ_M extracted from (34) and on standard results about composition of wavefront sets [Hö89].

Proposition 4.3. *In the hypotheses of Theorem 4.3, consider a standard domain $N \subset M$. If $\lambda_M^{(N)}$ is the restriction of the distribution $\lambda_M \in \mathcal{D}'(M \times M)$ to $C_0^\infty(N \times N)$, then*

$$WF(\lambda_M^{(N)}) \subset \{((x,k_x),(y,-k_y)) \in T^*N \setminus \mathbf{0} \times T^*N \setminus \mathbf{0} \mid (x,k_x) \sim (y,k_y)\,, k_x \rhd 0\}\ . \tag{49}$$

Next using the fact that $\lambda_M^{(N)}$ satisfies (Com) and (KG), using the theorem of propagation of singularities [Hör71, Ra96a] one can make stronger the proposition above obtaining the following result.

Proposition 4.4. *In the hypotheses of Theorem 4.3, consider a standard domain $N \subset M$. If $\lambda_M^{(N)}$ is the restriction of the distribution $\lambda_M \in \mathcal{D}'(M \times M)$ to $C_0^\infty(N \times N)$, then*

$$WF(\lambda_M^{(N)}) = \{((x,k_x),(y,-k_y)) \in T^*N \setminus \mathbf{0} \times T^*N \setminus \mathbf{0} \mid (x,k_x) \sim (y,k_y)\,, k_x \rhd 0\}\ . \tag{50}$$

Proof. Let $E_N \in \mathcal{D}'(N \times N)$ be the causal propagator associated with Klein-Gordon equation in the globally hyperbolic spacetime N and, in the following we denote by sing supp(T) the singular support of a distribution T. *In this proof $p \sim q$ means that there is, in the considered spacetime, at least one null geodesic joining p and q.*
We prove the thesis by *a reductio ad absurdum.* Our *per absurdum* claim is that there be $p, q \in N$ with $p \sim q$ but $((p,k_p),(q,-k_q)) \notin WF(\lambda_M^{(N)})$, where k_p and k_q are the cotangent vectors to a null geodesic joining p and q with $k_p \rhd 0$. Actually that geodesic is *uniquely determined* – for both N and M – by p and q, from the very definition of standard domain N. (Notice that the wavefront set is conic and thus the vectors k_p, k_q are determined up to a common, strictly positive, factor completely irrelevant in our discussion.) Since the singular support of a distribution of $\mathcal{D}'(N \times N)$ is the projection on $N \times N$ the of the wavefront set of the distribution, we must conclude that, in view of (49), $(p,q) \notin$ sing supp$(\lambda_M^{(N)})$. However, as $p \sim q$, (p,q) must belong to sing suppE_N (this is because E_N is the difference of the advanced and the retarded fundamentals solutions whose known wavefronts and causal properties of supports [Ra96a] entails that sing supp(E_N) is made exactly by the pairs $(p,q) \in N \times N$ with $p \sim q$). Since (Com) holds true, we conclude that $(q,p) \in$ sing supp$(\lambda_M^{(N)})$, and thus there are $k_p \in T_p^*N$ and $k_q \in T_q^*N$ such that $((q,k_q),(p,-k_p)) \in WF(\lambda_M^{(N)})$, and so, via proposition 4.3, k_p and k_q are vectors cotangent to the null geodesic joining p and q (the same as before since it is unique) and, finally, $k_q \rhd 0$.
The distribution $\lambda_M^{(N)}$ satisfies Klein-Gordon equation in both arguments (in other words (KG) holds), therefore the *propagation of singularities theorem* [Hör71] (see discussion after Theorem 4.6 in [Ra96a]) implies that the wavefront set of $\lambda_M^{(N)}$ is the union of sets of the form $B(x,k_x) \times B(y,k_y)$. Here $B(z,h)$ is the unique null complete geodesic (viewed as a curve in T^*N) passing through $z \in N$ with co-tangent vector $h \in T_x^*N$. As $((q,k_q),(p,-k_p)) \in WF(\lambda_M^{(N)})$ and since q and p belong to the same null geodesic, we are committed to conclude that $((p,k'_p),(q,-k'_q)) \in WF(\lambda_M^{(N)})$ where the cotangent vector k'_p has the same time orientation as k_q, so that $k'_p \rhd 0$, and the vector k'_q is cotangent to the geodesic at q. In other words, changing the used names for cotangent vectors: $((p,k_p),(q,-k_q)) \in WF(\lambda_M^{(N)})$ where $k_p \rhd 0$. This is in contradiction with our initial claim. □

We are now in place to take advantage of Radzikowski's results illustrated in Section 4.1. Since $\lambda_M^{(N)}$ determines a quasifree state for the Klein-Gordon field confined to the globally hyperbolic subspace N, the result established in (50) entails that $\lambda_M^{(N)}$ is Hadamard on N in view of Proposition 4.1. Therefore it verifies the local Hadamard condition (LH) in a neighborhood of every point $p \in N$. Since the sets N form a base of the topology of M and the Schwartz kernel of $\lambda_M^{(N)}$ is nothing but the restriction of the kernel of λ_M to N, we conclude that the Schwartz kernel of λ_M satisfies (LH) in a neighborhood of every point p of M. Due to Proposition 4.2, we can conclude that λ_M is Hadamard on M. □

5 Final Comments: summary and open issues.

Let us summarize the main results achieved in this work. We started from the unique, positive energy, BMS-invariant, quasifree, pure state λ acting on a natural Weyl algebra defined on $\Im^+$. That state is

completely defined using the universal structure of the class of (vacuum) asymptotic flat spacetimes at null infinity, no reference to any particular spacetime is necessary. In this sense λ is *universal*. It is the vacuum state for a representation of BMS group with vanishing BMS mass. Afterwords, we have seen that λ induces in any fixed (globally hyperbolic) bulk spacetime M, a preferred state λ_M. This happens if M admits future infinity i^+ (and the unphysical spacetime $\tilde{M}$ is globally hyperbolic as well). The induction of a state takes place by means of an injective isometric $*$ homomorphism $\imath : \mathcal{W}(M) \to \mathcal{W}(\Im^+)$ which identifies Weyl observables of the bulk with some Weyl observables of the boundary $\Im^+$.

$$\lambda_M(a) := \lambda(\imath(a)) \quad \text{for all } a \in \mathcal{W}(M)\,.$$

Using a very inflated term, we may say that this is a *holographic correspondence*.
The picked out state λ_M enjoys quite natural, as well as interesting, properties. These properties (barring the first one) have been established in this paper:

(i) λ_M coincides with Minkowski vacuum when M is Minkowski spacetime,

(ii) λ_M is invariant under every isometry of M (if any);

(iii) λ_M fulfills the requirement of energy positivity with respect to every timelike Killing field in M and, in the one-particle space, there are no zero modes for the self-adjoint generator of Killing-time displacements,

(iv) λ_M is Hadamard and therefore the state may be used as background for perturbative procedures (renormalization in particular).

The statement (ii) holds as it stands replacing λ_M with any other state λ'_M uniquely defined by assuming that $\lambda'_M(a) := \lambda'(\imath(a))$ for all $a \in \mathcal{W}(M)$ provided that λ' be a BMS-invariant state (not necessarily quasifree or pure or satisfying some positivity-energy condition) defined on $\mathcal{W}(\Im^+)$.
The state λ_M may have the natural interpretation of *outgoing scattering vacuum*, but also it provides a natural and preferred notion of massless particle in the absence of Poincaré symmetry. Indeed, all the construction works for massless conformally coupled scalar fields propagating in M. Notice that the two notions of mass arising in our picture, that in the bulk based on properties of Klein-Gordon operator (and on Wigner analysis if M is Minkowski spacetime) and that referred to the extent on $\Im^+$ relying upon Mackey-McCarthy analysis of BMS group unitary representations, are in perfect agreement: both vanishes. A natural question which deserves future investigation is now: what about massive fields? How to connect, if possible, massive particle defined in M to fields on $\Im^+$ associated with known unitary BMS representations with positive BMS mass?

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A Asymptotically flat spacetime with future time infinity

Definition A.1. *A time-oriented four-dimensional smooth spacetime (M, g) is called* **asymptotically flat vacuum spacetime with future time infinity** i^+*, if there is a smooth spacetime $(\tilde{M}, \tilde{g})$ with a preferred point i^+, a diffeomorphism $\psi : M \to \psi(M) \subset \tilde{M}$ and a map $\Omega : \psi(M) \to [0, +\infty)$ so that $\tilde{g} = \Omega^2 \psi^* g$ and the following facts hold. (We omit to write explicitly ψ and ψ^* in the following).*
(1) *$J^-(i^+; \tilde{M})$ is closed and $M = J^-(i^+) \setminus \partial J^-(i^+; \tilde{M})$. (Thus $M = I^-(i^+; \tilde{M})$, i^+ is in the future of and time-like related with all the points of M and $\Im^+ \cap J^-(M; \tilde{M}) = \emptyset$.) Moreover $\partial M = \Im^+ \cup \{i^+\}$ where $\Im^+ := \partial J_-(i^+; \tilde{M}) \setminus \{i^+\}$ is the* **future null infinity**.
(2) *M is strongly causal and satisfies vacuum Einstein solutions in a neighborhood of $\Im^+$ at least.*

(3) *Ω can be extended to a smooth function on $\tilde{M}$.*
(4) *$\Omega\restriction_{\partial J_-(i^+;\tilde{M})}= 0$, but $d\Omega(x) \neq 0$ for $x \in \Im^+$, and $d\Omega(i^+) = 0$, but $\tilde{\nabla}_\mu\tilde{\nabla}_\nu\Omega(i^+) = -2\tilde{g}_{\mu\nu}(i^+)$.*
(5) *If $n^\mu := \tilde{g}^{\mu\nu}\tilde{\nabla}_\nu\Omega$, for a strictly positive smooth function ω, defined in a neighborhood of $\Im^+$ and satisfying $\tilde{\nabla}_\mu(\omega^4 n^\mu) = 0$ on $\Im^+$, the integral curves of $\omega^{-1}n$ are complete on $\Im^+$.*

Remark A.1. Notice that ω in (5) can be fixed to be the factor ω_B mentioned in Section 1.2.

B Proofs of some technical propositions.

Proof of (c) in Proposition 3.1. Consider a one-parameter subgroup of G_{BMS}, $\{g_t\}_{t\in\mathbb{R}} \subset \Sigma$. Suppose that $\{g_t\}_{t\in\mathbb{R}}$ arises from the integral curves of a complete smooth vector $\tilde{\xi}$ tangent to $\Im^+$. In every Bondi frame $(u, \zeta, \overline{\zeta})$ one finds: $g_t : \mathbb{R} \times \mathbb{S}^2 \ni (u, \zeta, \overline{\zeta}) \mapsto \big(u + f_t(\zeta, \overline{\zeta}), \zeta, \overline{\zeta}\big)$, where, due to smoothness of $\tilde{\xi}$ and because of standard theorems of dynamical system theory, the function $(t, u, \zeta, \overline{\zeta}) \mapsto u + f_t(\zeta, \overline{\zeta})$ is jointly smooth. In particular f is jointly smooth and thus continuous in the parameter t, satisfies $f_t \in C^\infty(\mathbb{S}^2) \equiv \Sigma$ and verifies $f_t(\zeta, \overline{\zeta}) + f_{t'}(\zeta, \overline{\zeta}) = f_{t+t'}(\zeta, \overline{\zeta})$ for all $t, t' \in \mathbb{R}$ and $(\zeta, \overline{\zeta}) \in \mathbb{S}^2$. The relation above entails $f_{\frac{p}{q}t}(\zeta, \overline{\zeta}) = \frac{p}{q} f_t(\zeta, \overline{\zeta})$ for all $t \in \mathbb{R}$, $p, q \in \mathbb{Z}$, $q \neq 0$ and $(\zeta, \overline{\zeta}) \in \mathbb{S}^2$. Using continuity in t one finally gets: $af_t(\zeta, \overline{\zeta}) + bf_{t'}(\zeta, \overline{\zeta}) = f_{at+bt'}(\zeta, \overline{\zeta})$ for all $t, t' \in \mathbb{R}$, $a, b \in \mathbb{R}$ and $(\zeta, \overline{\zeta}) \in \mathbb{S}^2$. Therefore it holds: $f_t(\zeta, \overline{\zeta}) = tf_1(\zeta, \overline{\zeta})$. We conclude that if the one-parameter sub-group $\{g_t\}_{t\in\mathbb{R}} \subset \Sigma \subset G_{BMS}$ arises from the complete integral curves of a smooth vector $\tilde{\xi}$ tangent to $\Im^+$, in any fixed Bondy frame:

$$g_t : \mathbb{R} \times \mathbb{S}^2 \ni (u, \zeta, \overline{\zeta}) \mapsto \big(u + tf_1(\zeta, \overline{\zeta}), \zeta, \overline{\zeta}\big) \ ,$$

where the function $f_1 \in C^\infty(\mathbb{S}^2) \equiv \Sigma$ individuates completely the subgroup. $\square$

Proof of Proposition 3.3. (a) is an immediate consequence of (a) of proposition 3.1 and the definition of asymptotic symmetry. (b) Since the extension of ξ to $\Im^+$, $\tilde{\xi}$, has to be tangent to $\Im^+$, referring to a fixed Bondi frame, it must hold

$$\tilde{\xi} = \alpha\partial/\partial u + \beta\partial/\partial\zeta + \overline{\beta}\partial/\partial\overline{\zeta} \ .$$

Since the angular part of the degenerate metric on $\Im^+$ is positive, whereas that on the space spanned by $\partial/\partial u$ (which is orthogonal to the angular part) vanishes, one has $\tilde{g}(\tilde{\xi}, \tilde{\xi}) \geq 0$ – with $\tilde{g}(\tilde{\xi}, \tilde{\xi}) = 0$ if and only if $\beta = \overline{\beta} = 0$. On the other hand we know that $g(\xi, \xi) \leq 0$ in M by hypotheses and thus $\tilde{g}(\xi, \xi) \leq 0$ as well. Hence approaching $\Im^+$, by continuity, it must be $\tilde{g}(\tilde{\xi}, \tilde{\xi}) = 0$. We have found that: $\tilde{\xi}(u, \zeta, \overline{\zeta}) = \alpha(u, \zeta, \overline{\zeta})\partial/\partial u$. The (generally local) one-parameter group of transformations g_t obtained by integration of $\tilde{\xi}$ acts only on the variable u: $u \mapsto u_t$ and so it has to hold

$$\frac{du_t(u, \zeta, \overline{\zeta})}{dt}|_{t=0} = \alpha(u, \zeta, \overline{\zeta}) \ . \tag{51}$$

On the other hand this one-parameter group must coincide with a suitable one-parameter subgroup of BMS group because $\tilde{\xi}$ is a one-parameter generator of such an action by (a) in proposition 3.1. By comparison with the action (2)-(3) of BMS group on coordinates $(u, \zeta, \overline{\zeta})$, noticing that the subgroup leaves fixed the angular coordinates, the only possible action is $u_t = u + f(t, \zeta, \overline{\zeta})$ for some smooth class of functions $\{f(t, \cdot, \cdot)\}_{t\in\mathbb{R}} \subset C^\infty(\mathbb{S}^2)$. Therefore

$$\frac{du_t(u, \zeta, \overline{\zeta})}{dt}|_{t=0} = \frac{\partial f(t, \zeta, \overline{\zeta})}{\partial t}|_{t=0} \ .$$

Comparing with (51) we conclude that α cannot depend on u. (b) in proposition 3.1 also entails that α cannot vanish identically on $\Im^+$. In other words, $\tilde{\xi}$ is a generator of a nontrivial subgroup of Σ. Next, by (b) in proposition 3.1 we conclude that $\tilde{\xi}$ is a generator of a nontrivial subgroup of T^4. That is equivalent to say that $\alpha \in T^4 \setminus \{0\}$. To conclude, as a consequence of by (c) of proposition 3.2, it is sufficient to prove that α cannot attain both signs. Since ξ is future directed with respect to g and $\tilde{g}$, the limit values of ξ toward $\Im^+$, $\alpha\partial/\partial u$ must either vanish or be future directed. Since $\partial/\partial u$ is future directed with respect to $\tilde{g}$ too, the factor given by the smooth function α cannot be negative anywhere. □

Proof of Proposition 3.4. In this proof $\Omega_B := \omega_B \Omega$. Consider a smooth vector field v defined on $\tilde{M}$ which reduces to ξ in M and reduces to $\tilde{\xi}$ on $\Im^+$. By construction the jointly smooth one-parameter subgroup generated by v reduces to those generated by the relevant restrictions. The orbits of v in $M \cup \Im^+$ are complete by hypotheses. Indeed, if an orbit starts in M it remains in M and it is complete by hypotheses, if it starts on $\Im^+$ it must remain in $\Im^+$ and must be complete anyway, since $\tilde{\xi}$ generates a (complete) one-parameter subgroup of G_{BMS}. This fact entails, in turn, that the one-parameter group of diffeomorphisms generated by v in $M \cup \Im^+$ is global and thus its pull-back action on functions defined over $M \cup \Im^+$ is well defined. If $y \in \Im^+$ and $x \in M$ one has, by continuity of the flux of v:

$$\lim_{x \to y} g_\tau^{(\xi)}(x) = \lim_{x \to y} g_\tau^{(v)}(x) = g_\tau^{(v)}(y) = g_\tau^{(\tilde{\xi})}(y) \,.$$

In the proof of Proposition 2.7 in [DMP06] (within a more generalized context) we have found that, referring to a Bondi-frame where $g_\tau^{(\tilde{\xi})} = \left(\Lambda_\tau^{(\tilde{\xi})}, f_\tau^{(\tilde{\xi})}\right)$ and $y \equiv (u, \zeta, \overline{\zeta})$,

$$\lim_{x \to (u,\zeta,\overline{\zeta})} \frac{\Omega_B\left(g_{-t}^{(\xi)}(x)\right)}{\Omega_B(x)} = K_{\Lambda_{-t}^{(\tilde{\xi})}}\left(g_{-t}^{(\tilde{\xi})}(u,\zeta,\overline{\zeta})\right)^{-1} \,.$$

Therefore one has trivially that $\Gamma(\phi \circ g_{-t}^{(\xi)})(y)$ coincides with

$$\lim_{x \to y} \frac{\phi\left(g_{-t}^{(\xi)}(x)\right)}{\Omega_B(x)} = \lim_{x \to y} \frac{\phi\left(g_{-t}^{(\xi)}(x)\right)}{\Omega_B\left(g_{-t}^{(\xi)}(x)\right)} \lim_{x \to y} \frac{\Omega_B\left(g_{-t}^{(\xi)}(x)\right)}{\Omega_B(x)} = K_{\Lambda_{-t}^{(\tilde{\xi})}}\left(g_{-t}^{(\tilde{\xi})}(u,\zeta,\overline{\zeta})\right)^{-1} \psi\left(g_{-t}^{(\tilde{\xi})}(y)\right) \,.$$

Comparing with defined in (7), we have finally established that:

$$\Gamma(\phi \circ g_{-t}^{(\xi)}) = A_{g_t^{(\tilde{\xi})}}(\psi) \,.$$

This concludes the proof. □

Proof of Theorem 4.1. (a) and (b) the statements can be proved with the same procedure used in $\mathbb{R}^m$ in Theorem IX.1 in [RS75] with trivial changes, passing $\zeta, \overline{\zeta}$-derivatives under the relevant symbols of integration in dk and du since it is allowed by compactness of $\mathbb{S}^2$ and fast $\zeta, \overline{\zeta}$-uniform decaying for large $|u|$. (30) is a trivial consequence of the analogous statement in $\mathbb{R}^1$ noticing that if $f \in \mathscr{S}(\mathbb{R} \times \mathbb{S}^2)$ then, form fixed $\zeta, \overline{\zeta}$, the restriction $u \mapsto f(u, \zeta, \overline{\zeta})$ is a function of $\mathscr{S}(\mathbb{R})$. Hence (31) follows from (30) via Fubini-Tonelli theorem using the $\zeta, \overline{\zeta}$-uniform decaying for large u of the integrands in both sides of (30) and the fact that $\mathbb{S}^2$ has finite measure. (d) has the same proof as the analog in $\mathbb{R}^n$ in Theorem IX.2 [RS75]. (e) Has the same proof as in the $\mathbb{R}^n$ case (Theorem IX.6 in [RS75]) noticing that (31) holds true and that $\mathscr{S}(\Im^+)$ is dense in the Hilbert space $L^2(\mathbb{R} \times \mathbb{S}^2, du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}))$. The identity (32)

in (f) is trivially fulfilled for $\psi \in \mathscr{S}(\mathbb{R}\times\mathbb{S}^2)$ by construction. Moreover, by Plancherel theorem on $\mathbb{R}$, if $\psi \in L^2(\mathbb{R}\times\mathbb{S}^2, du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}))$ (so that its restrictions at $\zeta,\overline{\zeta}$ fixed belongs to $L^2(\mathbb{R},du)$ by Fubini-Tonelli theorem), one has

$$\int_{\mathbb{R}} |\tilde{\mathscr{F}}_\pm[\psi(\cdot,\zeta,\overline{\zeta})](k)|^2 dk = \int_{\mathbb{R}} |\psi(u,\zeta,\overline{\zeta})|^2 du$$

almost everywhere in $\zeta,\overline{\zeta}$. By Fubini-Tonelli theorem the right-hand side, and thus also the left-hand side is $\zeta,\overline{\zeta}$ integrable. By Fubini-Tonelli theorem one finally has that the integrands are $u,\zeta,\overline{\zeta}$ jointly integrable so that:

$$\int_{\mathbb{R}\times\mathbb{S}^2} |\tilde{\mathscr{F}}_\pm(\psi(\cdot,\zeta,\overline{\zeta}))(k)|^2 dk\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}) = \int_{\mathbb{R}\times\mathbb{S}^2} |\psi(u,\zeta,\overline{\zeta})|^2 du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta})\,.$$

We conclude that the map that associates every $\psi \in L^2(\mathbb{R}\times\mathbb{S}^2, du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}))$ with the function (in the same space) $(k,\zeta,\overline{\zeta}) \mapsto \tilde{\mathscr{F}}_\pm(\psi(\cdot,\zeta,\overline{\zeta}))(k)$ is continuous and isometric and coincides with $\mathscr{F}_\pm$ in the dense subspace $\mathscr{S}(\mathbb{R}\times\mathbb{S}^2)$, therefore it must coincide with $\mathscr{F}_\pm$ extended to $L^2(\mathbb{R}\times\mathbb{S}^2, du\wedge\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta}))$. In other words (32) holds true. Now (33) can be re-written replacing $\mathscr{F}_\pm$ by $\tilde{\mathscr{F}}_\pm$ and in this form is nothing but Plancherel theorem on the real line. The proof of (g) is immediate from (d) and (e). □

Proof of Lemma 4.1. (a) Using the definition of Γ_M (see Proposition 2.1) and the fact that E maps compactly-supported smooth functions to smooth solutions of Klein-Gordon equation with compactly-supported Cauchy data, it arises:

$$\psi_h(u,\zeta,\overline{\zeta}) = \omega_B(u,\zeta,\overline{\zeta})^{-1}\left(\lim_{\to\Im^+}\Omega^{-1}E(h)\right)\restriction^+_{\Im}(u,\zeta,\overline{\zeta})\,. \tag{52}$$

On the other hand since also $(\tilde{M},\tilde{g})$ is globally hyperbolic, the causal propagator $\tilde{E}$ for the massless conformally coupled Klein-Gordon operator $\tilde{P}$ in $(\tilde{M},\tilde{g})$ is well defined. Using the following facts: (1) that E and $\tilde{E}$ are the difference of the advanced and retarded fundamental solutions in the corresponding spaces (M,g) and $(\tilde{M},\tilde{g}=\Omega^2 g)$, and (2) that the following identity holds

$$\tilde{P}(\Omega^{-1}\phi) = \Omega^{-3}P\phi$$

and (3) that the causality relations are preserved under (positive) rescaling of the metric, one achieves the following identity valid on M

$$\Omega^{-1}E(h) = \tilde{E}(\Omega^{-3}h)\,, \quad \text{if } h\in C_0^\infty(M)\,.$$

The right-hand side is anyhow smoothly defined also in the larger manifold $\tilde{M}$ and on $\Im^+$ in particular. Therefore, exploiting Eq. (52), the expression of $\psi_h(u,\zeta,\overline{\zeta})$ found above can be re-written into a more suitable form given by (35). The singularity of Ω^{-3} on $\Im^+$ is harmless because the support of h does not intersect $\Im^+$ by construction and $\Omega>0$ in M. Notice that supp $(\Omega^{-3}h) =$ supp h if $h\in C_0^\infty(M)$.
(b) To prove the thesis take $h\in C_0^\infty(M)$ and a compact $K\subset M$ with supp $h\subset K$. We use Definition A.1 from now on. By definition $\tilde{E}(\Omega^{-3}h)$ (equivalently $\tilde{E}(h)$ since supp $(\Omega^{-3}h) =$ supp h) is supported in $J^+(\text{supp } h;\tilde{M})\cup J^-(\text{supp } h;\tilde{M})$ and thus in $J^+(K;\tilde{M})\cup J^-(K;\tilde{M})$. However $J^-(K;\tilde{M})$ has no intersection with $\Im^+$ since $K\subset I^-(i^+;\tilde{M})$ and $\Im^+\subset\partial J^-(i^+;\tilde{M})$, we conclude that the support of the solution $\tilde{E}(\Omega^{-3}h)$ intersects $\Im^+$ in a set completely included in $J^+(K;\tilde{M})$ and thus in $\Im^+\cap J^+(K;\tilde{M})$. As a consequence

$$\text{the support of } \psi_h = \omega_B(u,\zeta,\overline{\zeta})^{-1}\tilde{E}(\Omega^{-3}h)\restriction^+_{\Im} \text{ is included in } \Im^+\cap J^+(K;\tilde{M}). \tag{53}$$

Now consider a spacelike Cauchy surface S of $(\tilde{M}, \tilde{g})$ with K completely contained in the chronological future of S (such a Cauchy surface does exists due to global hyperbolicity of $(\tilde{M}, \tilde{g})$ and because K is compact, it is sufficient to use any Cauchy foliation of $\mathbb{R} \times S \equiv \tilde{M}$ taking the value of the smooth global time function $t \in \mathbb{R}$ far enough in the past). By standard properties of causal sets (e.g. see [Wa84]) it arises that $J^+(K; \tilde{M}) \subset I^+(S; \tilde{M}) \subset J^+(S, \tilde{M})$. Notice that the set $C := S \cap (\Im^+ \cup \{i^+\}) = S \cap \partial J^-(i^+; \tilde{M})$ is compact because it is a closed subset of $J^-(i^+; \tilde{M}) \cap J^+(S; \tilde{M})$ which is compact since $(\tilde{M}, \tilde{g})$ is globally hyperbolic (e.g. see [Wa84]). C cannot contain i^+ because $i^+ \in I^+(K; \tilde{M})$, $K \subset I^+(S, \tilde{M})$ and S is achronal. Let $u_0 = \min_C u$, which is finite because the coordinate $u : \Im^+ \to \mathbb{R}$ is smooth and $C \subset \Im^+$ is compact. By construction, and since u increases towards the future, we have

$$u(\Im^+ \cap J^+(\operatorname{supp} h; \tilde{M})) \subset [u_0, +\infty) \,. \tag{54}$$

Therefore, by (53), we have that ψ_h vanishes for $u < u_0$ due to (54). □

Proof of Proposition 4.3. Consider fixed Bondi frame $(u, \zeta, \overline{\zeta})$ on $\Im^+ \equiv \mathbb{R} \times \mathbb{S}^2$ and suppose that $\Im^+$ is equipped with the measure $du \wedge \epsilon_{\mathbb{S}^2}$, $\epsilon_{\mathbb{S}^2}$ being the standard volume form of the unit 2-sphere referred to the coordinates (θ, φ) with $\zeta = e^{i\varphi} \cot(\theta/2)$. In the following we view the measure $du \wedge \epsilon_{\mathbb{S}^2}$ as that induced by the Riemannian metric given by $g_{\mathbb{S}^2} \oplus g_{\mathbb{R}}$, $g_{\mathbb{S}^2}$ being the standard Riemannian metric on the unit 2-sphere and $g_{\mathbb{R}}$ represented in coordinates (θ, φ) the usual Riemannian metric on $\mathbb{R}$ referred to the coordinate u. In this way we can exploit the definition of distribution on manifolds equipped with a nondegenerate metric as working on scalar fields. As stressed above this choice is matter of convention. One may fix a different nonsingular smooth metric or define distributions as operating on scalar densities (see discussion on [Hö89]) and it does not affect the wavefront sets. It is because different choices change distributions by smooth nonvanishing factors and directly from the definition of wavefront set, on a given smooth manifold M, $\mathrm{WF}(au) \subset \mathrm{WF}(u)$ if $u \in \mathscr{D}'(M)$ and $a \in C^\infty(M)$ (notice that $1/a$ is smooth as well in the considered case, a being nonvanishing.

The proof of Proposition 4.3 relies upon the following preliminary pair of results.

Lemma B.1. *Consider the distribution $T \in \mathscr{D}'(\Im^+ \times \Im^+)$ defined as*

$$T = F \otimes D \quad \text{with } F(u, u') := \frac{1}{(u - u' - i0^+)^2} \text{ and } D(\omega, \omega') := \delta(\omega, \omega') \,., \tag{55}$$

where $u \in \mathbb{R}$ with covectors $k \in T^*_u\mathbb{R}$, ω is a point on $\mathbb{S}^2$ with covectors $\mathbf{k} \in T^*_\omega\mathbb{S}^2$ and similar notations are valid for primed variables. With those hypotheses it holds

$$WF(T) = A \cup B \tag{56}$$

where

$$A := \left\{((u, \omega, k, \mathbf{k}), (u', \omega', k', \mathbf{k}')) \in T^*\Im^+ \setminus 0 \times T^*\Im^+ \setminus 0 \;\middle|\; u = u', \omega = \omega', 0 < k = -k', \mathbf{k} = -\mathbf{k}'\right\}$$

$$B := \left\{((u, \omega, k, \mathbf{k}), (u', \omega', k', \mathbf{k}')) \in T^*\Im^+ \setminus 0 \times T^*\Im^+ \setminus 0 \;\middle|\; \omega = \omega', k = k' = 0, \mathbf{k} = -\mathbf{k}'\right\} \,.$$

Proof. It is is a straightforward consequence of the discussion after Theorem 8.2.14 in [Hö89] and the known wavefront sets of the delta distribution and $1/(k \pm i0^+)$ (e.g. see [RS75]). □

Lemma B.2. *Assume that the spacetime (M, g) is an asymptotically flat vacuum spacetime with future time infinity i^+ (Definition A.1) and that both (M, g) and the unphysical spacetime $(\tilde{M}, \tilde{g})$ are globally hyperbolic. Consider a Bondi frame $u, \zeta, \overline{\zeta}$ on $\Im^+$.*
If $N \subset M$ is a standard domain, all the null geodesics (of $(\tilde{M}, \tilde{g})$) joining points of $\overline{N}$ and points of $\Im^+$ intersect $\Im^+$ in a set contained in the compact $[u_0, u_f] \times \mathbb{S}^2$ for suitable $u_0, u_f \in \mathbb{R}$.
u_0 can be taken as the value u_0 determined in (b) of Lemma 4.1 for $K := \overline{N}$.

Proof. We use here the geometric structure defined in Definition A.1. The fact that all the null geodesics joining points of N and points of $\Im^+$ intersect $\Im^+$ in a set contained in a set of the shape $[u_0, +\infty) \times \mathbb{S}^2$, with a suitable $u_0 \in \mathbb{R}$, is straightforward since the pairs of points on each of the considered geodesics are contained in the (singular) support of the causal propagator $\tilde{E}$ (viewed as distribution on $C_0^\infty(M \times M)$ due to Schwartz kernel theorem) when it is restricted to N in the right argument. In fact, defining $K := \overline{N}$, we know that the support of the kernel of $\tilde{E} \restriction_{C_0^\infty(N)}$ is included in the set $(J^+(K; \tilde{M}) \cup J^-(K; \tilde{M})) \times K$. Therefore the considered geodesics meet $\Im^+$ in a subset of $J^+(K; \tilde{M}) \cap \Im^+$ (it being $J^-(K; \tilde{M}) \cap \Im^+ = \emptyset$). We know by the proof of (b) in Lemma 4.1 that $J^+(K; \tilde{M}) \cap \Im^+$ is contained in a set of the form $(-\infty, u_0'] \times \mathbb{S}^2$. Therefore we may fix the claimed value u_0 as the very value u_0'.
Let us prove the existence of u_f. First of all we notice that the following holds:
(A) *If $p \in M$, there is no null geodesic (with respect to $(\tilde{M}, \tilde{g})$) joining p and i^+.*
Indeed, suppose there is such a geodesic γ for some $p \in M$. As is known from the general theory of causal sets in globally hyperbolic spacetimes and the structure of the boundary of $J^\pm(x)$ (e.g [Wa84]), after starting from i^+, γ must belong to $\partial J^-(i^+; \tilde{M}) \setminus \{i^+\} = \Im^+$ till it encounters its cut locus $c \in \Im^+$ where $\partial J^-(i^+; \tilde{M})$ terminates along the direction of γ. We conclude, in particular, that c is the end point on $\Im^+$ of one of the null geodesics forming $\partial J^-(i^+; \tilde{M})$. Afterwords γ leaves $\partial J^-(i^+; \tilde{M})$, enters M and reaches p. In the portion of its trip which lies on $\Im^+$, with a corresponding subset of the domain for its affine parameter $t \in (0, b]$, one has $\Omega(\gamma(t)) = 0$ for definition of $\Im^+$. Therefore $\dot{\gamma}^\mu(t) \nabla_\mu \Omega(\gamma(t)) = \dot{\gamma}^\mu(t) n_\mu(\gamma(t)) = 0$. Finally, since $\dot{\gamma}$ is null as n (and both do not vanish anywhere), it has to be $\dot{\gamma}(t) = f(t) n(\gamma(t))$ for some non vanishing smooth function f. In other words, the portion of γ contained in $\Im^+$ is, up to a re-parametrization, an integral line of n. Therefore c is the (past) end point on $\Im^+$ of one of the integral lines of n forming $\Im^+$. This is in contradiction with the requirement (5) in Definition A.1 which implies that the integral lines of n cannot have endpoints on $\Im^+$.
We pass to conclude the proof of existence of u_f. Suppose *per absurdum* that, for the compact set $K := \overline{N} \subset M$, u_f does not exist, so that the null geodesics starting from K can intersect $\Im^+$ arbitrarily close to i^+. In this case we can consider a sequence $\{\gamma_n\}$ of null geodesics through K which intersect $\Im^+$ in the corresponding points $p_n \to i^+$ as $n \to +\infty$. However the following statement[9] holds:
(B) *The existence of the mentioned geodesics $\{\gamma_n\}$ implies that there is a null geodesic γ from $K \subset M$ to i^+.*
Statement (B) is in contradiction with the statement (A), hence there is no sequence $\{\gamma_n\}$ with the claimed properties and thus u_f must exist.
To demonstrate the statement (B) consider the sequence $\{\gamma_n\}$ where the geodesics are extended maximally after i^+ and before K. Choose a $(\tilde{M}, \tilde{g})$ spacelike Cauchy surface C through i^+, and normalize the null-geodesic tangents so that they have unit inner product with the normal to C. Let x_n denote the intersection point of the null geodesic with C and and let k_n denote the normalized tangent at x_n. Then $\{(x_n, k_n)\}$ is a sequence in a compact subset of the tangent bundle, so there is a subsequence that converges to a point (x, k). Clearly $x = i^+$. Let γ be the maximally extended null geodesic individuated by (p, k) and we assume that all the used geodesics start from C with the value of the affine parameter

[9] The kind of argument to prove the statement (B) was suggested to the author by R. M. Wald.

$s_0 = 0$. Moreover, since $\tilde{M}$ is globally hyperbolic, rescaling the metric $\tilde{g}$ with a strictly positive smooth factor, we can make complete every null geodesic (Theorem 6.5 in [BGP96]), without affecting the causal structure of $\tilde{M}$. In this way we ignore problems of domains of the parameters of the geodesics. Let C' be a second Cauchy surface in the past of K. Since γ is causal, one has $\gamma(s_1) \in C'$ for some $s > 0$. Consider an auxiliary Riemannian smooth metric defined on $\tilde{M}$ and denote by d the distance associated with that metric – whose metric balls, as is known, form a base of the pre-existent topology of $\tilde{M}$ –. Using the jointly continuous dependence of maximal solutions of differential equations (in this case on $T\tilde{M}$) from the parameter describing the curves and the initial data, and exploiting the fact that continuous functions defined on a compact set are uniformly continuous, we gets easily the following statement: For every $\epsilon > 0$, there is a natural N_ϵ with $d(\gamma(s), \gamma_n(s)) < \epsilon$ for all $s \in [0, s_1]$ if $n > N_\epsilon$. It is clear that, in this way, if γ does not intersect K, one can fix ϵ in order that no γ_n meets K if $n > N_\epsilon$. This is in contradiction with the hypotheses on the curves γ_n. $\square$

Let us pass to the main statement of Proposition 4.3. Fix the standard domain $N \subset M$ and notice that, from (34), $\lambda^{(N)}(f, g)$, for $f, g \in C_0^\infty(N)$ can be written as

$$\lambda_M^{(N)}(f, g) = -\frac{1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^2} \frac{\tilde{E}(\Omega^{-3} f)(u, \zeta, \overline{\zeta}) \tilde{E}(\Omega^{-3} g)(u', \zeta, \overline{\zeta})}{\omega_B(u, \zeta, \overline{\zeta}) \omega_B(u', \zeta', \overline{\zeta}) \left(u - u' - i0^+\right)^2} \, du \wedge du' \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}) \,, \tag{57}$$

We want to rearrange (57) into a more useful expression. To this end, consider two Cauchy surfaces: S_1 in the past of $\overline{N}$ such that, considering the compact set $S_1 \cap \Im^+$, $\max_{S_1 \cap \Im^+} u \leq u_0$, S_2 in the future of $\overline{N}$ but in the past of i^+ and such that, considering the compact set $S_2 \cap \Im^+$, it holds $\min_{S_1 \cap \Im^+} u \geq u_1$. u_0 and u_f are those individuated in Lemma B.2 for the fixed standard domain N. By construction no future-directed null geodesics starting from $\overline{N}$ can meet S_1 and S_2. Let H the compact region in $J^-(i^+; \tilde{M})$ bounded by S_1 in the past and by S_2 in the future and let $\chi \in C_0^\infty(\tilde{M})$ with $1 \geq \chi \geq 0$ and $\chi(p) = 1$ in a neighborhood of H. Finally define $\chi' := 1 - \chi$. If $\tilde{E}(x, y)$ is the Schwartz kernel of $\tilde{E}$, decompose $\tilde{E}$ as

$$\tilde{E}(x, y) = \chi(x) \tilde{E}(x, y) + \chi'(x) \tilde{E}(x, y) \,. \tag{58}$$

By construction $\chi(x) E(x, y)$ has a nonempty singular support, whereas $\chi'(x) \tilde{E}(x, y)$ is a smooth kernel when $y \in N$ and $x \in J^+(\overline{N})$. Therefore $\chi' \tilde{E}$ can be restricted to $\Im^+ \times N$ without problems and it determines a smooth function. Let us consider the same issue for $\chi \tilde{E}$. $\chi \tilde{E}$ can in fact be restricted to $\Im^+ \times N$ producing distribution of $\mathcal{D}'(\Im^+ \times N)$. To show it with a judicious choice of Ω, define a coordinate system about $\Im^+$ given by coordinates $\Omega, u, \zeta, \overline{\zeta}$ [Wa84]. In these coordinates, exactly for $\Omega = 0$, i.e. on $\Im^+$, the metric of $\tilde{M}$ reads

$$-d\Omega \otimes du - du \otimes d\Omega + d\Sigma_{\mathbb{S}^2}(\zeta, \overline{\zeta}) \,, \tag{59}$$

$d\Sigma_{\mathbb{S}^2}(\zeta, \overline{\zeta})$ being the standard metric on a 2-sphere. Let $j : \Im^+ \times N \to \tilde{M} \times \tilde{M}$ be the immersion map of $\Im^+ \times N$ in $\tilde{M} \times \tilde{M}$. It reads simply $j : (\Omega, u, \zeta, \overline{\zeta}, y) \mapsto (0, u, \zeta, \overline{\zeta}, y)$ about $\Im^+$ and for $y \in N$. Therefore, the *set of normals of the map* j in the sense of Theorem 8.2.4 in [Hö89]) is (using notations as in Lemma B.1)

$$N_j = \{((x, k_x), (y, k_y)) \in T^*M \times T^*M \mid (y, k_y) \in T^*N \,, \; x \in \Im^+ \,, \; k_x = (k_x)_\Omega d\Omega \,, \; (k_x)_\Omega \in \mathbb{R}\} \,.$$

On the other hand [Ra96a]:

$$WF(\tilde{E}) = \{((x, k_x), (y, -k_y)) \in T^*M \setminus 0 \times T^*M \setminus 0 \mid (x, k_x) \sim (y, k_y)\} \,. \tag{60}$$

The condition that $k_x \in T^*M$ for $x \in \Im^+$ is null and not tangent to $\Im^+$ – because $y \in N \subset M$ and there are no null geodesics tangent to $\Im^+$ and connecting $\Im^+$ (and thus i^+) with a point in M for proposition (A) in the proof of Lemma B.2 – implies that the component $(k_x)_u$ of k_x cannot vanishing in coordinates $(\Omega, u, \zeta, \overline{\zeta})$. The proof is immediate by the expression of the metric on $\Im^+$ given above. Therefore $N_j \cap WF(\tilde{E}) = \emptyset$ and $\tilde{E}$ can be restricted to $\Im^+ \times N$ as stated in Theorem 8.2.4 in [Hö89]. The same theorem states that $WF(\tilde{E} \restriction_{\Im^+ \times N})$ is made of the pairs $((x, k_x), (y, -k_y)) \in T^*\Im^+ \setminus 0 \times T^*N \setminus 0$ such that $(k_x, k_y) = {}^t\jmath'(x, y)(h_x, h_y)$ and $(x, h_x) \sim (y, h_y)$. Using once again the form of the metric (59), and the fact that h_x is null, one sees that it must be $k_x = (k_x)_u du$ the remaining components being zero, whereas there is no restriction on the covector $k_y = h_y$. The presence of the smooth factor χ does not affect the result by the very definition of wavefront set, so that $\chi\tilde{E}$ can be restricted to $\Im^+ \times N$ and

$$WF(\chi E|_{\Im^+ \times N}) \subset \left\{ ((x, k_x), (y, -k_y)) \in T^*\Im^+ \setminus 0 \times T^*N \setminus 0 \ \mid \ (x, \widehat{k_x}) \sim (y, k_y) \ , \ (k_x)_u \neq 0 \right\} \ , \tag{61}$$

where, referring to the basis $d\Omega_x, du_x, d\zeta_x, d\overline{\zeta}_x$ of $T^*_x\tilde{M}$ and $du_x, d\zeta_x, d\overline{\zeta}_x$ of $T^*_x\Im^+$, the covector $\widehat{k_x} \in T^*_x\tilde{M}$ is that uniquely determined by $k_x \in T^*_x\Im^+ \setminus \{0\}$ and the condition $\tilde{g}(\widehat{k_x}, \widehat{k_x}) = 0$. $\widehat{k_x}$ is in fact the generic tangent vector in x of a null geodesic starting in N and reaching $\Im^+$ in x.
Let us came back to (57), it is convenient to introduce the distributions $\mathbb{E} \in \mathcal{D}'(\Im^+ \times N)$ and $\mathcal{E} \in \mathcal{D}'(\Im^+ \times N) \cap C^\infty(\Im^+ \times N)$ individuated via Schwartz kernel theorem by

$$\mathbb{E}(f)(u, \zeta, \overline{\zeta}) := \omega_B(u, \zeta, \overline{\zeta})^{-1} \chi \tilde{E}\restriction_{\Im^+ \times N} (\Omega^{-3} f)(u, \zeta, \overline{\zeta}) \ , \quad \text{for } f \in C^\infty_0(N) \ , \tag{62}$$

$$\mathcal{E}(f)(u, \zeta, \overline{\zeta}) := \omega_B(u, \zeta, \overline{\zeta})^{-1} \chi' \tilde{E}\restriction_{\Im^+ \times N} (\Omega^{-3} f)(u, \zeta, \overline{\zeta}) \ , \quad \text{for } f \in C^\infty_0(N). \tag{63}$$

The wavefront set of $\mathcal{E}$ is obviously empty, whereas as ω_B^{-1} and Ω^{-3} are smooth, from (61), we get again

$$WF(\mathbb{E}) \subset \left\{ ((x, k_x), (y, -k_y)) \in T^*\Im^+ \setminus 0 \times T^*N \setminus 0 \ \mid \ (x, \widehat{k_x}) \sim (y, k_y) \ , \ (k_x)_u \neq 0 \right\} \ . \tag{64}$$

Indicating by ω the angular coordinates $\zeta, \overline{\zeta}$ on $\Im^+$, and with $dud\omega$ the measure on $\Im^+$, $du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta})$ one has, for $h \in C^\infty_0(N)$,

$$\begin{aligned} \lambda^{(N)}_M(f, g) = & -\frac{1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^2} \frac{\mathbb{E}(f)(u, \omega)\mathbb{E}(g)(u', \omega)}{(u - u' - i0^+)^2} \, dud\omega du' - \frac{1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^2} \frac{\mathcal{E}(f)(u, \omega)\mathbb{E}(g)(u', \omega)}{(u - u' - i0^+)^2} \, dud\omega du' \\ & - \frac{1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^2} \frac{\mathbb{E}(f)(u, \omega)\mathcal{E}(g)(u', \omega)}{(u - u' - i0^+)^2} \, dud\omega du' - \frac{1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^2} \frac{\mathcal{E}(f)(u, \omega)\mathcal{E}(g)(u', \omega)}{(u - u' - i0^+)^2} \, dud\omega du' \ . \end{aligned} \tag{65}$$

Let us focus on the first term in the right-hand side of (65). First of all we notice that it is possible to replace, *without affecting the final result*, the kernel $(u - u' - i\epsilon)^{-2}$ with the compactly supported kernel $\tilde{\chi}(u, , u') \, (u - u' - i\epsilon)^{-2}$ where $\tilde{\chi} \in C^\infty_0(\mathbb{R}^2)$ attains the value constant 1 on the compact $[\min_{S_1 \cap \Im^+} u, \max_{S_2 \cap \Im^+} u] \times [\min_{S_1 \cap \Im^+} u, \max_{S_2 \cap \Im^+} u]$. The first term in the right-hand side of (65) can be re-written, formally speaking and barring the factor $-1/\pi$ as

$$\left\langle \int_{\mathbb{R} \times \mathbb{R} \times \mathbb{S}^2 \times \mathbb{S}^2} du du' d\omega d\omega' \, \frac{\tilde{\chi}(u, u')\delta(\omega, \omega')}{(u - u' - i0^+)^2} \ \mathbb{E}(\cdot)(u, \omega)\mathbb{E}(\cdot)(u', \omega') \ , \ f \otimes g \right\rangle$$

The (bi)linear functional in the left entry looks like a distribution of $\mathcal{D}'(N \times N)$ obtained by the action of the Schwartz kernel $\mathbb{E} \otimes \mathbb{E} \in \mathcal{D}'(\Im^+ \times N \times \Im^+ \times N)$ on the distribution $\tilde{\chi}T$:

$$\Delta_1(x, x') := \int_{\mathbb{R} \times \mathbb{R} \times \mathbb{S}^2 \times \mathbb{S}^2} du du' d\omega d\omega' \, (\tilde{\chi}T)(u, u', \omega, \omega')(\mathbb{E} \otimes \mathbb{E})(u, u', \omega, \omega', x, x') \ , \tag{66}$$

where $T \in \mathcal{D}'(\Im^+ \times \Im^+)$ has been introduced in Lemma B.1.
By Theorem 8.2.13 in [Hö89], such an interpretation *makes rigorous sense* and Δ_1 exists as an element of $\mathcal{D}'(N \times N)$ provided that (a) $\tilde{\chi}T$ has compact support – and this is assured by the introduction of the function $\tilde{\chi}$ which in turn may exist due to Lemma B.2 – and (b):

$$WF'(\mathbb{E} \otimes \mathbb{E})_{\Im^+ \times \Im^+} \bigcap WF(\tilde{\chi}T) = \emptyset \tag{67}$$

where, if $K \in \mathcal{D}'(X \times Y)$,

$$WF'(K)_X := \{(x, k_y) \mid ((x, -k_x), (y, 0)) \in WF(K) \quad \text{for some } x \in X\} .$$

Notice that *differently from the case examined in Theorem 8.2.13 in [Hö89], in our case the kernel $K := \mathbb{E} \otimes \mathbb{E}$ acts on the compact support function $u := \tilde{\chi}T$ via the left entry rather than the right one.* However the prof for this case is the same as that considered in the mentioned theorem with very trivial adaptations.
Let us prove that this condition (67) is fulfilled in our case. By Theorem 8.2.9 in [Hö89] we know that

$$WF(\mathbb{E} \otimes \mathbb{E}) \subset (WF(\mathbb{E}) \otimes WF(\mathbb{E})) \bigcup ((\operatorname{supp} \mathbb{E} \times \{0\}) \times WF(\mathbb{E}))$$

$$\bigcup (WF(\mathbb{E}) \times (\operatorname{supp} \mathbb{E} \times \{0\})) . \tag{68}$$

Since there are no null geodesics with vanishing tangent vector in $y \in N$ joining $x \in \Im^+$, $WF'(\mathbb{E} \otimes \mathbb{E})_{\Im^+ \times \Im^+} = \emptyset$ and so (67) turns out to be fulfilled.
Finally Theorem 8.2.13 in [Hö89] gives an inclusion of $WF(\Delta_1)$ with Δ_1 defined in (66), establishing that:

$$\begin{aligned} WF(\Delta_1) \subset \{((x, k_x), (x', k'_x)) \mid ((x, k_x), (u, \omega, -k_u, -\mathbf{k}), (x', k'_x), (u', \omega', -k'_u, -\mathbf{k}')) \in WF(\mathbb{E} \otimes \mathbb{E}) , \\ \text{for some } (u, \omega, k_u, \mathbf{k}), (u', \omega', k'_u, \mathbf{k}') \in WF(\tilde{\chi}T)\} \bigcup WF(\mathbb{E} \otimes \mathbb{E})_{N \times N} . \end{aligned} \tag{69}$$

Similarly to $WF'(\mathbb{E} \otimes \mathbb{E})_{\Im^+ \times \Im^+}$ one finds (with the same argument) $WF'(\mathbb{E} \otimes \mathbb{E})_{N \times N} = \emptyset$. Whereas the remaining part in the right hand side of (69), taking into account the inclusions (68), $WF(\tilde{\chi}T) \subset WF(T)$ and (64), exploiting (56), produces straightforwardly the final result:

$WF(\Delta_1)$ is contained in the set G of pairs $((x, k_x), (x', -k'_x)) \in T^*N \setminus \mathbf{0} \times T^*N \setminus \mathbf{0}$ such that:
(a) (x, k_x) and (x', k'_x) are points and associated cotangent vectors of the same maximal null geodesic which reaches $\Im^+$ in some point p, and
(b) k_x is future directed.

(Since the coordinate u is future directed and (59) holds, k_x is future directed if and only if, considering the geodesic γ with initial conditions $(x, k_x) \in T^*_x N$, the opposite of the covector tangent to γ in the point p where γ meets $\Im^+$ has component $(k_p)_u$ *positive*. This agrees with the condition $k > 0$ in the definition of the wavefront set of the distribution T (56) concerning the subset A, the subset B gives no contribution to G.)
Notice that, if $(x, k_x), (x', k'_x) \in G$, by construction it holds: $(x, k_x) \sim (x', k'_x)$ with $k_x \rhd 0$. Conversely, consider a pair $(x, k_x) \sim (y, k_y)$ with $k_x \rhd 0$. The maximal null geodesic passing through x and y with respective cotangent vectors k_x and k_y must achieves $\Im^+$ in some point: By known theorems the complete null geodesic cannot remain confined in the compact $J^-(i^+; \tilde{M}) \cap D^+(S)$ where S is a spacelike

Cauchy surface of $\tilde{M}$ which intersects p or q and lies in the past of the other point, so it must get out intersecting $\partial(J^-(i^+;\tilde{M})\cap D^+(S))$ in some point p. Since it cannot intersect twice S, the geodesic has to meet $\partial J^-(i^+;\tilde{M})$ somewhere. The point i^+ is forbidden as established in the proof of Lemma B.2. We conclude that the geodesics must intercept some point of $\Im^+$. We have found that if, for $x,y\in N$, $(x,k_x)\sim(y,k_y)$ with $k_x\rhd 0$, then it also holds $(x,k_x),(x',k'_x)\in G$. In other words, changing the names of the points and covectors:

$$WF(\Delta_1)\subset\{((x,k_x),(y,-k_y))\in T^*N\setminus\mathbf{0}\times T^*N\setminus\mathbf{0}\ \mid\ (x,k_x)\sim(y,k_y)\,,k_x\rhd 0\}\ . \tag{70}$$

To go on, we remind the reader that $\mathcal{E}\in C^\infty(\Im^+\times N)$ by construction. Furthermore for $\mathcal{E}(u,\omega,x)=0$ smoothly $u<u_0$. Moreover by (63) recalling that $\chi'(x,y)\tilde{E}(x,y)$ has smooth kernel when $y\in N$ and $x\in J^+(\overline{N})$ so that it is smooth for x varying in a neighborhood of i^+ when $y\in N$, we can control the behaviour as $u\to+\infty$ of $\partial_x^\alpha\mathcal{E}(u,\omega,x)$ and $\partial_x^\alpha\partial_u\mathcal{E}(u,\omega,x)$ by Lemma 4.2 for every multi-index α and for any fixed $\beta\in[1,2)$: $\partial_x^\alpha\mathcal{E}(u,\omega,x)$ and $\partial_x^\alpha\partial_u\mathcal{E}(u,\omega,x)$ are bounded, respectively, by functions of the form $M_\alpha(x)/|u-b|$ and $M_{\alpha\beta}(x)/|u-b|^\beta$. The bounds $M_\alpha(x),M_{\alpha\beta}(x)$ can be made locally uniform in x taking the sup in (44) over $B\times B'$, B' being a relatively compact neighborhood of every fixed point $x_0\in N$.
By integration by parts, the last term in the right-hand side of (65) can be re-written (omitting a constant overall factor):

$$\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^2\times\mathbb{S}^2\times N\times N}du\,du'\,d\omega\,d\mu_{\tilde{g}}(x)\,d\mu_{\tilde{g}}(x')\ \frac{\partial_u\mathcal{E}(u,\omega,x)\mathcal{E}(u',\omega,x')}{u-u'-i\epsilon}\,f(x)g(x')\,. \tag{71}$$

The functional in (71) can be rearranged by using Fubini-Tonelli and Lebesgue's dominate convergence and computing the limit under the symbol of integration, obtaining that the last term in the right-hand side of (65) is, in fact, up to an overall factor:

$$\int_{N\times N}K(x,x')f(x)g(x')\,d\omega\,d\mu_{\tilde{g}}(x)$$

where the *smooth* kernel $K(x,x')$ reads:

$$i\pi\int_{\mathbb{R}\times\mathbb{S}^2}du\,d\omega\ \mathcal{E}(u,\omega,x)\partial_u\mathcal{E}(u,\omega,x')-\int_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2}du\,du'\,d\omega\ \rho(u-u')\,\frac{\mathcal{E}(u',\omega,x')-\mathcal{E}(u,\omega,x')}{u-u'}\,\partial_u\mathcal{E}(u,\omega,x)$$
$$-\int_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2}du\,du'\,d\omega\ \rho'(u-u')\,\frac{\mathcal{E}(u',\omega,x')}{u-u'}\,\partial_u\mathcal{E}(u,\omega,x)\,,$$

$\rho':=1-\rho$ and $\rho\in C_0^\infty(\mathbb{R})$ being any, arbitrarily fixed, function which attains the value 1 constantly in a neighborhood of 0. Absolute convergence of the integrals and smoothness of $K(x,x')$ can be checked by direct inspection taking derivatives under the symbol of integration by standard theorems based on dominate convergence theorem together with the uniform bounds on the behaviour as $u\to+\infty$ mentioned above. We conclude that the last term in right-hand side of (65) gives no contribution to the wavefront set of the two-point function of $\lambda_M^{(N)}$.
To conclude let us examine the third term in the right-hand side of (65), the second can be analyzed with the same procedure obtaining the same result. As before this term can be re-arranged and the limit can be explicitly computed obtaining that third term in the right-hand side of (65) equals (up to a constant overall factor)

$$\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^2\times\mathbb{S}^2\times N}du\,du'\,d\omega\,d\mu_{\tilde{g}}(x')\,\frac{\partial_u\mathbb{E}(f)(u,\omega)\,\mathcal{E}(u',\omega,x')}{u-u'-i\epsilon}\,g(x')=\int_N d\mu_{\tilde{g}}(x')\,H(f,x')g(x')\,,$$

with, for every fixed $f \in C_0^\infty(N)$, the *smooth* function $H(f,x')$ given by:

$$i\pi \int_{\mathbb{R}\times\mathbb{S}^2\times N} du\, d\omega \quad \mathcal{E}(u,\omega,x')\partial_u \mathbb{E}(f)(u,\omega) - \oint_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2} du\, du'\, d\omega \;\; \rho(u-u')\frac{\mathcal{E}(u',\omega,x') - \mathcal{E}(u,\omega,x')}{u-u'}\partial_u \mathbb{E}(f)(u,\omega)$$

$$- \oint_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2} du\, du'\, d\omega \;\; \rho'(u-u')\frac{\mathcal{E}(u',\omega,x')}{u-u'}\partial_u \mathbb{E}(f)(u,\omega)\,.$$

As before, the function $\rho \in C_0^\infty(\mathbb{R})$ is any function with $\rho = 1$ in a neighborhood of 0. Each of the three integrals in the expression of H have form, with a suitable $\mathcal{S} \in C^\infty(\Im^+ \times N)$,

$$F(f)(x') := \int_{\mathbb{R}\times\mathbb{S}^2\times N} du\, d\omega \quad \mathcal{S}(u,\omega,x')\partial_u \mathbb{E}(f)(u,\omega)\,.$$

At least formally, one may think of $F : C_0^\infty(N) \to \mathcal{D}'(N)$ as individuated by the Schwartz kernel $F(x,x')$ composition of Schwartz kernels:

$$F(x,x') = \int_{\mathbb{R}\times\mathbb{S}^2} du\, d\omega \quad \mathcal{S}(u,\omega,x')\partial_u \mathbb{E}(u,\omega,x) \tag{72}$$

This interpretation makes rigorous sense in view of Theorem 8.2.14 of [Hö89] provided (a) the projection $\mathrm{supp}(\mathbb{E}) \ni (u,\omega,x) \mapsto x \in N$ is proper – and this can be straightforwardly verified true by the properties of the support of $\mathbb{E}$ – and (b) $WF'(\mathcal{S})_{\Im^+_{(u',\omega')}} \cap WF(\partial_u \mathbb{E})_{\Im^+_{(u',\omega')}}$ – and this is also true because $WF'(\mathcal{S})_{\Im^+_{(u',\omega')}}$ is empty since $\mathcal{S}$ is smooth whereas $WF(\partial_u \mathbb{E})_{\Im^+_{(u',\omega')}} \subset WF(\mathbb{E})_{\Im^+_{(u',\omega')}}$ which is empty as can be found by direct inspection using (64) (there are no null geodesics from N to $\Im^+$ with zero tangent vector). The inclusion given in Theorem 8.2.14 in [Hö89] states that $WF(F)$ is a subset of the union of the following sets: (1) $WF'(\mathcal{S}) \circ WF'(\partial_u \mathbb{E})$, which is empty because $WF'(\mathcal{S})$ is empty, (2) $WF(\mathcal{S})_N \times N \times \{0\}$, which is empty due to the same reason, and (3) $N \times \{0\} \times WF'(\partial_u \mathbb{E})_N$, which is empty because $WF'(\partial_u \mathbb{E})_N \subset WF'(\mathbb{E})_N$, and referring to (64), there are no null geodesics from N to $\Im^+$ with zero tangent vector.
We conclude that the second and the third term in right-hand side of (65) give no contribution to the wavefront set of the two-point function of $\lambda_M^{(N)}$. The only contribution comes from the first term and thus (49) follows from (70) □

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